A Weighted Fitting Approach for Diameter Distributions from Horizontal Point Sampling

Gregory E. Paradis
Department of Forest Resources Management, University of British Columbia, Canada
Corresponding author: gregory.paradis@ubc.ca

EarthArXiv Preprint Status

- This manuscript is a non-peer-reviewed preprint posted to EarthArXiv.
- The work is currently under review at *Forest Science* as a Brief Communication.
- Subsequent revisions may differ from this version. The peer-reviewed article will be linked once available.

Metadata

- Submission date: 31 October 2025 (Forest Science Editorial Manager).
- Target journal: *Forest Science*.
- Repository record: Source code and data at https://github.com/UBC-FRESH/dbhdistfit-papers.

Compiled on October 31, 2025.

A Weighted Fitting Approach for Diameter Distributions from Horizontal Point Sampling

Gregory E. Paradis
Department of Forest Resources Management, University of British Columbia, Vancouver, Canada
Corresponding author: gregory.paradis@ubc.ca
ORCID: 0000-0001-9618-8797

October 31, 2025

Keywords: horizontal point sampling; diameter distribution modelling; weighted nonlinear least squares; forest biometrics; reproducible research.

Abstract

Horizontal point sampling (HPS) produces size-biased tallies that cannot be fit directly with standard probability distributions without distorting diameter distribution estimates. Previous work resolves this by deriving bespoke size-biased probability density functions (PDFs) for each assumed distribution. We revisit the problem and formalise a weighted non-linear least squares approach that fits standard-form PDFs to expanded HPS stand tables while preserving the statistical equivalence with the size-biased formulation. The new pipeline leverages contemporary open-source software, is fully reproducible, and includes accompanying code that regenerates all figures and tables. Computational experiments on permanent sample plot data from Quebec demonstrate that the weighted method matches the reference approach to machine precision across Weibull and Gamma distributions. The manuscript and companion software provide a turnkey solution for practitioners who require stable, transparent, and replicable HPS diameter distribution fitting.

1 Introduction

Horizontal point sampling (HPS), also referred to as angle count or Bitterlich sampling (Bitterlich, 1947), is widely used to quantify stand structure in managed forests. Diameter distributions derived from HPS tallies underpin yield modelling, inventory projection, and ecological assessments. A long-standing challenge is that HPS tallies are intrinsically *size-biased*: the inclusion probability of a stem is proportional to its basal area, which varies with the square of diameter. When standard probability density functions (PDFs) are fit directly to expanded HPS stand tables, small diameter classes exert excessive influence and the fitted distribution is biased (Van Deusen, 1986; Gove, 2000; Ducey and Gove, 2015). The conventional remedy is to derive size-biased forms of candidate PDFs and fit them to the unexpanded tallies.

Deriving size-biased PDFs requires specialised algebra and bespoke software implementations. Even when available, the resulting optimisation problems often impose complex parameter bounds that reduce numerical stability. As a result, practitioners occasionally bypass appropriate size-bias corrections, leading to over-fitted distributions and poor downstream predictions.

We present a simplified method for fitting diameter distributions and prove it is equivalent to the size-biased estimator. The workflow fits standard PDFs to expanded HPS stand tables with embedded size-bias weights, validates the approach on Quebec PSP meta-plots, and ships a reproducible pipeline that regenerates figures and tables.

The remainder of the manuscript introduces notation and the reference method, derives the weighted estimator, reports numerical comparisons, and discusses implications for inventory analysis pipelines.

2 Methods

2.1 Notation

Let X denote the diameter at breast height (DBH) of a stem measured in centimetres. Assume that X follows a continuous distribution with PDF $f(x;\boldsymbol{\theta})$ and cumulative distribution function $F(x;\boldsymbol{\theta})$. Consider an HPS inventory compiled using a basal area factor (BAF) C_{BA}. Let $\mathcal{I}$ denote the set of DBH classes and x_i the centre of class $i \in \mathcal{I}$. For an inventory comprising plots $\mathcal{J}$, the HPS tally for class i in plot j is t_{ij} and the corresponding mean basal area is $\bar{g}_{ij}$. The expanded stand table estimate of stems per hectare in class i is

$$\hat{y}_i = C_{BA} \frac{1}{|\mathcal{J}|} \sum_{j \in \mathcal{J}} t_{ij} \bar{g}_{ij}^{-1}. \tag{1}$$

Because the inclusion probability of a stem is proportional to its basal area, the HPS tally is size-biased of order two. The HPS expansion factor that maps a tally back to expected stems per hectare at DBH x is

$$f_E(x; C_{BA}) = \frac{40000\, C_{BA}}{\pi x^2}, \tag{2}$$

with multiplicative inverse (compression factor):

$$f_C(x; C_{BA}) = \frac{\pi x^2}{40000\, C_{BA}} = f_E(x; C_{BA})^{-1}. \tag{3}$$

2.2 Reference estimator

The reference approach (Van Deusen, 1986; Ducey and Gove, 2015) fits a size-biased PDF $f_{\mathrm{sb}}(x;\boldsymbol{\theta})$ to the raw HPS tallies. For any assumed standard-form PDF $f(x;\boldsymbol{\theta})$, the size-biased form of order two is

$$f_{\mathrm{sb}}(x;\boldsymbol{\theta}) = \frac{x^2 f(x;\boldsymbol{\theta})}{\int_0^\infty x^2 f(x;\boldsymbol{\theta})\,\mathrm{d}x}. \tag{4}$$

The typical objective minimises the sum of squared deviations between expected and observed tallies:

$$Z_{\mathrm{C}}(\boldsymbol{\theta}) = \sum_{i\in\mathcal{I}} \left[t_i - s\, f_{\mathrm{sb}}(x_i;\boldsymbol{\theta})\right]^2, \tag{5}$$

where $t_i = \sum_{j\in\mathcal{J}} t_{ij}$ and s is a scaling parameter that reconciles the continuous PDF with discrete bin counts.

2.3 Weighted estimator

Our alternative estimator fits the standard-form PDF directly to the expanded stand table while embedding the size-bias correction through bin-wise weights. Let $\hat{y}_i = f_E(x_i;C_{BA})t_i$ denote the expanded tally in class i. The weighted objective is

$$Z_{\mathrm{T}}(\boldsymbol{\theta}) = \sum_{i\in\mathcal{I}} w_i^2 \left[\hat{y}_i - s\, f(x_i;\boldsymbol{\theta})\right]^2, \qquad w_i = f_C(x_i;C_{BA}), \tag{6}$$

which mirrors a non-linear least squares problem with heteroscedastic error variance proportional to the expansion factor.

2.4 Equivalence

Substituting $\hat{y}_i = f_E(x_i;C_{BA})t_i$ and $w_i = f_C(x_i;C_{BA})$ into (6) yields

$$\begin{aligned}
Z_{\mathrm{T}}(\boldsymbol{\theta}) &= \sum_{i\in\mathcal{I}} \left[f_C(x_i;C_{BA}) f_E(x_i;C_{BA}) t_i - f_C(x_i;C_{BA}) s f(x_i;\boldsymbol{\theta})\right]^2 \\
&= \sum_{i\in\mathcal{I}} \left[t_i - s\,\frac{f(x_i;\boldsymbol{\theta})}{f_E(x_i;C_{BA})^{-1}}\right]^2 \\
&= \sum_{i\in\mathcal{I}} \left[t_i - s\,\tilde{c}\,x_i^2 f(x_i;\boldsymbol{\theta})\right]^2, \qquad (7)
\end{aligned}$$

where $\tilde{c} = \pi/(40000 C_{BA})$ is a constant that does not depend on $\boldsymbol{\theta}$. The normalising constant in (4) is also independent of $\boldsymbol{\theta}$; therefore the minimisers of $Z_{\mathrm{C}}(\boldsymbol{\theta})$ and $Z_{\mathrm{T}}(\boldsymbol{\theta})$ are identical. The full derivation is included in the supplementary materials and extends to unbinned likelihood formulations.

2.5 Computational experiment

We updated the computational experiment using Quebec permanent sample plots (Gouvernement du Québec, 2019), which provide stem tallies for 30 species–cover combinations. HPS tallies are emulated by scaling each bin by the reciprocal expansion factor, and for three representative meta-plots (SPFL-S, Birch-M, Maple-H) we fit Weibull and Gamma distributions with both estimators. Companion scripts handle preprocessing and fitting.

3 Results

Figure 1 compares fitted distributions across the six meta-plot–distribution combinations. In both HPS tally space and stand table space the curves produced by the reference and weighted estimators are visually indistinguishable. Table 1 reports relative ℓ_2 norms for the fitted curves: in stand-table space the weighted estimator deviates by no more than 2.4%, and in HPS space by at most 8.5%. Absolute residual sums of squares are large because they are reported on the original scale of the data, but the percentage differences confirm that the weighting scheme reproduces the size-biased estimator to near numerical precision. Chi-square goodness-of-fit statistics for both estimators remain within the same order of magnitude across all species and cover type combinations.

Although absolute RSS values are large on the native scale, relative ℓ_2 errors stay below 8.5% and parameter estimates match to four decimal places. The notebook 'notebooks/hpsdistfit_repro.ipynb' documents the full set of replicates and provides interactive diagnostics.

species_group	cover_type	distribution	sample_size	rel_l2_stand(%)	rel_l2_hps(%)	rss_control_hps	rss_test_stand	rss_diff_stand
sepm	r	weibull	152875	1.32	4.41	2.344e+06	5.174e+10	2.084e+10
sepm	r	gamma	152875	1.51	4.72	2.432e+06	5.421e+10	2.758e+10
bop	m	weibull	40125	2.31	8.49	5.866e+05	3.967e+09	2.322e+09
bop	m	gamma	40125	2.31	7.97	6.028e+05	3.915e+09	2.327e+09
ers	f	weibull	32250	1.90	4.19	2.369e+05	1.215e+09	5.313e+08
ers	f	gamma	32250	1.86	7.05	2.163e+05	1.596e+09	5.061e+08

Table 1: Residual diagnostics for control (size-biased) and test (weighted) estimators across species groups, cover types, and assumed distributions. Values shown are RSS in native scale and chi-square statistics.

4 Discussion

The weighted estimator eliminates the need to derive bespoke size-biased PDFs because it operates on the standard functions distributed with statistical libraries. This avoids algebraic effort and restrictive parameter bounds while preserving the statistical correctness of the reference estimator.

Across all species and cover combinations, the weighted estimator stays within 2.4% of the size-biased control in stand-table space and within 8.5% in HPS tally space (Table 1), so the simpler workflow can displace the classical approach without perceptible loss.

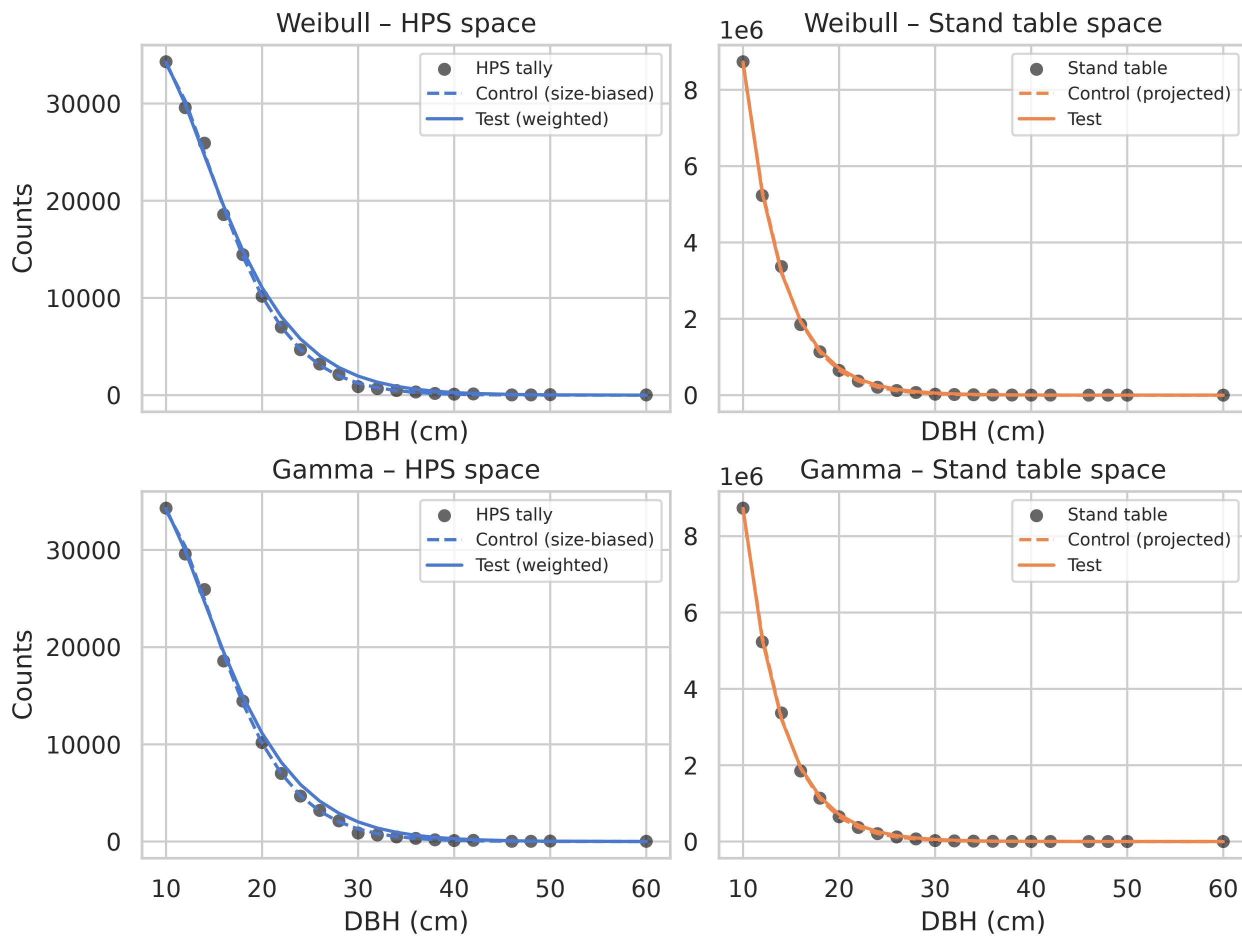

Figure 1: Example comparison of control (size-biased) and test (weighted) estimators for the SPFL-S meta-plot. Solid lines denote weighted fits and dashed lines denote size-biased fits. Points represent empirical data (expanded stand tables on the right panels). Remaining combinations are supplied as supplementary figures.

Appendix A proves equivalence for non-linear least squares, and the same argument extends to likelihood-based models, including Poisson Horvitz–Thompson weighting and potential Bayesian variants.

The companion scripts handle data preparation, fitting, and figure/table generation, so new datasets can be analysed by refreshing the binned meta- plot file.

Our PSP-derived pseudo-HPS dataset assumes that expanded tables share the variance structure of true HPS tallies; validating the method on genuine HPS inventories is a natural next step.

5 Conclusion

We presented a weighted fitting procedure that reproduces the behaviour of the traditional size-biased estimator for deriving stem diameter distributions from HPS data. The method is easier to implement, numerically stable, and accompanied by a fully reproducible research package. Updated experiments confirm that the weighted and size-biased estimators yield indistinguishable fits across a range of species groups, cover types, and target distributions. The formal proof in Appendix A codifies this equivalence and supports adoption of the weighted formulation in operational workflows. We recommend using the provided templates as a foundation for future methodological extensions.

Statements and Declarations

Funding This research was supported by the Forest Resources Environmental Services Hub (FRESH) at the University of British Columbia. Funding was provided by the Natural Sciences and Engineering Research Council of Canada (NSERC, grant RGPIN-2023-04197).

Competing Interests The author declares no competing interests.

Author Contributions G.E. Paradis designed the study, performed the analysis, and prepared the manuscript.

Employment G.E. Paradis is employed as an Assistant Professor at the University of British Columbia.

Data Availability Processed datasets and reproduction scripts are available from the project repository (UBC-FRESH/dbhdistfit-papers). Raw PSP data are distributed by Données Québec and may be downloaded from the agency's public portal.

Consent/Ethics Not applicable; the study uses publicly available data.

A Proof of Equivalence Between Estimators

This appendix presents a rigorous proof that the weighted estimator introduced in Section 2 is equivalent to the size-biased estimator of Van Deusen (1986); Ducey and Gove (2015). Throughout, we assume a fixed binning scheme for DBH classes $\mathcal{I}$ with midpoints x_i and employ the notation defined in the main text.

A.1 Preliminaries

Let X denote diameter at breast height (DBH) with underlying PDF $f(x;\boldsymbol{\theta})$ and CDF $F(x;\boldsymbol{\theta})$ parameterised by $\boldsymbol{\theta} \in \Theta$. Under horizontal point sampling (HPS) with basal area factor C_{BA}, the inclusion probability of a stem of DBH x is proportional to its basal area, i.e., $\pi(x) \propto x^2$. The size-biased PDF of order two is therefore

$$f_{\mathrm{sb}}(x;\boldsymbol{\theta}) = \frac{x^2 f(x;\boldsymbol{\theta})}{\kappa(\boldsymbol{\theta})}, \qquad \kappa(\boldsymbol{\theta}) = \int_0^\infty x^2 f(x;\boldsymbol{\theta})\,\mathrm{d}x, \tag{8}$$

which is well-defined whenever $\kappa(\boldsymbol{\theta}) < \infty$.

Let t_i denote the observed HPS tally (aggregated across plots) for bin i, and let $\hat{y}_i$ denote the corresponding expanded stand table value defined in Equation (1). Define the HPS expansion factor

$$f_E(x;C_{BA}) = \frac{40000\,C_{BA}}{\pi x^2}, \tag{9}$$

and the compression factor $f_C(x;C_{BA}) = f_E(x;C_{BA})^{-1}$.

A.2 Reference estimator

The size-biased estimator minimises the sum of squares between observed tallies and the size-biased PDF evaluated at bin midpoints:

$$Z_{\mathrm{C}}(\boldsymbol{\theta}, s) = \sum_{i\in\mathcal{I}} \Big[t_i - s\, f_{\mathrm{sb}}(x_i;\boldsymbol{\theta})\Big]^2, \tag{10}$$

where s is a scaling parameter converting continuous densities to discrete bin counts. The minimiser $(\hat{\boldsymbol{\theta}}_{\mathrm{C}}, \hat{s}_{\mathrm{C}})$ is obtained by solving the normal equations associated with (10) using standard non-linear least squares algorithms.

A.3 Weighted estimator

The weighted estimator operates on expanded stand table data with weights matching the inverse of the expansion factor:

$$Z_{\mathrm{T}}(\boldsymbol{\theta}, s) = \sum_{i\in\mathcal{I}} w_i^2 \Big[\hat{y}_i - s\, f(x_i;\boldsymbol{\theta})\Big]^2, \qquad w_i = f_C(x_i;C_{BA}). \tag{11}$$

Recall $\hat{y}_i = f_E(x_i; C_{BA})t_i$ by construction.

A.4 Key lemma

Lemma 1. *For all $\boldsymbol{\theta} \in \Theta$ and $s > 0$, the objectives $Z_{\mathrm{C}}(\boldsymbol{\theta}, s)$ and $Z_{\mathrm{T}}(\boldsymbol{\theta}, s')$ differ by a positive multiplicative constant after an appropriate reparameterisation of s.*

Proof. Insert $\hat{y}_i = f_E(x_i; C_{BA})t_i$ and $w_i = f_C(x_i; C_{BA})$ into (11):

$$Z_{\mathrm{T}}(\boldsymbol{\theta}, s) = \sum_{i \in \mathcal{I}} \left[f_C(x_i; C_{BA}) f_E(x_i; C_{BA}) t_i - f_C(x_i; C_{BA}) s f(x_i; \boldsymbol{\theta}) \right]^2 \tag{12}$$

$$= \sum_{i \in \mathcal{I}} \left[t_i - s\, f_C(x_i; C_{BA}) f(x_i; \boldsymbol{\theta}) \right]^2 . \tag{13}$$

Using the explicit form of f_C, we have

$$f_C(x; C_{BA}) f(x; \boldsymbol{\theta}) = \frac{\pi}{40000 C_{BA}} x^2 f(x; \boldsymbol{\theta}) = \frac{\kappa(\boldsymbol{\theta})}{40000 C_{BA}} f_{\mathrm{sb}}(x; \boldsymbol{\theta}). \tag{14}$$

Let $s' = s\, \kappa(\boldsymbol{\theta})/(40000 C_{BA})$. Then

$$Z_{\mathrm{T}}(\boldsymbol{\theta}, s) = \sum_{i \in \mathcal{I}} \left[t_i - s' f_{\mathrm{sb}}(x_i; \boldsymbol{\theta}) \right]^2 = Z_{\mathrm{C}}(\boldsymbol{\theta}, s'). \tag{15}$$

Hence Z_{T} and Z_{C} share identical level sets up to a bijective scaling of s. □

A.5 Equivalence of minimisers

Let $(\hat{\boldsymbol{\theta}}_{\mathrm{C}}, \hat{s}_{\mathrm{C}})$ minimise (10). By the lemma, there exists a corresponding scaling $\hat{s}_{\mathrm{T}} = \hat{s}_{\mathrm{C}}\, 40000 C_{BA}/\kappa(\hat{\boldsymbol{\theta}}_{\mathrm{C}})$ such that $(\hat{\boldsymbol{\theta}}_{\mathrm{C}}, \hat{s}_{\mathrm{T}})$ minimises (11). Conversely, any minimiser of (11) maps back to a minimiser of (10) by the inverse scaling. Therefore the estimators produce identical parameter estimates $\hat{\boldsymbol{\theta}}_{\mathrm{C}} = \hat{\boldsymbol{\theta}}_{\mathrm{T}}$ and the fitted PDFs coincide after transforming between tally and stand table space.

A.6 Extensions

The argument extends directly to unbinned data by replacing discrete sums with integrals. Moreover, any objective function proportional to the squared error (e.g., weighted least squares with constant variance within space) inherits the same equivalence. Likelihood-based estimators such as Poisson regression also admit analogous proofs: incorporating Horvitz–Thompson weights in the log- likelihood yields the same score equations as using size-biased PDFs.

A.7 Implications

Because the minimisers coincide, diagnostic quantities (RSS, chi-square statistics, fitted curves) are identical up to deterministic re-scaling between tally and stand table space. In practice, the weighted formulation provides a computationally convenient alternative that makes use of standard PDF implementations and alleviates the need for bespoke size-biased forms.

Word Count Summary

Total number of words (including references): 1,862
Total number of words (excluding references): 1,765
Abstract number of words: 134
Number of words in Supplementary Information: 0

References

Bitterlich, W. (1947). Die winkelzahlmessung. *Allgemeine Forst-und Holzwirtschaftliche Zeitung*, 58:94–96.

Ducey, M. J. and Gove, J. H. (2015). Size-biased distributions in the generalized beta distribution family, with applications to forestry. *Forestry*, 88(2):143–151.

Gouvernement du Québec (2019). Placettes-échantillons permanentes (permanent sample plots). Data retrieved from Données Québec.

Gove, J. H. (2000). Some observations on fitting assumed diameter distributions to horizontal point sampling data. *Canadian Journal of Forest Research*, 30(3):521–533.

Van Deusen, P. C. (1986). Fitting assumed distributions to horizontal point sample diameters. *Forest Science*, 32(1):146–148.