Gutenberg-Richter-type earthquake size distributions: maximum likelihood estimation, unbiased estimation, and Bayesian forecasting

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Abstract

Characterizing earthquake size distributions using the Gutenberg-Richter (GR) law is ubiquitous in seismology. According to the GR law, earthquake magnitudes follow an exponential distribution, with a rate parameter commonly represented by the b-value. For many applications, including seismic hazard and risk assessment, estimating the b-value is therefore a common procedure. However, the parameter estimation is usually only a means to an end, as ultimately, the earthquake size distribution itself is the quantity of interest. We show that the typical approach of estimating the b-value parameter and plugging the estimate into the GR law is suboptimal and yields a biased size distribution. As an alternative, we derive an unbiased estimator for the distribution itself that also has a lower variance than the plug-in distribution. Both estimation approaches may be characterized as point estimations. From a forecasting perspective, however, and for hazard and risk assessment in particular, using point estimates is inadequate, as that does not properly account for model uncertainty. Ideally, one would take into account all plausible size distributions to the extent that they are consistent with the observations. Therefore, in a forecasting context we advise to use Bayesian inference to obtain the posterior predictive size distribution. We show that if the prior belief for the b-value is expressed as a Gamma distribution, a conjugate Gamma posterior distribution and closed-form expression for the posterior predictive earthquake size distribution are available, negating the need for numerically evaluating likelihood expressions and integrating over a posterior.

Key words

Earthquake parameterization; earthquake interaction, forecasting, and prediction; probabilistic forecasting, Bayesian inference

Introduction

Characterizing earthquake size distributions using a Gutenberg-Richter (GR) law (Gutenberg & Richter, 1941, 1944) is ubiquitous in seismology. According to the GR law, earthquake magnitudes are distributed exponentially, with survival function:

$$S(m) = P(M \geq m) = 10^{-b(m-m_0)} = e^{-b^*(m-m_0)}, m \geq m_0 \tag{1}$$

with $b^* = b \ln 10$, and m_0 a minimum magnitude, typically a magnitude of catalogue completeness or minimum magnitude of interest. Its corresponding probability density function:

$$f(m) = -\frac{dS}{dm} = b^* e^{-b^*(m-m_0)} \tag{2}$$

In many seismicity studies, it is common to estimate the parameter b^* (or equivalently the 'b-value', where $b = b^* \log_{10}(e)$) based on the observed earthquake catalogue, and to use this parameter estimate to forecast the expected future behavior of the system (e.g. Weichert, 1980; Schorlemmer et al. 2004; Hirose and Maeda, 2011; Taroni et al., 2021; Rollins et al., 2024). Most commonly, this is done using the maximum-likelihood estimator (MLE) of the b-value given as (Aki, 1965; Utsu, 1965):

$$b_{MLE} = \frac{\log_{10}(e) \times n}{T}, \qquad \text{or} \quad b^*_{MLE} = \frac{n}{T} \tag{3}$$

which is then 'plugged' into the expression for the earthquake size distribution:

$$S^{plug-in}(m) = e^{-\,b^*_{MLE}\,(m-m_0)} = e^{-\frac{n}{T}(m-m_0)} \tag{4}$$

where $T = \sum_{i=1}^{n}(M_i - m_0)$ the sum of all observed magnitudes $\{M_1, \dots, M_n\}$, relative to the minimum magnitude m_0. The parameter point estimate b^*_{MLE} is asymptotically unbiased for infinite catalogue sizes, meaning that $\lim_{n\to\infty} \mathbb{E}(b^*_{MLE}) = b^*_{true}$. For real catalogues of finite size, the bias in b^*_{MLE} is known, and can easily be corrected for such that $\mathbb{E}(\widetilde{b^*}) = b^*_{true}$ for all n (Ogata and Yamachina, 1998):

$$\widetilde{b^*} = {b^*}_{MLE} \frac{n-1}{n} = \frac{n-1}{T} \tag{5}$$

giving:

$$S^{plug-in\,corrected}(m) = e^{-\,\widetilde{b^*}\,(m-m_0)} = e^{-\frac{n-1}{T}(m-m_0)} \tag{6}$$

Crucially, in many instances, and particularly in the context of seismic hazard and risk analysis (SHRA), estimating b^* is only a means to an end. The earthquake size distribution itself, i.e. $S(m)$ or equivalently $f(m)$, is the true quantity of interest, since it is the distribution of earthquakes of the magnitudes that determines seismic hazard and risk. Therefore, we turn our attention to the question of how to best describe the earthquake distribution itself, rather than its defining parameter.

Characterization of GR distribution: estimation and prediction

Identically to the parameter estimation problem described before, we assume that the earthquake-generating system of interest produces earthquakes $\{M_1, \dots, M_n\}$ which are characterized by a distribution with a known form (exponential) and with a fixed, but unknown, parameter b^*_{true}. The data is assumed to be complete above m_0, unrounded, and free of measuring error. In the context of seismic hazard and risk assessment, our primary objective is to minimize the discrepancy between an estimated magnitude distribution, $S^{est}(m)$, and the underlying true distribution $S^{true(m)}$. To quantify this performance, we define an estimation risk R^{est}, based on a squared error loss function. This risk represents the expected loss over an infinite ensemble of potential datasets generated by the same physical system:

$$R^{est}\left(S^{est}(m)\right) = \mathbb{E}\left(\left(S^{est}(m) - S^{true}(m)\right)^2\right) = \mathrm{Bias}^2\left(S^{est}(m)\right) + \mathrm{Var}\left(S^{est}(m)\right) \quad (7)$$

This "risk-minimization" framework, decomposes the total error into two distinct and desirable statistical properties: accuracy (lack of bias) and precision (low variance). Under this paradigm, a "sensible" estimator for earthquake size distributions is one that balances these two components. Ideally, the estimator should be unbiased, where the expectation of the estimate equals the true value for all magnitudes: $\mathbb{E}\left(S^{estimated}(m)\right) - S^{true}(m) = 0$. Simultaneously, it should be efficient, meaning the variance is minimized as much as possible. This approach assumes a frequentist perspective: we treat the earthquake-generating system as having a fixed functional form governed by a true but unknown parameter. The inherent aleatory variability of the system implies that if we were to repeatedly sample this distribution, we would obtain an ensemble of datasets; our goal is to ensure that, across this ensemble, the estimation risk is minimized.

Using the desired property of unbiasedness, we derive an unbiased estimator of the earthquake size distribution itself (see Appendix A for the derivation):

$$S^{unbiased}(m) = \left(1 - \frac{m - m_0}{T}\right)^{n-1} = \left(\frac{n - b^*_{MLE}(m - m_0)}{n}\right)^{n-1} \quad (8)$$

with support $T \geq m - m_0 \geq 0$, and $T = \sum_{i=1}^{n}(M_i - m_0)$ as before. For the exponential distribution, T is the complete sufficient statistic (Casella & Berger, 2002). Sufficiency refers to the fact that the likelihood for b^* depends on the data only through T. Because T is sufficient, and the unbiased estimator is a function of T, it is a Minimum-Variance Unbiased Estimator (MVUE) of $S^{true}(m)$; no other unbiased estimator exists that has lower variance (Casella & Berger, 2002). In fact, because T is also complete (a stronger property than sufficiency and not guaranteed even when the likelihood depends on the data only through a single summary statistic, see Casella & Berger (2002) for details) by the Lehmann-Scheffé theorem (Lehmann & Scheffé, 1950) it is also unique, i.e., the Unique Minimum-Variance Unbiased Estimator (UMVUE). This means that the unbiased estimator in Eq. 8 has the lowest possible pointwise estimation risk under squared error among all unbiased estimators. Note that this doesn't mean that there are no estimators which do not have lower estimation risk for some values of m and b^*_{true}. For example, the 'estimator' $S^{alt} = \exp\left(-\ln 10 \ (m - m_0)\right)$ has zero squared error loss for all m, so long as b^*_{true} happens to be $\ln 10$. However, if we require our estimator to be unbiased for all values of m and b^*_{true} , the estimator in Eq. 8 is optimal. While it is theoretically possible to reduce total estimation risk by accepting some bias in exchange for lower variance, in the next section we will show that the unbiased estimator derived here also has (much) lower variance than the plug-in estimators, particularly in the high-magnitude tail.

A different way to think about characterization of the earthquake distribution is to consider the task of forecasting future observations: we want to describe $P(M_{new} > m \,|\{M_1, \ldots, M_n\})$. Forecasting inherently requires accounting for two sources of uncertainty: 1) the (aleatory) randomness that the exponential imposes on M_{new}, even if we knew b^*_{true}, 2) the (epistemic) uncertainty that comes from not knowing b^*_{true}. Only accounting for aleatoric variability, the best forecaster under any reasonable loss function would of course be $S^{forecast}(m) = \exp(-b^*_{true}(m - m_0)$. However, we cannot use this because we don't know b^*_{true}. The next natural step is therefore to find the distribution that performs best on average over all plausible values of b^*. In other words, we're searching for $S^{forecast}$ which minimizes $R^{forecast}$ over the weighted distribution b^*:

$$S^{forecast}_{best}(m) = \underset{S(m)}{\arg\min} \int R^{est}(S(m); b^*) w(b^*) db^* \quad (9)$$

where $w(b^*)$ is a weighting function assigning weights to each plausible value of b^*. With prior distribution $w(b^*) = \pi(b^*)$ this is Bayes risk. Under a given prior belief, $S^{forecast}_{best}$ is in fact the Bayesian posterior predictive (Aitchison & Dunsmore, 1975). The posterior predictive can be numerically evaluated for any choice of prior. However, since we're considering an exponentially distributed quantity, if we – for convenience – choose our prior to be distributed according to a Gamma distribution, the posterior distribution is also Gamma distributed (conjugate prior/posterior pair) and the posterior predictive is given by a Pareto type II distribution. This posterior predictive distribution can be described analytically (Galanis et al., 2002). Specifically, if we choose a prior for b^* according to:

$$b^* \sim \mathrm{Gamma}(\mathrm{shape} = a_0, \mathrm{rate} = \lambda_0) \quad (10)$$

the posterior after observing n events with $T = \sum_{i=1}^{n}(M_i - m_0)$ becomes:

$$b^* | \, T \sim \mathrm{Gamma}(a_0 + n, \lambda_0 + T) \quad (11)$$

and the posterior predictive is given by the closed form expression:

$$S^{posterior\_predictive}_{Gamma}(m) = \left(\frac{\lambda_0 + T}{\lambda_0 + T + m - m_0}\right)^{a_0 + n} \quad (12)$$

The transformation from prior distribution (Eq. 10) to posterior distribution (Eq. 11) shows that the parameters a_0 and λ_0 effectively represent pseudo-observations. In the special limit of zero pseudo-observations ($a_0 = 0, \lambda_0 = 0$) we obtain Jeffreys prior (Jeffreys, 1946), which are often referred to as 'uninformative' although this term is somewhat misleading since it clearly carries information about

b^*. ‘Uninformative’ in this sense simply means that the prior is unaffected by parameter transformation. For this special case (see Appendix B for detailed derivation):

$$b^* \sim \text{Gamma}(0, 0), (\text{improper prior}) \tag{13}$$

$$b^*|T \sim \text{Gamma}(n, T) \tag{14}$$

and posterior predictive:

$$S_{Jeffreys}^{posterior\_predictive}(m) = \left(\frac{T}{T + m - m_0}\right)^n = \left(1 + \frac{m - m_0}{T}\right)^{-n} \tag{15}$$

Comparing behavior of the different earthquake size predictive distributions

We have defined four expressions to describe the earthquake size distribution, conditional on a dataset characterized by $\{n, T\}$, all of which are based on the underlying assumption that the data is generated by a Gutenberg-Richter process with a parameter b_{true}^*:

1. Plug-in MLE estimator: $S^{plug-in}(m) = e^{-\frac{n}{T}(m-m_0)}$.
2. Corrected plug-in MLE estimator: $S^{plug-in\ corrected}(m) = e^{-\frac{n-1}{T}(m-m_0)}$.
3. Unbiased estimator: $S^{unbiased}(m) = \left(1 - \frac{m-m_0}{T}\right)^{n-1}$
4. Bayesian posterior predictive: $S_{Gamma}^{posterior\_predictive}(m) = \left(\frac{\lambda_0+T}{\lambda_0+T+m-m_0}\right)^{a_0+n}$

Throughout, we use the term ‘predictive distribution’ to denote any probability distribution over future earthquake magnitudes conditional on past observations, regardless of whether it is derived via plug-in estimation, unbiased estimation, or Bayesian posterior prediction. Here ‘prediction’ refers exclusively to probabilistic forecasting of future observations, not to deterministic prediction of individual earthquake events. For the Bayesian posterior predictive we employ the Jeffreys prior, setting $a_0 = 0, \lambda_0 = 0$, which means all predictive distributions only rely on n, T, m_0, which are the same for any given catalogue.

Both the bias (Appendix C) and the variance (Appendix D) of our different predictive distributions can be expressed analytically or obtained through simulations of catalogues (Figure 2). We see that for magnitudes sufficiently above m_0, the unbiased estimator also has the lowest variance. At very low magnitudes, the variance of all predictive distributions is low, and the unbiased estimator does not always have the lowest variance. We derive the 'cross-over' magnitude above which the unbiased estimator is in fact also has the lowest variance (Appendix D).

For $S^{plug-in}(m)$, it is simple to show that the expected magnitude distribution, $\mathbb{E}\left(S^{plug-in}(m)\right) \geq S^{true}(m)$, for $m > m_{crit}$ where $m_{crit} \approx \frac{2}{b^*_{true}}\frac{(n-2)}{(n-1)} + m_0$ (which rapidly goes to $m_{crit} \approx \frac{2}{b^*_{true}} + m_0$ for increasing n). This means that the often employed inference of the earthquake size distribution based on obtaining b^*_{MLE} and plugging it into the GR relation is expected to result in overestimating the probability of larger events (in fact, for a 'typical' $b_{true} = 1.0 \rightarrow b^*_{true} = \ln\ 10$, the overestimation starts at $m_0 + 0.87$). This can also be confirmed using simulations from a known ground truth (Figure 1). Interestingly, using the bias-corrected distribution parameter $\widetilde{b^*}$ only exacerbates this overestimation, since the correction term $\frac{n-1}{n}$ decreases the slope of the survival function and therefore leads to further overestimation of the true magnitude exceedance probability (Figure 1 and Appendix C).

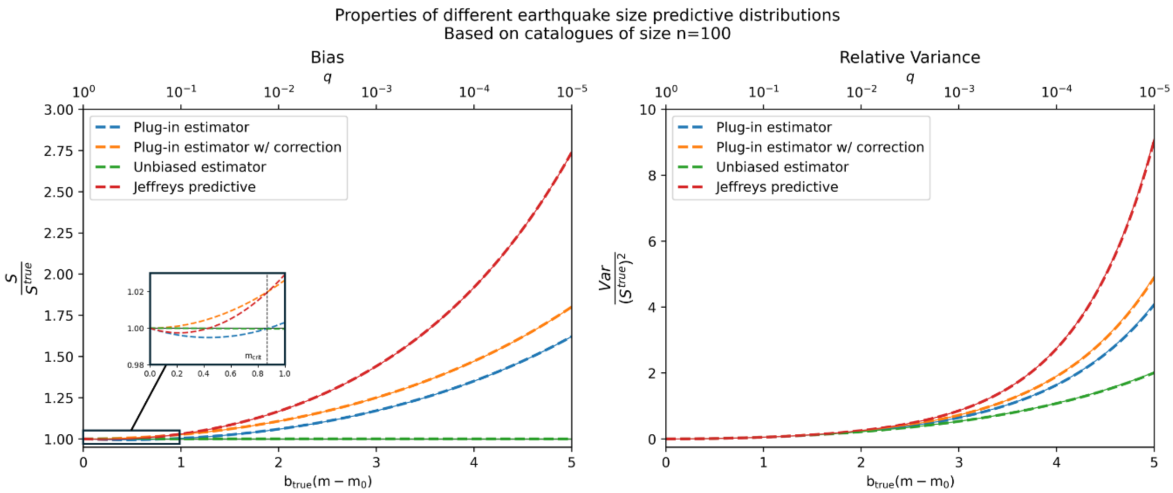

Figure 1: Bias and variance behavior of the different predictive distributions. The thin lines are based on the analytical expressions while the thick dashed lines are based on 1 million simulated catalogues. Note that the x-axis is defined as $b_{true}(m - m_0)$, or equivalently at the top, in terms of true single-event survival probability q.

In other words, obtaining an unbiased estimate of the distribution parameter does not result in an unbiased estimate of the distribution of interest when used as a plug-in. This bias is a specific example of a more general fact: unbiasedness does not survive nonlinear transformations. The exponential survival function $e^{-b^*(m-m_0)}$ is a convex function of the parameter b^*. By Jensen's inequality (Jensen, 1906) for a convex function:

$$\mathbb{E}\big(f(\theta)\big) \geq f\big(\mathbb{E}(\theta)\big) \tag{16}$$

and therefore

$$\mathbb{E}\left(e^{-b^*_{MLE}(m-m_0)}\right) \geq e^{-\mathbb{E}(b^{*MLE})(m-m_0)} \tag{17}$$

This means that even if our parameter estimate is unbiased, i.e. $\mathbb{E}(b^*_{estimated}) = b^*_{true}$ as in Ogata and Yamachina (1998), we expect our function to overestimate the true function, because the function applied to the parameter is convex.

Both the bias and the variance of a predictive distribution depend on both b^*_{true} and the magnitude of interest. This poses a potential problem, since b^*_{true} is not known in practice. However, this requirement of knowledge about the ground truth can be avoided by (instead of considering a fixed magnitude of interest) considering a fixed true *per-event survival probability* $q = S^{true}(m_{unknown})$ (or equivalently, the *expected number of events* above m per catalogue: $\Lambda^{true} = nq$). For a given value of q we can derive analytical probability distributions for q^{est} for all predictive distributions, independent of the value of b^*_{true} (see Appendix E). This means that while we cannot assess how well our predictive distribution works at a given magnitude – at least not without knowing the ground truth – we can assess how well it works at a given 'rarity'.

For example, consider the case where we have a catalogue of 100 events, and we want to know how well we can estimate the survival probability of events that have a true survival probability such that they would occur only once every 10 catalogues ($\Lambda^{true} = 0.1$). This means that we're interested in events with a true single-event survival probability $q = \frac{\Lambda^{true}}{n} = 10^{-3}$. Using the analytical probability distributions for q^{est} for each predictive distribution, we can obtain the mean (expected value, which we can relate to the bias/accuracy) and the variance (which relates to the precision). For our example, with $q = 10^{-3}$ and $n = 100$, we obtain (see Figure 1 and Figure 2a for a visual representation) that $\mathbb{E}(q^{est})_{plug-in} = 1.17 \times 10^{-3}$, $\mathbb{E}(q^{est})_{plug-in\ corrected} = 1.25 \times 10^{-3}$ and $\mathbb{E}(q^{est})_{Jeffreys} = 1.44 \times 10^{-3}$, while $\mathbb{E}(q^{est})_{unbiased} = 1.00 \times 10^{-3}$. These results mean that if our $n = 100$ catalogue was drawn from a ground truth with $b^*_{true} = \ln 10$ ($b_{true} = 1.0$), this is the performance of our predictive distributions at $m - m_0 = \frac{\ln q}{-b^*_{true}} = 3.0$. But for a exactly this same performance is expected at $m - m_0 = 4.0$ and $b^*_{true} = 0.75 \ln 10$. Note that the unbiased estimator is indeed unbiased ($\mathbb{E}(q^{est}) = q$) and that its variance at this value of q is indeed the lowest of all predictive distributions: $\mathrm{Var}(q^{est})_{unbiased} < \mathrm{Var}(q^{est})_{plug-in} < \mathrm{Var}(q^{est})_{plug-in\ corrected} < \mathrm{Var}(q^{est})_{Jeffreys}$ (see Figure 1). This ranking in terms of variance is identical for $q < e^{-(2+\sqrt{2})} \approx 0.032$ (see Appendix D for cross-overs between variance of different predictive distributions). For a catalogue with $n = 1000$, estimates become more accurate (less bias, with the obvious exception of the unbiased estimator, which is unbiased for any n by its very nature), more precise (less variance) and the difference between the estimators becomes less pronounced.

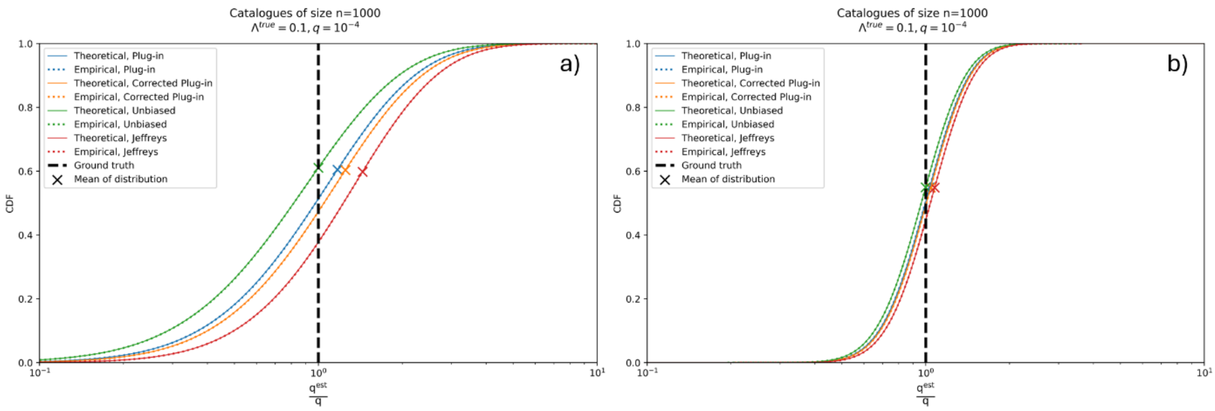

Figure 2: Probability distribution of q^{est} for the different predictive distributions, for events with a magnitude which is expected to be exceeded once every 10 catalogues $\Lambda^{true} = 0.1$. a) For a catalogue of 100 events, b) for a catalogue of 1000 events. With increasing catalogue size, the estimates become more precise (less variance) and the difference between the estimators becomes less pronounced.

Discussion

We have derived a new unbiased estimator for earthquake size distributions under the Gutenberg-Richter law. Just like the typically employed plug-in estimator, this new estimator is purely based on the observed data, summarized in $\{n, T\}$. However, while the plug-in estimator is expected to overestimate the probability of large events, the unbiased estimator is, by design, unbiased. We have shown that this improved accuracy does not come at the expense of higher variance, but rather, that the variance of the unbiased estimator is lower than that of both of the MLE plug-in estimators when considering events with per-event survival probability $q < 0.032$. In workflows that use the plug-in estimator, this unbiased estimator can serve as a 'drop-in' replacement with superior statistical properties. The additional property for the unbiased estimator, $S(m) = 0$ when $m - m_0 \geq T$, does not have practical implications for all magnitudes of practical interest for catalogue sizes of $n \geq 50$; even at $b^*_{true} = 2\ln 10$ (meaning $b = 2.0$), $n = 50$, $S(m) > 0$ for $m_0 + 6$ for virtually all catalogues ($P(T > 6) = \frac{\Gamma(50,\ 6\times 2\times \ln 10)}{\Gamma(50)} \approx 0.99992$).

We have also shown a direct closed-form Bayesian posterior predictive magnitude distribution, based on a conjugate prior/posterior pair of the Gamma family. For the special case of Jeffreys prior, we show that this Bayesian magnitude distribution has larger variance than both the plug-in estimators and the unbiased estimator for all magnitudes sufficiently above m_0 and all magnitudes of practical interest. However, for informative priors, this behavior can be different. It should be noted that while the Gamma distribution may appear to be an unusual choice for a prior distribution, its parameters can easily be chosen in such a way that it represents prior belief appropriately. The distribution $b^* \sim \mathrm{Gamma}(a_0, \lambda_0)$ has properties:

$$\mathrm{Mean} = \mathbb{E}(b^*) = \frac{a_0}{\lambda_0}, \qquad \mathrm{Var}(b^*) = \frac{a_0}{\lambda_0^2} \tag{18}$$

This means that if, for example, we want to define a prior with $b^*_{mean} = \ln 10$ and a 95% confidence interval of $b^*_{CI95} = [1.85, 2.75]$ ($b_{mean} = 1.0$ and $b_{CI95} = [0.8, 1.2]$), we can do so with either $\mathrm{Normal}(\ln 10, 0.23)$ or (using Eq. 18) with $\mathrm{Gamma}\left(100, \frac{100}{\ln 10}\right)$. The benefits of expressing prior belief in terms of a Gamma distribution are the mathematically convenient conjugate properties and the ability to directly express the posterior predictive earthquake size distribution in terms of the prior parameters $\{a_0, \lambda_0\}$ and the data-derived $\{n, T\}$.

For a catalogue sizes $n = 1000$, even for relatively rare events ($\Lambda^{true} = 0.1$), all predictive distributions perform remarkably well (for $n = 1000$ and $\Lambda^{true} = 0.1$, the central 95% confidence interval is within a factor 2 of the ground truth) and similar to one another. However, especially in cases with smaller catalogues and where no prior information is available or required, the unbiased estimator presented in this manuscript performs particularly well compared to the alternatives considered here. As shown in Figure 1 (for $n = 100$), it exhibits more desirable properties in terms of bias and variance than either plug-in estimators (both with and without bias correction b^*-space), as well as the Bayesian posterior predictive based on Jeffreys prior. In situations where the goal is only to obtain an accurate and precise point estimate under the idealized assumption of an exponential earthquake size distribution, the unbiased estimator is the best option.

More generally however, SHRA applications require more than minimizing the bias and variance of an estimator in an idealized parametric world. The exponential magnitude distribution is only an approximation of a complex physical system; the underlying properties of which are likely to vary across space and evolve in time. In addition, substantial epistemic uncertainty exists regarding which models or parameter values best represent reality, and valuable insights may be available from physics, analogue regions, or previous studies. The unbiased estimator does not account for these considerations. Therefore, we believe that in the context of seismic hazard and risk analysis, a Bayesian approach is typically preferable. Bayesian inference naturally fits into the broader probabilistic SHRA philosophy of acknowledging and accounting for (epistemic) uncertainty, and it provides more flexibility for model extensions, such as allowing for parameters like b^* be a function of some spatially or temporally varying predictor variables, or embedding the earthquake size distribution within an hierarchical model.

When prior information is available, using a Gamma prior is a useful alternative to the more typically applied uniform or normally distributed priors. In cases without prior information, Jeffreys prior remains an option, although it does lead to a bias and higher variance compared to the unbiased estimator as discussed before.

The commonly employed plug-in estimator does not appear to have a valid use-case, since better alternatives are available for probabilistic forecasting of the earthquake size distribution, both in the point-estimation context and in the more general Bayesian forecasting context.

Conclusion

We propose a direct unbiased estimator for the exponential earthquake size distribution; a result which, to our knowledge, has not been previously documented in the seismological literature. This estimator provides a practical and statistically well-behaved alternative to the commonly used approach of estimating b_{MLE}^* and directly plugging it into the exponential distribution. Our results show that, especially for small catalogues, the unbiased estimator outperforms both plug-in estimators (with or without bias correction) and the Bayesian posterior predictive estimator based on Jeffreys prior in terms of bias and variance. This unbiased estimator extends the methodological toolkit for modelling earthquake size distributions and provides a principled alternative to plug-in estimators for cases where prior information is unavailable or where an unbiased, minimum-variance estimation is desired. In the context of seismic hazard and risk analysis, however, a Bayesian approach is typically preferable to point estimation. If prior belief for b^* is expressed as a Gamma distribution, a conjugate Gamma posterior distribution and closed form expression for the posterior predictive earthquake size distribution are available, negating the need for (numerically) evaluating likelihood expressions and integrating over a posterior.

Data and resources

All data used in this paper has been synthetically generated using Python/Jupyter Notebooks. The code at github.com/TNO/earthquake-size-estimation can be used to (re-)generate this data and the figures in this manuscript. Claude (Anthropic, 2025) and ChatGPT (OpenAI, 2025) were used to explore statistical approaches and derive equations. These derivations have been independently verified by the authors.

Acknowledgments

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Appendix A – Derivation of unbiased estimator

The Gutenberg-Richter scaling relation for earthquake sizes (Gutenberg & Richter, 1941, 1944) can, for individual earthquakes, be interpreted as a probability distribution. When the size is expressed in terms of magnitude m, the distribution is exponential. The corresponding survival function (also exceedance probability, or complementary cumulative distribution function) S is defined as

$$S(m) = 10^{-b(m-m_0)}, m \geq m_0 \tag{A1}$$

with m_0 an arbitrary lower magnitude threshold, such as the magnitude of completeness of the seismic catalogue, and b the parameter controlling the scaling, referred to as the b-value.

Changing to base-e allows many of the derivations to be simplified:

$$S(m) = e^{-b^*(m-m_0)}, m \geq m_0 \tag{A2}$$

with $b^* = b \ln 10$.

The (minimal) complete sufficient statistic for an exponential distribution is:

$$T = \sum_{i=1}^{n} M_i - m_0 \tag{A3}$$

T can be determined on a per-catalogue basis. Conveniently, the distribution for T is known:

$$T \sim \mathrm{Gamma}(n, \mathrm{rate} = b^*) \tag{A4}$$

The corresponding probability density function:

$$p_T(t) = \frac{{b^*}^n}{\Gamma(n)} t^{n-1} e^{-b^* t}, t > 0 \tag{A5}$$

where $\Gamma(n)$ is the gamma function:

$$\Gamma(\alpha) = \int_0^\infty u^{\alpha-1} e^{-t} du \tag{A6}$$

We want to find an unbiased estimator $S(T)$, with the property $\mathbb{E}(S(T)) = e^{-b^*_{true}(m-m0)}$. We again use the fact that T is Gamma-distributed. Using A5, we can set:

$$\int_0^\infty S(t)\frac{b^{*n}}{\Gamma(n)}t^{n-1}e^{-b^*t}dt = e^{-b^*_{true}(m-m0)}, \qquad \forall\beta > 0 \tag{A7}$$

We can rearrange this to get all terms that do not depend on s to the LHS, leaving the RHS a Laplace transform in variable s:

$$\int_0^\infty (t^{n-1}S(t))e^{-b^*t}dt = \frac{\Gamma(n)}{b^*}e^{-b^*_{true}(m-m0)} \tag{A8}$$

Defining $w(t) = t^{n-1}S(t)$ we get:

$$\mathcal{L}\{w\}(b^*) = \int_0^\infty w(s)\,e^{-b^*t}dt = \frac{\Gamma(n)}{b^*}e^{-b^*_{true}(m-m0)} \tag{A9}$$

Taking the inverse Laplace transform requires two standard identities:

$$\mathcal{L}\{u^{\alpha-1}\}(\alpha) = \frac{\Gamma(\alpha)}{b^{*\alpha}} \tag{A10}$$

$$\mathcal{L}\{f(t-s)H(t-s)\}(\alpha) = e^{-\alpha s}\mathcal{L}\{f\}(\alpha) \tag{A11}$$

where H is the Heaviside function: 1 if t≥ s, 0 otherwise.

We can recognize the RHS of A9 as the combination A10 and A11 and therefore:

$$w(t) = \big(t-(m-m_0)\big)^{n-1}, t \geq s \tag{A12}$$

Combining this with the original definition of $w(t)$:

$$t^{n-1}S(t) = \big(t-(m-m_0)\big)^{n-1} \tag{A13}$$

$$S(t) = \frac{\big(t-(m-m_0)\big)^{n-1}}{t^{n-1}} = \left(1-\frac{m-m_0}{t}\right)^{n-1}, t \geq m-m_0 \tag{A14}$$

$$S^{unbiased}(m) = \left(1-\frac{m-m_0}{T}\right)^{n-1}, T \geq m-m_0 \geq 0 \tag{A15}$$

Appendix B – Jeffreys prior for exponential distribution

Jeffreys prior is defined as (Jeffreys, 1946):

$$\pi_J(\theta) \propto \sqrt{I(\theta} \quad \text{with} \quad I(\theta) = -\mathbb{E}\left[\frac{\partial^2 \log\ L(\theta, X)}{\partial \theta^2}\right] \tag{B1}$$

where $L(\theta, X)$ is the likelihood based on the data $X = \{X_1, \ldots, X_n\}$.

For an exponentially distributed random variable (i.i.d.) with rate parameter $b^* > 0$:

$$f(x|b^*) = b^* e^{-b^* x}, x \geq 0 \tag{B2}$$

The corresponding log-likelihood:

$$\ell(b^*) = n \log(b^*) - \ b^* \sum_{i=1}^{n} x_i \tag{B3}$$

Taking the derivative w.r.t. b^* twice:

$$\frac{\partial \ell}{\partial b^*} = \frac{n}{b^*} - \sum_{i=1}^{n} x_i \tag{B4}$$

$$\frac{\partial^2 \ell}{\partial {b^*}^2} = -\frac{n}{{b^*}^2} \tag{B5}$$

This gives Fisher information:

$$I(b^*) = -\mathbb{E}\left[\frac{\partial^2 \ell}{\partial {b^*}^2}\right] = \frac{n}{{b^*}^2} \tag{B6}$$

Jeffreys prior then becomes:

$$\pi_J(b^*) \propto \sqrt{I(b^*)} \propto \frac{1}{b^*} \tag{B7}$$

This is an improper prior (if does not integrate to 1) on the domain $(0, \infty)$. This prior can also be written as an improper limiting member of the Gamma family: $\mathrm{Gamma}(0{,}0)$:

$$\pi_J(b^*) \propto \lim_{\alpha_0, \lambda_0 \to 0^+} \mathrm{Gamma}(b^*; \alpha_0, \lambda_0) \propto {b^*}^{\alpha - 1} e^{-\lambda_0 b^*} \propto \frac{1}{b^*} \tag{B8}$$

Appendix C – Bias of different predictive distributions for a known ground truth distribution

The plug-in MLE-based estimator is given by:

$$S^{plug-in}(m) = e^{-b^{*}{}_{MLE}(m-m_0)} = e^{-\frac{n}{T}(m-m_0)} \tag{C1}$$

It's bias with respect to the ground truth:

$$Bias(m) = \mathbb{E}\left(S^{plug-in}(m)\right) - S^{true}(m) \tag{C2}$$

Using A4 and A5:

$$\mathbb{E}\left(S^{plug-in}(m)\right) = \int_0^\infty S^{plug-in}(m)\, p_T(t)dt = \int_0^\infty e^{-\frac{n}{t}(m-m_0)} \frac{(b^*_{true})^n}{\Gamma(n)} t^{n-1} e^{-b^*_{true}t} dt \tag{C3}$$

$$\mathbb{E}\left(S^{plug-in}(m)\right) = \frac{(b^*_{true})^n}{\Gamma(n)} \int_0^\infty t^{n-1} \exp\,(-b^*_{true}t - \frac{n}{t}\,(m-m_0)\, dt \tag{C4}$$

We can recognize this integral to be of the standard type $\int_0^\infty t^{\nu-1} \exp\left(-pt - \frac{q}{t}\right) dt$ with positive p, q.

This expression has a known solution (Gradshteyn & Ryzhik, 2015) for $\mathrm{Re}(p) > 0$, $\mathrm{Re}(q) > 0$, $\nu \in \mathbb{R}$:

$$\int_0^\infty t^{\nu-1} \exp\left(-pt - \frac{q}{t}\right) dt = 2\left(\frac{q}{p}\right)^{\frac{\nu}{2}} K_\nu\left(2\sqrt{pq}\right) \tag{C5}$$

Where K_ν is the modified Bessel function of the second kind. In our case, $p = b^*_{true} > 0$, $\nu = n$, and $q = n(m-m_0) > 0$ (and by continuity $q = n(m-m_0) \geq 0$). This means we can apply C5 to C4:

$$\mathbb{E}\left(S^{plug-in}(m)\right) = \frac{(b^*_{true})^n}{\Gamma(n)} \times 2\left(\frac{n(m-m_0)}{b^*_{true}}\right)^{\frac{n}{2}} K_n\left(2\sqrt{b^*_{true}n(m-m_0)}\right) \tag{C6}$$

$$\mathbb{E}\left(S^{plug-in}(m)\right) = \frac{2}{\Gamma(n)}(b^*_{true}n(m-m_0))^{\frac{n}{2}} \times K_n\left(2\sqrt{b^*_{true}n(m-m_0)}\right) \tag{C7}$$

And using C2:

$$Bias^{plug-in}(m) = \frac{2}{\Gamma(n)}(b^*_{true}n(m-m_0))^{\frac{n}{2}} \times K_n\left(2\sqrt{b^*_{true}n(m-m_0)}\right) - e^{-b^*_{true}(m-m_0)} \tag{C8}$$

The exact same derivation can be followed for the estimator that uses the bias-corrected $\widetilde{b^*} = b^*\frac{n-1}{n}$. We start with:

$$S^{plug-in\_corrected}(m) = e^{-\widetilde{b^*}_{MLE}(m-m_0)} = e^{-\frac{n-1}{T}(m-m_0)} \tag{C9}$$

and obtain:

$$Bias^{plug-in\_corrected}(m) = \frac{2}{\Gamma(n)}\left(b^*_{true}(n-1)(m-m_0)\right)^{\frac{n}{2}} \times K_n\left(2\sqrt{b^*_{true}(n-1)(m-m_0)}\right) - e^{-b^*_{true}(m-m_0)} \tag{C10}$$

Finally, for the predictive distribution based on Jeffreys prior we start with:

$$S^{Jeffreys}(m) = \left(\frac{T}{T+m-m_0}\right)^n = \left(1+\frac{m-m_0}{T}\right)^{-n} \tag{C11}$$

We recognize the standard identity (Gradshteyn & Ryzhik, 2015):

$$(1+z)^{-n} = \frac{1}{\Gamma(n)}\int_0^\infty t^{n-1}e^{-t(1+z)}dt, \quad \mathrm{Re}(n) > 0, \mathrm{Re}(z) > -1 \tag{C12}$$

So:

$$\left(1+\frac{m-m_0}{T}\right)^{-n} = \frac{1}{\Gamma(n)}\int_0^\infty s^{n-1}e^{-\frac{s(m-m_0)}{T}}ds \tag{C13}$$

Taking the expectation over T and swapping integrals (Fubini's theorem, integrand is positive):

$$\mathbb{E}\left(S^{Jeffreys}(m)\right) = \int_0^\infty S^{Jeffreys}(m)\,p_T(t)dt = \frac{1}{\Gamma(n)}\int_0^\infty s^{n-1}\mathbb{E}(e^{-\frac{s(m-m_0)}{T}})dt \tag{C14}$$

The term $\mathbb{E}(e^{-\frac{t(m-m_0)}{T}})$ again has a standard solution (C5) so:

$$\mathbb{E}(e^{-\frac{s(m-m_0)}{T}}) = \frac{2}{\Gamma(n)}\left(b^*_{true}s(m-m_0)\right)^{\frac{n}{2}}K_n(2\sqrt{b^*_{true}s(m-m_0)}) \tag{C15}$$

Combine C14 and C15:

$$\mathbb{E}\left(S^{Jeffreys}(m)\right) = \frac{2\left(b^*_{true}(m-m_0)\right)^{\frac{n}{2}}}{\left(\Gamma(n)\right)^2}\int_0^\infty s^{\frac{3n}{2}-1}e^{-t}K_n(2\sqrt{b^*_{true}s(m-m_0)})\;ds \tag{C16}$$

Appendix D – Expressions for the variance of all predictive distributions

For all predictive distributions covered in this manuscript, we can derive an expression for the variance. In general, the variance of the survival function $S(m)$ is given by:

$$\mathrm{Var}\big(S(m)\big) = \mathbb{E}[S(m)^2] - \mathbb{E}[S(m)]^2 \tag{D1}$$

which means that for each predictive distribution we need to find both $\mathbb{E}[S(m)^2]$ and $\mathbb{E}[S(m)]^2$. In Appendix C, we've derived $\mathbb{E}[S(m)]$ for each predictive distribution, which we simply need to square to obtain $\mathbb{E}[S(m)]^2$. Here, we therefore focus on obtaining $\mathbb{E}[S(m)^2]$. We start with the plug-in estimator and follow essentially C1 through C7, replacing n with $2n$:

$$\begin{aligned}\mathbb{E}\big[S^{plug-in}(m)^2\big] &= \mathbb{E}\left[e^{-\frac{2n}{T}(m-m_0)}\right] \\ &= \frac{2}{\Gamma(n)}\left(b^*_{true}2n(m-m_0)\right)^{\frac{n}{2}} \times K_n\left(2\sqrt{b^*_{true}2n(m-m_0)}\right)\end{aligned} \tag{D2}$$

Giving:

$$\begin{aligned}\mathrm{Var}\big[S^{plug-in}(m)\big] &= \mathbb{E}\left[e^{-\frac{2n}{T}(m-m_0)}\right] - \mathbb{E}\left[e^{-\frac{n}{T}(m-m_0)}\right]^2 \\ &= \frac{2}{\Gamma(n)}\left(b^*_{true}2n(m-m_0)\right)^{\frac{n}{2}} \times K_n\left(2\sqrt{b^*_{true}2n(m-m_0)}\right) \\ &\quad - \left[\frac{2}{\Gamma(n)}\left(b^*_{true}n(m-m_0)\right)^{\frac{n}{2}} \times K_n\left(2\sqrt{b^*_{true}n(m-m_0)}\right)\right]^2\end{aligned} \tag{D3}$$

Similarly for the corrected plug-in estimator:

$$\begin{aligned}\mathrm{Var}\big[S^{plug-in\ corrected}(m)\big] &= \mathbb{E}\left[e^{-\frac{2(n-1)}{T}(m-m_0)}\right] - \mathbb{E}\left[e^{-\frac{n-1}{T}(m-m_0)}\right]^2 \\ &= \frac{2}{\Gamma(n)}\left(b^*_{true}2(n-1)(m-m_0)\right)^{\frac{n}{2}} \times K_n\left(2\sqrt{b^*_{true}2(n-1)(m-m_0)}\right) \\ &\quad - \left[\frac{2}{\Gamma(n)}\left(b^*_{true}(n-1)(m-m_0)\right)^{\frac{n}{2}} \times K_n\left(2\sqrt{b^*_{true}(n-1)(m-m_0)}\right)\right]^2\end{aligned} \tag{D4}$$

For the unbiased estimator:

$$\mathbb{E}\big[S^{unbiased}(m)^2\big] = \mathbb{E}\left[\left(1-\frac{m-m_0}{T}\right)^{2(n-1)}\right] = \int_{m-m_0}^{\infty}\left(1-\frac{m-m_0}{t}\right)^{2(n-1)} f_T(t)dt \tag{D5}$$

Inserting A5 and using that for integer n, $\Gamma(n) = (n-1)!$ we get:

$$\mathbb{E}\left[S^{unbiased}(m)^2\right] = \frac{b^*_{true}}{(n-1)!} \int_{m-m_0}^{\infty} \left(1 - \frac{m-m_0}{t}\right)^{2(n-1)} t^{n-1} e^{-b^*_{true}t} dt \qquad \text{(D6)}$$

Expanding the power:

$$\left(1 - \frac{m-m_0}{t}\right)^{2(n-1)} = \sum_{k=0}^{2n-2} \binom{2n-2}{k} (-(m-m_0))^k t^{-k} \qquad \text{(D6)}$$

Combining D5 and D6:

$$\mathbb{E}\left[S^{unbiased}(m)^2\right] = \frac{b^*_{true}}{(n-1)!} \sum_{k=0}^{2n-2} \binom{2n-2}{k} (-(m-m_0))^k \int_{m-m_0}^{\infty} t^{n-1-k} e^{-b^*_{true}t} dt \qquad \text{(D7)}$$

Use:

$$\int_{m-m_0}^{\infty} t^{n-1-k} e^{-b^*_{true}t} dt = (b^*_{true})^{k-n} \Gamma_{\mathrm{ui}}(n-k, b^*_{true}(m-m_0)) \qquad \text{(D8)}$$

where Γ_{ui} is the upper incomplete gamma function to get:

$$\mathbb{E}\left[S^{unbiased}(m)^2\right] =$$

$$\frac{1}{(n-1)!} \sum_{k=0}^{2n-2} \binom{2n-2}{k} (-b^*_{true}(m-m_0))^k \, \Gamma_{\mathrm{ui}}(n-k, b^*_{true}(m-m_0)) \qquad \text{(D9)}$$

and since for unbiased estimator, $\mathbb{E}[(S^{unbiased}(m)] = e^{-b^*_{true}(m-m0)}$, we obtain:

$$\mathrm{Var}[(S^{unbiased}(m)] =$$

$$\frac{1}{(n-1)!} \sum_{k=0}^{2n-2} \binom{2n-2}{k} \left(-b^*_{true}(m-m_0)\right)^k \Gamma_{\mathrm{ui}}\left(n-k, b^*_{true}(m-m_0)\right) - e^{-2b^*_{true}(m-m0)} \qquad \text{(D10)}$$

For the Jeffreys posterior predictive:

$$\mathbb{E}\left[S^{Jeffreys}(m)\right] = \mathbb{E}\left[\left(\frac{T}{T+m-m_0}\right)^n\right] = \frac{b^*_{true}}{\Gamma(n)} \int_0^{\infty} \left(\frac{t}{t+m-m_0}\right)^n t^{n-1} e^{-b^*_{true}t} dt \qquad \text{(D11)}$$

and

$$\mathbb{E}\left[S^{Jeffreys}(m)^2\right] = \mathbb{E}\left[\left(\frac{T}{T+m-m_0}\right)^{2n}\right] = \frac{b^*_{true}}{\Gamma(n)} \int_0^\infty \left(\frac{t}{t+m-m_0}\right)^{2n} t^{n-1} e^{-b^*_{true} t} dt \quad \text{(D12)}$$

Both D11 and D12 are integrals belonging to the Meijer-G family. In the standard notation for this function:

$$\mathbb{E}\left[S^{Jeffreys}(m)\right] = \frac{\left(b^*_{true}(m-m_0)\right)^n}{\Gamma(n)^2} G^{1,2}_{2,2}\left(b^*_{true}(m-m_0)\middle|\begin{matrix}1-2n,-\\0,-n\end{matrix}\right) \quad \text{(D13)}$$

and

$$\mathbb{E}\left[S^{Jeffreys}(m)^2\right] = \frac{\left(b^*_{true}(m-m_0)\right)^n}{\Gamma(n)\Gamma(2\text{n})} G^{1,2}_{2,2}\left(b^*_{true}(m-m_0)\middle|\begin{matrix}1-3n,-\\0,-n\end{matrix}\right) \quad \text{(D14)}$$

Giving the variance:

$$\begin{aligned}\text{Var}[\left(S^{Jeffreys}(m)\right] = \\ \frac{\left(b^*_{true}(m-m_0)\right)^n}{\Gamma(n)\Gamma(2\text{n})} G^{1,2}_{2,2}\left(b^*_{true}(m-m_0)\middle|\begin{matrix}1-3n,-\\0,-n\end{matrix}\right) \\ -\left[\frac{\left(b^*_{true}(m-m_0)\right)^n}{\Gamma(n)^2} G^{1,2}_{2,2}\left(b^*_{true}(m-m_0)\middle|\begin{matrix}1-2n,-\\0,-n\end{matrix}\right)\right]^2\end{aligned} \quad \text{(D15)}$$

For all predictive distributions , the variance is a function of the same terms: b^{*true}, $(m - m_0)$, and n. The intersection between any two of these variance functions can be found numerically, or via the delta method. Given that $T = M_1 + \cdots + M_n \sim Gamma(n, rate = b^*_{true})$, as n increases, T becomes a sharply concentrated around its mean:

$$\mu = \mathbb{E}[T] = \frac{n}{b^*_{true}} \quad \text{(D16)}$$

$$T = \mu + O_p(\sqrt{n}), \qquad \text{Var}(T) = \frac{n}{(b^*_{true})^2} \quad \text{(D17)}$$

The delta method tells us that under these conditions, we can write our predictive distributions (for example $S^{plug-in}(m) = \exp\left(-\frac{n(m-m_0)}{T}\right)$ as:

$$g(t) = \exp\left(-\frac{n(m-m_0)}{t}\right) \quad \text{(D18)}$$

and since $T = \mu + O_p(\sqrt{n})$, we can approximate $S^{plug-in}(m)$ by a first-order Taylor expansion around μ:

$$S^{plug-in}(m) = g(\mu) + g'(\mu)(T-\mu) \quad \text{(D18)}$$

and compute its variance:

$$\mathrm{Var}[(S^{plug-in}(m)] \approx [g'(\mu)]^2 \mathrm{Var}(T) \quad \text{(D19)}$$

We derive $g'(s)$:

$$g'(t) = \exp\left(-\frac{n(m-m_0)}{t}\right) \cdot \left(\frac{n(m-m_0)}{t^2}\right) \quad \text{(D20)}$$

Evaluate at $t = \mu = \frac{n}{b_{true}^*}$

$$g'(\mu) = e^{-b_{true}^*(m-m_0)} \cdot \frac{(b_{true}^*)^2 \ (m-m_0)}{n} \quad \text{(D21)}$$

and use D17:

$$[g'(\mu)]^2 \mathrm{Var}(T) = e^{-2b_{true}^*(m-m_0)} \cdot \frac{(b_{true}^*)^4 \ (m-m_0)^2}{n^2} \cdot \frac{n}{(b_{true}^*)^2} \quad \text{(D22)}$$

giving:

$$\mathrm{Var}[(S^{plug-in}(m)] \approx e^{-2b_{true}^*(m-m_0)} \cdot \frac{(b_{true}^*)^2 \ (m-m_0)^2}{n} \quad \text{(D23)}$$

Similarly for the variance of the unbiased estimator, we get:

$$\mathrm{Var}[(S^{unbiased}(m)] \approx (b_{true}^*)^2 \ (m-m_0)^2 \frac{(n-1)^2}{n^3} \left(1 - \frac{b_{true}^*(m-m_0)}{n}\right)^{2n-4} \quad \text{(D24)}$$

Setting the ratio $R = \frac{\mathrm{Var}[(S^{unbiased}(m)]}{\mathrm{Var}[(S^{plug-in}(m)]}$ and then finding $\log(R) = 0$ yields:

$$b_{true}^*(m-m_0) \ \approx 2 \pm \sqrt{2} \quad \text{(D25)}$$

And so the variance of the unbiased estimator is lower than the variance of the plug-in estimator when $b^*_{true}(m - m_0) > 2 + \sqrt{2}$ (or, using E1, $q < e^{-2+\sqrt{2}}$), which is the result mentioned in the main text.

Appendix E – Probability distribution at q for all predictive distributions

We want to find expressions for the probability distributions at some *per-event* survival probability q for the predictive distributions $S^{plug-in}(m), S^{plug-in\ corrected}(m), S^{unbiased}(m)$, $S^{posterior\_predictive}_{Jeffreys}(m)$. The logic for all rules is the same, so we start with some common preliminaries. We first define:

$$L = -\ln q = b^*_{true}(m - m_0) \tag{E1}$$

Like previously, we use the fact that sufficient statistic T for the exponential distribution is given by a Gamma distribution $\mathrm{Gamma}(n, rate = b^*)$ with pdf:

$$f_T(t) = \frac{b^{*n}}{\Gamma(n)} t^{n-1} e^{-b^* t}, t > 0 \tag{E2}$$

If we now define $u = b^* t$, we can instead write:

$$f_U(u) = \frac{1}{\Gamma(n)} u^{n-1} e^{-u}, u > 0 \tag{E3}$$

Next, we'll express our general estimator $\hat{q}$ as a function $Y = g(U)$ of which we want to find the distribution with $U \sim \mathrm{Gamma}(n, 1)$. Since we want to find the distribution of Y, we can use (Casella & Berger, 2002):

$$f_Y(y) = f_U(u(y)) \left|\frac{du}{dy}\right| \tag{E4}$$

$$F_Y(y) = P(Y \le y) = P\big(U \le u(y)\big) = \frac{\gamma\big(n, u(y)\big)}{\Gamma(n)} \tag{E5}$$

where γ is the lower incomplete gamma function.

These expressions give us the distribution of an estimator $\hat{q}$ at value y. To find the distribution for a given estimator, we need to derive $u(y)$ for the CDF and both $u(y)$ and $\frac{du(y)}{dy}$ for the PDF.

For the plug-in estimator we have:

$$Y = \hat{q}^{plug-in} = \exp\,(-b^*_{MLE}(m - m_0) = \exp\left(-\frac{nL}{u}\right) \tag{E6}$$

We invert to find:

$$u(y) = \frac{nL}{-\ln y} \tag{E7}$$

And differentiate:

$$\frac{du}{dy} = \frac{nL}{y(\ln y)^2} \tag{E8}$$

Inserting into E4 and E5 gives:

$$f_Y^{plug-in}(y) = \frac{(nL)^n}{\Gamma(n)} \frac{\exp\left(-\frac{nL}{-\ln y}\right)}{y\,(-\ln y)^{n+1}} \tag{E9}$$

$$S_Y^{plug-in}(y) = \frac{\gamma\left(n, \frac{nL}{-\ln y}\right)}{\Gamma(n)} \tag{E10}$$

For the corrected plug-in estimator we have:

$$Y = \hat{q}^{plug-in\ corrected} = \exp\left(-\frac{(n-1)L}{u}\right) \tag{E11}$$

We invert to find:

$$u(y) = \frac{(n-1)L}{-\ln y} \tag{E12}$$

And differentiate:

$$\frac{du}{dy} = \frac{(n-1)L}{y(\ln y)^2} \tag{E13}$$

Inserting into E4 and E5 gives:

$$f_Y^{plug-in\ corrected}(y) = \frac{((n-1)L)^n}{\Gamma(n)} \frac{\exp\left(-\frac{(n-1)L}{-\ln y}\right)}{y\,(-\ln y)^{n+1}} \tag{E14}$$

$$S_Y^{plug-in\ corrected}(y) = \frac{\gamma\left(n, \frac{(n-1)L}{-\ln y}\right)}{\Gamma(n)} \tag{E15}$$

For the unbiased estimator we have:

$$Y = \hat{q}^{unbiased} = \left(1 - \frac{m - m_0}{T}\right)^{n-1} = \left(1 - \frac{L}{u}\right)^{n-1} \tag{E16}$$

We invert to find:

$$u(y) = \frac{L}{1 - y^{\frac{1}{n-1}}} \quad \text{(E17)}$$

And differentiate:

$$\frac{du}{dy} = \frac{L}{n-1} \frac{y^{\frac{1}{n-1}-1}}{\left(1 - y^{\frac{1}{n-1}}\right)^2} \quad \text{(E18)}$$

Inserting into E4 and E5 gives:

$$f_Y^{unbiased}(y) = \frac{L^n}{(n-1)\Gamma(n)} \frac{y^{\frac{1}{n-1}-1}}{\left(1 - y^{\frac{1}{n-1}}\right)^{n+1}} \exp\left(-\frac{L}{1 - y^{\frac{1}{n-1}}}\right) \quad \text{(E19)}$$

$$S_Y^{unbiased}(y) = \frac{\gamma\left(n, \frac{L}{1 - y^{\frac{1}{n-1}}}\right)}{\Gamma(n)} \quad \text{(E20)}$$

For the predictor based on Jeffreys prior:

$$Y = \hat{q}^{Jeffreys} = \left(\frac{T}{T + m - m_0}\right)^n = \left(\frac{u}{u + L}\right)^n \quad \text{(E21)}$$

We invert to find:

$$u(y) = \frac{Ly^{\frac{1}{n}}}{1 - y^{\frac{1}{n}}} \quad \text{(E22)}$$

And differentiate:

$$\frac{du}{dy} = \frac{Ly^{\frac{1}{n}-1}}{n\left(1 - y^{\frac{1}{n}}\right)^2} \quad \text{(E23)}$$

Inserting into E4 and E5 gives:

$$f_Y^{Jeffreys}(y) = \frac{L^n}{n\,\Gamma(n)} \frac{y^{\frac{1}{n}-1}}{\left(1 - y^{\frac{1}{n}}\right)^{n+1}} \exp\left(-\frac{Ly^{\frac{1}{n}}}{1 - y^{\frac{1}{n}}}\right) \quad \text{(E24)}$$

$$S_Y^{Jeffreys}(y) = \frac{\gamma\left(n, \frac{Ly^{\frac{1}{n}}}{1 - y^{\frac{1}{n}}}\right)}{\Gamma(n)} \quad \text{(E25)}$$

In general, for all these predictive distributions, L encodes the true per-event exceedance probability q. Note that we do not need to know the ground truth b^*_{true} to obtain this. The PDF/CDFs given in E9, E10, E14, E15, E19, E20, E24, E25 give the distribution of estimated values for q, given a $\{q, n\}$. All these formulas hold for $0 < y < 1$.