

Static enthalpy equilibrium and the energetic stability of saturated granular systems (Version 2)

Corresponding Author:

Manfred Wittig

Affiliation: Former Lecturer in Applied Geotechnics,
Institute of Geotechnics, Technische Universität Bergakademie Freiberg
(University of Mining and Technology), Germany

Independent Researcher, Germany

Email: Wittig.geotechnics@icloud.com

Date: October 2025 – Version 5.0

Preprint notice: Prepared for submission to a peer-reviewed journal in geomechanics. This work is licensed under the Creative Commons Attribution 4.0 International Licence (CC BY 4.0)

Abstract

Instabilities and liquefaction phenomena in saturated sands are commonly described in terms of stress and strain, but the underlying energy balance between the granular framework and the pore water remains undefined. This study introduces a static enthalpy equilibrium as a physical condition governing the energetic stability of saturated granular systems. Equilibrium is expressed as the equality between the structural enthalpy released by the granular framework and the hydraulic work capacity of the pore water, providing a single energetic criterion that unifies spontaneous and induced liquefaction within one framework. The condition applies to all saturated granular systems where pore water forms a continuous phase, and the mechanical behaviour is governed by particle contacts. The formulation offers a concise energetic basis for interpreting stability, metastability and collapse in saturated soils.

This preprint (Version 2) is available at EarthArXiv: <https://doi.org/10.31223/X5NF3X.v2>

© 2025 Manfred Wittig – Licensed under Creative Commons Attribution 4.0 International (CC BY 4.0)

Keywords: static enthalpy equilibrium, energetic stability, saturated granular systems, energy balance, collapse initiation, metastable states, liquefaction limit

1 Introduction

The mechanical behaviour of saturated sands, from stable dense states to sudden liquefaction in loose deposits, has traditionally been described in terms of effective stress and strain.

While this approach captures the macroscopic response of soils, it does not explicitly account for the energy balance that governs the transition between stability and instability.

In particular, the interaction between the granular framework and the pore-water phase, and the way in which their energetic equilibrium determines mechanical stability, has remained unexplored.

This paper introduces a static enthalpy equilibrium as a general physical condition governing the stability of saturated sands.

The formulation defines mechanical equilibrium as the equality between the structural enthalpy stored in the grain contacts and the hydraulic work capacity of the pore water.

It thereby unifies spontaneous liquefaction, induced liquefaction and densification within a single energetic framework.

The resulting condition provides a consistent physical basis for understanding the behaviour of saturated granular systems and offers a bridge between classical soil mechanics and thermodynamic descriptions of granular materials.

2 Theoretical frameworks

The behaviour of saturated sands can be described as an exchange of energy between two coupled subsystems: the granular framework and the pore-water phase.

The granular structure stores structural energy in the form of contact and gravitational potential, while the pore water performs hydraulic work in response to changes in pore volume and pressure.

A state of static equilibrium is reached when the potential of one subsystem to perform work equals the capacity of the other to absorb it.

The energetic formulation can be derived directly from the first law of thermodynamics for an isothermal, quasi-static two-phase system.

In this case, the total energy balance reduces to the mechanical work of the solid skeleton and the pore fluid under gravity, including potential energy, hydraulic work and local dissipation.

The following expression defines the specific enthalpy $H(n)$ of the coupled system as the sum of potential and hydraulic energy per unit volume:

$$H(n) = (1 - n) \cdot \rho_s \cdot g \cdot h + n \cdot K_h \cdot \ln\left(\frac{n_0}{n}\right) \quad (1)$$

where n is the porosity, ρ_s the grain density, g the acceleration of gravity, h the representative vertical energy distance, and K_{eff} the effective bulk modulus of the saturated mixture (Houlsby and Puzrin, 2007; Collins, 2016).

All quantities have the dimension of energy per unit volume [$J\ m^{-3}$].

The first term represents the gravitational potential energy of the solid skeleton, and the second term describes the hydraulic work associated with the expulsion of water as the granular structure rearranges into a denser configuration.

In this context, the term “enthalpy” is used in its general thermodynamic sense to denote the total mechanical energy of the coupled solid–fluid system, including gravitational potential and hydraulic work, under isothermal and quasi-static conditions. It does not refer to thermal processes but expresses the combined ability of the system to perform or absorb mechanical work.

The natural logarithm expresses the relative change of porosity during this process, which is continuous and quasi-static under the given boundary conditions of the coupled fluid-solid system.

This expression represents the isothermal form of the total mechanical energy in the gravity field, in which the first term describes energy storage in the solid framework and the second term the work of the fluid phase.

Equilibrium between the two subsystems is obtained when

$$H(n_0) - H(n_f) = \rho_w \cdot g \cdot h \quad (2)$$

a condition consistent with the thermodynamic interpretation of soil instability proposed by Buscarnera & Einav (2012), who described liquefaction as an energy conversion between mechanical and hydraulic work.

The effective hydraulic modulus K_h controls the rate at which hydraulic work is performed during a change in pore volume, but it does not appear explicitly in the equilibrium condition because, at equilibrium, the total structural enthalpy and hydraulic work are already equalised.

The modulus therefore governs the evolution toward equilibrium, not the final energetic state itself.

The parameter K_h is not introduced to describe the mechanical stiffness of the granular skeleton but to represent the hydraulic coupling between the solid and fluid phases during the transition from the initial to the final equilibrium state.

While the static equilibrium condition $H(n_0) - H(n_f) = \rho_w \cdot g \cdot h$ specifies the total energy transfer between the two phases, it does not contain any information about the intermediate states through which this transfer occurs.

During collapse, the grains move downward, and the pore water is displaced upward; the pressure that develops in the fluid depends on how easily the pore volume can change, that is, on the hydraulic storage capacity of the system.

The effective hydraulic modulus K_h defines this coupling by relating incremental changes of porosity and pore pressure, $du = (K_h \cdot \frac{dn}{n})$, thereby determining the rate and spatial distribution of energy conversion and dissipation during the transient process.

Once equilibrium is re-established and the excess pore pressure has dissipated, K_h no longer contributes to the energy balance.

It thus characterises the path of the system between n_0 and n_f , not the stiffness of the solid skeleton, and ensures that the enthalpy formulation remains valid both for stable and for fluidised regimes.

The formulation assumes a homogeneous granular assembly with uniform grain density and geometry, such that the porosity n represents a representative elementary volume.

For mixtures of particles with contrasting densities or sizes, the equilibrium condition remains valid locally but may not describe the global behaviour of the heterogeneous system.

In this expression, h denotes the representative vertical energy distance, defined as the characteristic depth over which the potential energy of the granular structure is balanced by the hydraulic work of the pore water.

It is an energetic rather than a geometric measure and depends on both the grain size, which controls permeability and hydraulic response, and the grain density ρ_s , which determines the gravitational potential energy stored per unit pore volume.

Consequently, materials with higher grain density possess a larger effective energy distance, reflecting a greater capacity for gravitational work within the same porosity range.

The representative vertical energy distance h defines the integration path of the gravitational potential of the granular framework and the hydraulic work of the pore fluid within the control volume.

It represents the distance over which the gravitational body force acts and the corresponding potential difference of the pore water is integrated.

Formally,

$$W_g = \int_{z_0}^{z_0-h} (\rho_s - \rho_w) \cdot g \cdot dz = (\rho_s - \rho_w) \cdot g \cdot h \quad (3)$$

where z_0 is the reference level at the top boundary of the considered element.

The potential energy of the granular framework is therefore referenced to this level, and h acts as the characteristic vertical length over which the solid mass contributes to the total potential energy difference.

This explicit definition clarifies that h is not an empirical factor but a geometric measure of the potential drop of the coupled solid–fluid system relative to its reference elevation. In analogy to a thermodynamic phase transition, the static enthalpy equilibrium can be regarded as a stationary energetic state in which mechanical equilibrium corresponds to the condition that the first variation of the enthalpy functional is zero, while the stability of this equilibrium depends on the sign of the second variation.

The condition is general and applies to all saturated granular systems.

The formulation follows the classical variational framework of mechanics, in which equilibrium corresponds to the stationary condition of the total potential and stability is assessed through the second variation.

The enthalpy functional defined here can therefore be interpreted as the Lagrangian potential of the coupled solid–fluid system under quasi-static, isothermal conditions, consistent with the principle of virtual work and the standard energy criteria of continuum mechanics (Lagrange 1788; Love 1927; Zienkiewicz & Taylor 2000; Houlsby & Puzrin 2007; Collins 2005, 2016).

3 Limiting states – Energetic spectrum of behaviour

The static enthalpy equilibrium defines a continuous energetic spectrum that characterises all mechanical states of a saturated granular system.

Each state corresponds to a particular relationship between the structural enthalpy of the solid framework and the hydraulic work of the pore water.

Depending on whether the total energy is above, equal to, or below the equilibrium condition, the material exhibits different regimes.

Let

$$R(n) = [H(n_0) - H(n_f)] - \rho_w \cdot g \cdot h \quad (4)$$

denote the energetic residual.

3.1 Over-energetic state

If $R > 0$, the granular framework stores more potential energy than the pore-water system can absorb. Such configurations are metastable: small perturbations or hydraulic fluctuations may release the excess energy and trigger collapse.

The release of this structural enthalpy generates transient pore-pressure rises until the energetic equilibrium is restored.

3.2 Equilibrium state

For $R = 0$ the total enthalpy of the coupled system is stationary. At this point the first variation of the enthalpy functional is zero; mechanical equilibrium is satisfied, and perturbations are dissipated locally without progressive deformation.

This state corresponds to the critical porosity n_f and marks the energetic boundary between stability and instability.

3.3 Under-energetic state

If $R < 0$, the framework is energetically stable. The stored structural energy is insufficient to overcome hydraulic resistance; rearrangement requires external work, for example through vibration or loading.

Spontaneous and induced liquefaction therefore represent opposite directions of the same energetic process, distinguished only by the origin of the energy restoring equilibrium.

4 Physical interpretation – Metastability and relaxation

When the equilibrium condition $R=0$ is violated, the granular–fluid system enters a metastable state in which excess structural energy is temporarily balanced by contact friction and local pore-pressure variations.

Successive micro-collapses release energy and reduce R until the stationary condition is reached.

4.1 Second variation and stability

For the isothermal, quasi-static two-phase formulation

$$H(n) = (1 - n) \cdot \rho_s \cdot g \cdot h + n \cdot K_h \cdot \ln\left(\frac{n_0}{n}\right) \quad (5)$$

the derivatives are

$$\frac{dH}{dn} = -\rho_s \cdot g \cdot h + n \cdot K_h \cdot \left[\ln\left(\frac{n_0}{n}\right) - 1 \right], \quad \frac{dn^2}{dn^2} = -\frac{K_h}{n} \quad (6)$$

Hence

$$R'(n) = \frac{dH}{dn}, \quad R''(n) = -\frac{dn^2}{dn^2} = \frac{K_h}{n} > 0 \quad (7)$$

The positive curvature indicates local convexity of R and defines the stability range.

The three regimes follow directly from the sign of R :

$R > 0$: *over – energetic*, $R = 0$: *stationary*, $R < 0$: *under – energetic*.

4.2 Boundary conditions and latent instabilities

The boundary conditions determine the admissible variations δn : drained ($\delta n \neq 0$), undrained ($\delta n \approx 0$), displacement- or load-controlled. An energetically unstable configuration in the full space may appear stable if variations are restricted.

Buscarnera & Einav (2012) formalised this difference between material and boundary-condition-controlled instability.

In the present framework, the enthalpy equilibrium defines the stationary state, while the manifestation of instability depends on the sign of the second variation within the permitted variation space.

5 Discussion

The static enthalpy equilibrium provides a compact energetic framework that unifies spontaneous and induced liquefaction, densification and metastability of saturated granular materials.

It interprets soil behaviour as the redistribution of energy between solid and fluid phases rather than as a purely stress-path phenomenon.

5.1 Relation to classical soil mechanics

The static enthalpy equilibrium provides a unifying energetic interpretation of the classical concepts of soil mechanics.

It extends the early ideas of Terzaghi (1925) and Casagrande (1936), who implicitly considered hydraulic equilibrium and the critical void ratio as stability limits, by formulating the underlying energy balance explicitly.

At the critical porosity n_f the gravitational potential of the solid framework and the hydraulic work of the pore water are in exact balance.

The formulation therefore does not replace the traditional stress-based framework but provides its physical foundation, clarifying why stability or liquefaction occurs under given conditions of density and permeability.

Terzaghi's concept of effective stress defines how the total stress is shared between the pore water and the grain skeleton, providing the basis for all mechanical descriptions of saturated soils.

However, the effective stress alone does not determine whether a given state is stable.

Stability requires that the mechanical work stored in the granular framework remains balanced by the energy dissipated through friction and hydraulic flow.

The static enthalpy equilibrium extends Terzaghi's principle by formulating this balance explicitly and by defining the energetic condition under which the effective stress state can be maintained or lost.

Conventional soil-mechanical models describe the path toward failure based on predefined strength or critical-state parameters.

In this sense, they reproduce a condition that is already assumed within their formulation.

The static enthalpy equilibrium, by contrast, defines the limit state intrinsically, as the point where the gravitational potential of the solid framework equals the hydraulic work of the pore water.

It therefore replaces empirical calibration by a physical criterion derived directly from energy balance.

Recent advances in the energetic interpretation of soil stability have further formalised this connection between mechanics and thermodynamics.

Borja (2006) and Andrade (2009) developed predictive frameworks for liquefaction instability in saturated granular soils based on the loss of convexity of the energy potential.

Buscarnera and co-workers (Buscarnera & Di Prisco 2011, 2013; Buscarnera & Whittle 2013) generalised these concepts to drained and undrained conditions and introduced the notion of latent instabilities governed by the admissible boundary variations.

Most recently, Gajo (2025) provided a comprehensive derivation of the energy and entropy balance laws for porous media saturated by one or more non-miscible fluids at different temperatures, demonstrating that the isothermal single-fluid case considered here is a specific limit of a more general thermodynamic framework.

Within this continuum of developments, the static enthalpy equilibrium proposed in the present study can be regarded as the explicit energetic formulation of stability for saturated granular systems under isothermal, quasi-static conditions, consistent with and complementary to these earlier theories.

The present work isolates this principle and formulates it explicitly as a static enthalpy equilibrium, independent of constitutive assumptions, providing a universal energetic criterion for stability.

5.2 Worked example

Consider a 10 m saturated column of water ($\rho_w = 1000 \text{ kg m}^{-3}$).

The hydraulic work capacity is $\rho_w \cdot g \cdot h = 98.1 \cdot \text{kPa}$.

From controlled tests the structural energy release between n_0 and n is measured as $[H(n_0) - H(n)]$.

The energetic residual $R(n) = [H(n_0) - H(n)] - 98.1 \text{ kPa}$ defines the regime:

Table 1 Worked example for different cases

Case	$[H(n_0) - H(n)]$	R	Interpretation
	[kPa]	[kPa]	

A	120	+21.9	Over-energetic - collapse expected under permissive BCs
B	98.1	0	Stationary equilibrium - stability from 2nd variation
C	75	−23.1	Under-energetic - no collapse without external work

All numerical interpretations of n_f are constrained by attainable packing limits; values below n_{min} are not physically realisable and are therefore not reported.

Boundary-condition influence:

- Undrained ($\delta n \approx 0$) $\Rightarrow$ latent instabilities appear stable.
- Drained ($\delta n \neq 0$) $\Rightarrow$ metastable states become manifestly unstable

5.2.1 Effect of grain density (natural and heavy mineral sands)

In natural sands, where grain densities ρ_s typically range between 2.60 and 2.70 g cm^{−3}, the attainable porosity is mainly governed by particle shape, grading and packing fabric. Within this range, the energetic framework captures the observed behaviour by linking the gravitational energy term $(1 - n) \cdot \rho_s \cdot g \cdot h$ to the effective density contrast $(\rho_s - \rho_w)$.

The resulting equilibrium porosities predicted by the static enthalpy condition fall between approximately ($n_{min} \approx 0.33 - 0.44$), in good agreement with field and laboratory data for loose to medium-dense natural sands.

These materials represent the primary domain in which liquefaction and collapse phenomena occur under natural conditions.

For sands with substantially higher grain densities - such as heavy-mineral or industrial sands - the same energetic relationship remains valid, but the physically attainable porosity becomes limited by packing geometry rather than by the energy balance itself.

In these cases, the predicted equilibrium porosity should be interpreted within the practical bounds of random close packing ($n_{min} \approx 0.30 - 0.33$).

The framework therefore applies universally, but its practical relevance is greatest for natural, quartz-dominated sands, which occupy the energetic range most susceptible to

5.3 Dimensionality and variational formulation

Porosity and deformation are kinematically linked by $\delta n = (1 - n) \cdot \delta \varepsilon_v$.

For a vertical column (height H , cross-section A), the total energy per unit area is

$$E_{S_col}[n] = \int_0^H [(1 - n)] \cdot \rho_s \cdot g \cdot z + n \cdot K_h \cdot \ln\left(\frac{n_0}{n}\right) \cdot dz \quad (8)$$

Stationarity $E_{S_col}[n] = 0$ gives the 1D Euler–Lagrange condition; the local residual $R(z)$ and its curvature define the stability regimes.

In 3D the functional becomes

$$\Pi[u, n] = \int_{\Omega} [(1 - n)] \cdot \rho_s \cdot g \cdot z + n \cdot K_h \cdot \ln\left(\frac{n_0}{n}\right) \cdot d\Omega \quad (9)$$

subject to $\delta n = (1 - n) \cdot tr(\delta \varepsilon)$.

Here $tr(\delta \varepsilon)$ denotes the trace of the infinitesimal strain variation, corresponding to the incremental volumetric deformation $\delta\left(\frac{\Delta V}{V}\right)$.

In incompressible two-phase systems, where the solid and fluid phases are individually volume-preserving, the total volumetric strain represents the change of pore volume within the representative elementary volume.

The kinematic relation $\delta n = (1 - n) \cdot tr(\delta \varepsilon)$ therefore ensures that the porosity variation is directly coupled to the volumetric deformation, maintaining geometric and energetic consistency between the mechanical displacement field and the pore structure. Stationarity ($\delta \Pi = 0$) gives equilibrium; the second variation ($\delta^2 \Pi$) yields stability under specified boundary controls.

This provides a direct variational formulation compatible with numerical (FEM) implementation.

It should be noted that conventional numerical boundary conditions are only valid as long as the medium behaves as a continuous structure.

Once the static enthalpy equilibrium is violated and collapse initiates, the physical meaning of fixed or drained boundaries becomes questionable.

Beyond this point, the process can no longer be represented within a conventional finite-element framework; it requires a description based on

energy and momentum exchange rather than prescribed displacements or pressures.

Although advanced numerical formulations that permit large deformations can continue to produce results beyond the onset of liquefaction, such calculations no longer represent the physical process.

They merely extend the continuum assumption into a regime where the granular structure has already disintegrated.

The static enthalpy equilibrium defines precisely this transition: the point at which a mechanical description based on continuous stiffness ceases to be valid, and the process must instead be interpreted as an energetic release and reorganisation of phases.

5.4 Boundary conditions and latent instabilities

The stability of the system depends not only on energy but also on the boundary conditions defining admissible variations. A configuration that is unstable in the full space may appear stable under constraint (e.g. undrained or displacement-controlled).

The enthalpy equilibrium gives the stationary condition; the sign of the second variation under the permitted variation space defines the manifest or latent instability.

5.5 Broader implications

The static enthalpy equilibrium bridges equilibrium, metastability and instability within one consistent energetic principle, offering a physical framework applicable to liquefaction, densification and energy-based soil modelling.

The 3D functional of Eq. (12), together with the kinematic constraint, allows immediate implementation within a variational finite-element framework and establishes a direct connection between analytical theory and numerical simulation.

Although the formulation presented here is static and quasi-static in nature, it defines a general energetic framework that can also accommodate external excitations.

Dynamic or cyclic loadings, such as those generated by earthquakes or artificial vibrations, may be interpreted as external work that temporarily drives the system into the over-energetic regime ($R > 0$), from which it relaxes toward equilibrium.

The detailed analysis of such dynamic processes, including their time-dependent evolution within the enthalpy framework, will be the subject of future work.

6 Conclusions

This study introduces a static enthalpy equilibrium as a general energetic condition governing the stability of saturated granular systems.

Equilibrium is expressed as the equality between the structural enthalpy released by the granular framework and the hydraulic work capacity of the pore water.

The formulation represents the energetic foundation of soil stability and liquefaction processes and applies to all isothermal, quasi-static, saturated granular materials.

The energetic balance $H(n_0) - H(n_f) = \rho_w \cdot g \cdot h$ defines the total exchange of potential and hydraulic energy between solid and fluid phases.

Three characteristic regimes arise from the sign of the energetic residual R :

- over-energetic (loose, collapsible),
- stationary (critical),
- under-energetic (dense, stable).

Spontaneous and induced liquefaction are opposite directions of the same energetic process.

The effective hydraulic modulus K_h governs the transient coupling of porosity and pore pressure during collapse; it controls the rate and spatial distribution of energy conversion but vanishes from the final equilibrium condition.

The representative energy distance h defines the gravitational integration path, quantifying the potential drop of the coupled solid–fluid system.

The framework incorporates boundary-condition-dependent stability (latent vs. manifest instabilities) through the second variation of the enthalpy functional.

The variational form of the model provides direct implementation in numerical simulations of liquefaction and densification processes.

Ultimately, the static enthalpy equilibrium isolates the fundamental energetic principle that underlies the stability of saturated granular systems and establishes a concise physical basis for analytical, experimental and numerical developments in geomechanics.

References

- Andrade JE (2009) A predictive framework for liquefaction instability. *Géotechnique* 59(8): 673–682. <https://doi.org/10.1680/geot.7.00087>
- Borja RI (2006) Condition for liquefaction instability in fluid-saturated granular soils. *Acta Geotechnica* 1(4): 211–224. <https://doi.org/10.1007/s11440-006-0017-5>
- Buscarnera A, Einav I (2012) The yielding and liquefaction of saturated granular soils as thermodynamic instabilities. *Acta Geotechnica* 7(2): 83–106. <https://doi.org/10.1007/s11440-011-0154-3>
- Buscarnera G, Di Prisco C (2011) Stability criteria for unsaturated shallow slopes. *Géotechnique Letters* 1(4): 85–90. <https://doi.org/10.1680/geolett.11.00034>
- Buscarnera G, Di Prisco C (2013) Soil stability and flow slides in unsaturated shallow slopes: can saturation events trigger liquefaction processes? *Géotechnique* 63(10): 801–817. <https://doi.org/10.1680/geot.11.P.097>
- Buscarnera G, Whittle AJ (2013) Model prediction of static liquefaction: influence of the initial state on potential instabilities. *Journal of Geotechnical and Geoenvironmental Engineering* 139(3): 420–432. [https://doi.org/10.1061/\(ASCE\)GT.1943-5606.0000779](https://doi.org/10.1061/(ASCE)GT.1943-5606.0000779)
- Casagrande A (1936) Characteristics of cohesionless soils affecting the stability of slopes and earth fills. *Journal of the Boston Society of Civil Engineers* 23(1): 13–32
- Collins IF (2005) The concept of stored plastic work or ‘plastic potential’ in soil mechanics. *International Journal of Solids and Structures* 42(14): 4189–4207. <https://doi.org/10.1016/j.ijsolstr.2004.11.030>
- Collins IF (2016) An energy-based interpretation of rate-independent plasticity in geomaterials. *Acta Geotechnica* 11: 255–271. <https://doi.org/10.1007/s11440-015-0371-1>
- Gajo A (2025) Energy and entropy balance laws for porous media saturated by one or two non-miscible pore fluids at different temperatures. *Geomechanics for Energy and the Environment* 100731. <https://doi.org/10.1016/j.gete.2025.100731>

Houlsby GT, Puzrin AM (2007) Principles of thermomechanical energy methods in geomechanics. *Acta Geotechnica* 2: 143–160. <https://doi.org/10.1007/s11440-007-0037->

Lagrange JL (1788) *Mécanique analytique*. Paris: Veuve Dessaint (Reprinted by Dover, New York, 1987)

Love AEH (1927) *A Treatise on the Mathematical Theory of Elasticity*. Cambridge University Press, Cambridge

Terzaghi K (1925) *Erdbaumechanik auf bodenphysikalischer Grundlage*. Franz Deuticke, Vienna. ISBN 978-3-211-81214-7

Zienkiewicz OC, Taylor RL (2000) *The Finite Element Method*, 5th ed. Butterworth-Heinemann, Oxford