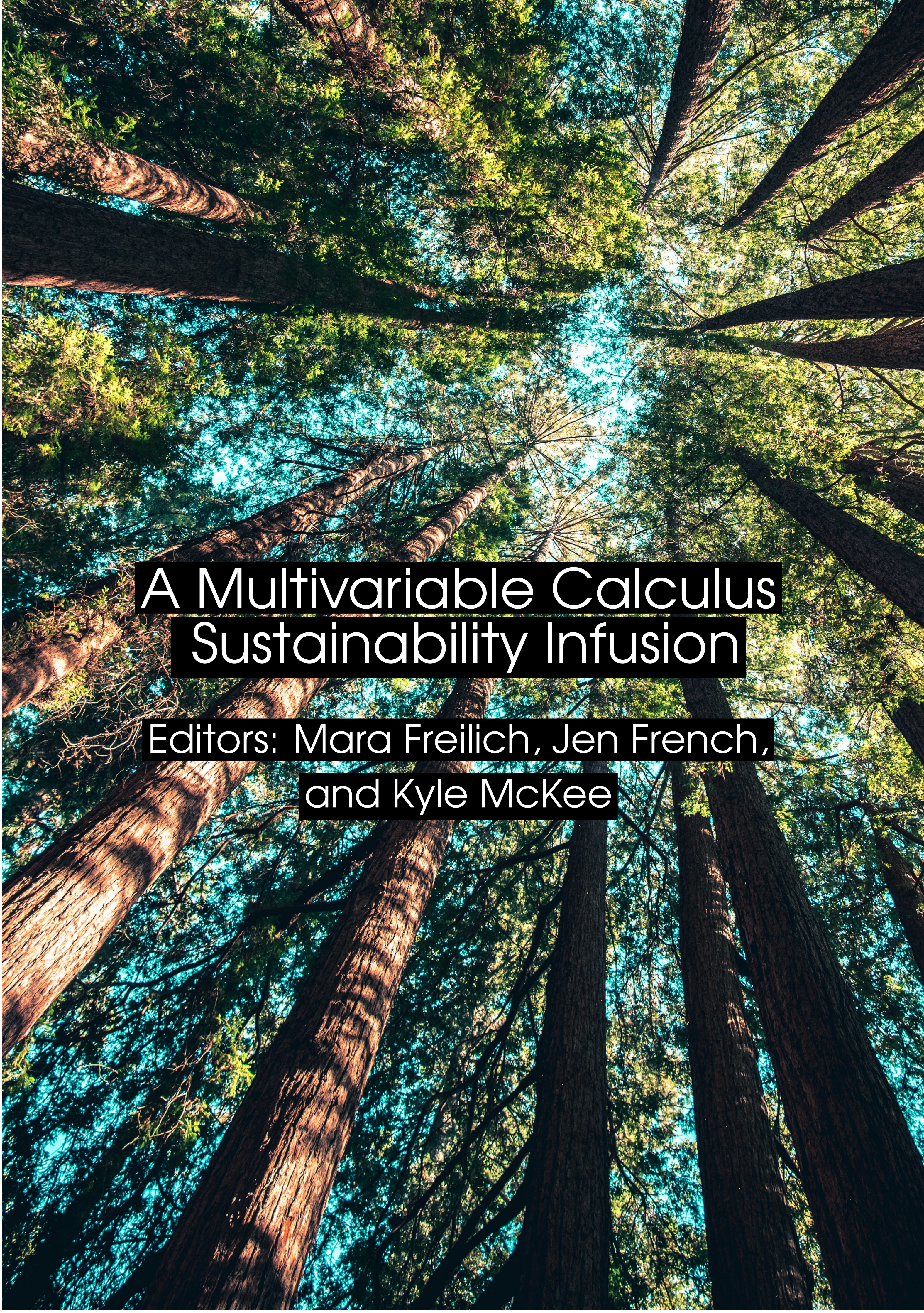
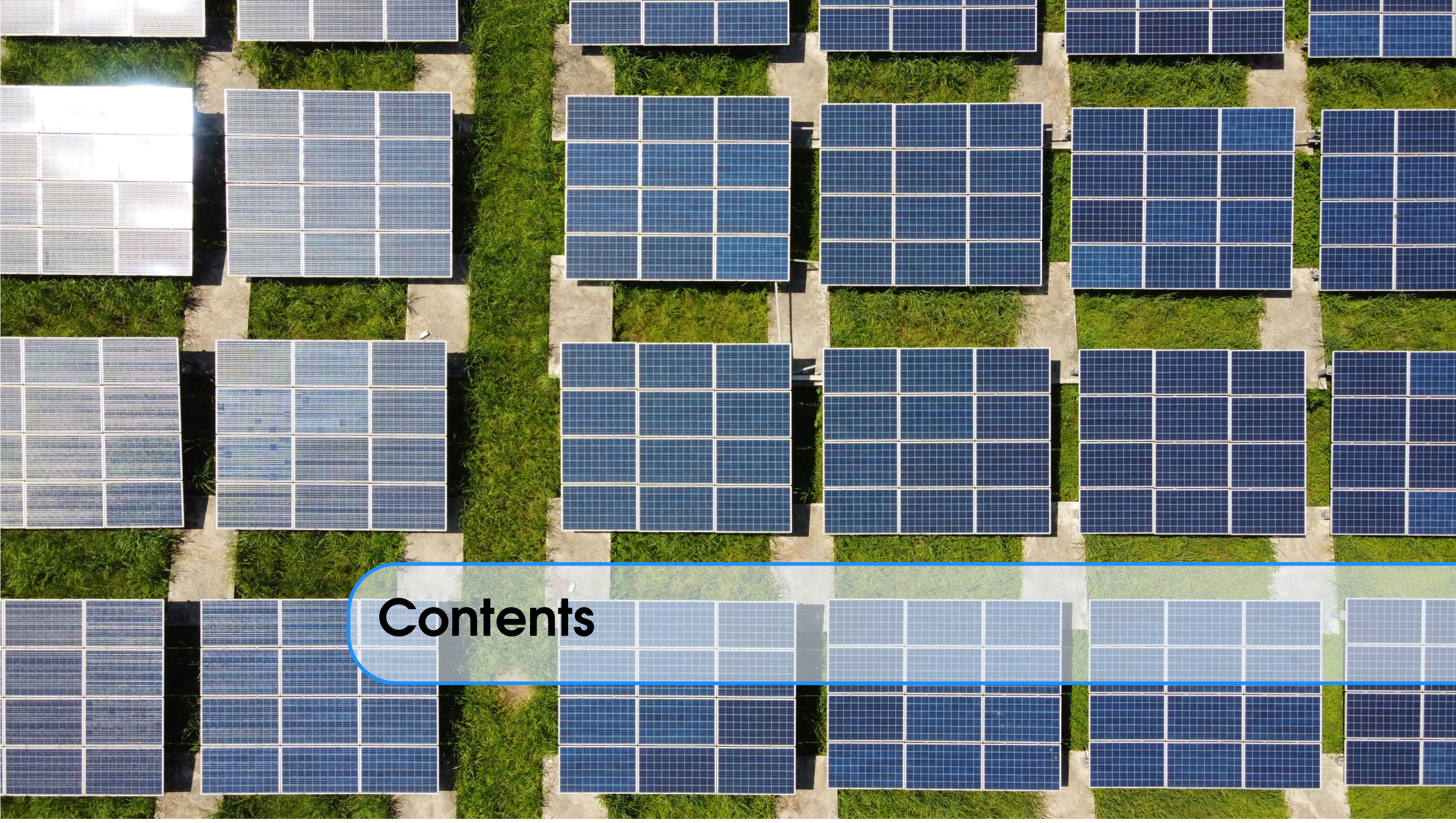


A Multivariable Calculus Sustainability Infusion

Editors: Mara Freilich, Jen French,
and Kyle McKee



Contents

1. Preface

1.1 Motivation

Climate change is one of the big challenges facing this generation. Given the threats faced by climate change, it is important that everyone have an education that allows them to engage with the issue as an informed member of society.

Cliff Freeman from the Young People's Project reminded us that math literacy and skills are part of a high quality education that prepares students to contribute to the challenges of their generation. Those challenges are numerous and multifaceted, often combining issues of environment, race, class, gender, and more.

Math skills are an important foundation for climate and environmental sciences (but other disciplines are essential too!). However, typical math courses rarely include examples from Earth science or ecology as part of the curriculum. Incorporating sustainability into pre-requisite math classes is one way to ensure that engineering majors, who often have very little time for electives, can meet the goal of becoming informed about climate change, environmental issues, and sustainability. Students increasingly share this goal.

This resource also meets an important learning objective of translating knowledge to its applications, which is an objective of many introductory math courses. Transferring math skills across disciplines is often very challenging for students. In order to support students in developing this skill, it is helpful to embed well-constructed applications word problems across a wide variety of disciplines to engage more diverse students in applications of math.

Math instructors understandably often don't have the expertise to create problems with applications to a wide variety of disciplines. For this reason, we convened Earth scientists and math instructors to write problems collaboratively for a multivariable calculus curriculum. We are sharing those problems with a creative commons license so that they can be used and modified for courses around the world.

1.2 Overview

The resource includes problems developed by graduate students and postdocs in Earth science and math instructors at MIT for use in multivariable calculus courses. The problems are arranged in an approximate order in which the topics might be introduced in a multivariable calculus course.

Each problem has four key parts:

1. An accompanying biography to illustrate the expertise of the author and provide an example of ways that researchers moved from math to its applications in Earth science.
2. Background material. The background material provides context for the problem and invites the students into the particular application of math including some of the relevant mathematical, scientific, and societal issues.
3. A multi-part word problem. Some of the problems have parts that build on each other, but some of the problems could easily be edited to be shorter or longer depending on the needs of the course.
4. References, which are both references for the background information and resources for interested students to explore further.

1.3 Method

These problems originated during a 3 hour virtual workshop at MIT in summer 2021 to build community and brainstorm topics. We found that the workshop contributed to building community across campus and with local groups and to training students and postdocs in interdisciplinary, socially engaged pedagogy. In addition to the problems themselves, a major outcome was developing new relationships between departments on campus among both students and faculty. The workshop had three parts.

1. **Introduction**: all attendees introduced themselves, we introduced the project, and the math instructors (led by Larry Guth) talked about the topics covered in multivariable calculus at MIT (18.02) and the topics that students find challenging, with time for conversation with attendees.
2. **Brainstorming**: we moved participants around in breakout rooms to brainstorm topics to write problems about. During and after this brainstorming, attendees filled out a worksheet with their ideas for topics (including a math topic, environmental topic, and a connection to social action). Participants then selected some of these topics to flesh out into full problems, which they developed throughout the summer into the problems that are contained in this volume. The resulting problems were edited by the editors of this volume and peer reviewed by the contributors to the volume.
3. **Human connections**: Two speakers from local social movements, Ben Thompson (350 Massachusetts, PhD in Math from Boston University) and Cliff Freeman (The Young People's Project, MSc in Technology Management from Wentworth Institute of Technology, PhD candidate in Math and Science Education at Boston University), joined for a panel discussion on math, social inequity, social change, and the ways that math can be used for social change. This was intended to build connections from math topics to social change and to prompt participants to share their stories and perspectives.

1.4 Contact

Please contact jfrench@mit.edu to request a solutions manual, the LaTeX source files, or to let us know if you use the problems in your course.

1.5 Acknowledgements

This project would not have happened without leadership from Sarah Meyers and the MIT Environmental Solutions Initiative. Larry Guth, Gigliola Staffilani, Glenn Flierl, Duncan Levear, Semyon Dyatlov, and Mason Rogers all contributed their time and expertise to this project and supported problem development.

1.5.1 Funding

Development of these problems was funded by ESI Curriculum Mini-Grants for Infusing Sustainability in STEM and the Arthur Vining Davis Foundation.

1.5.2 Image credits

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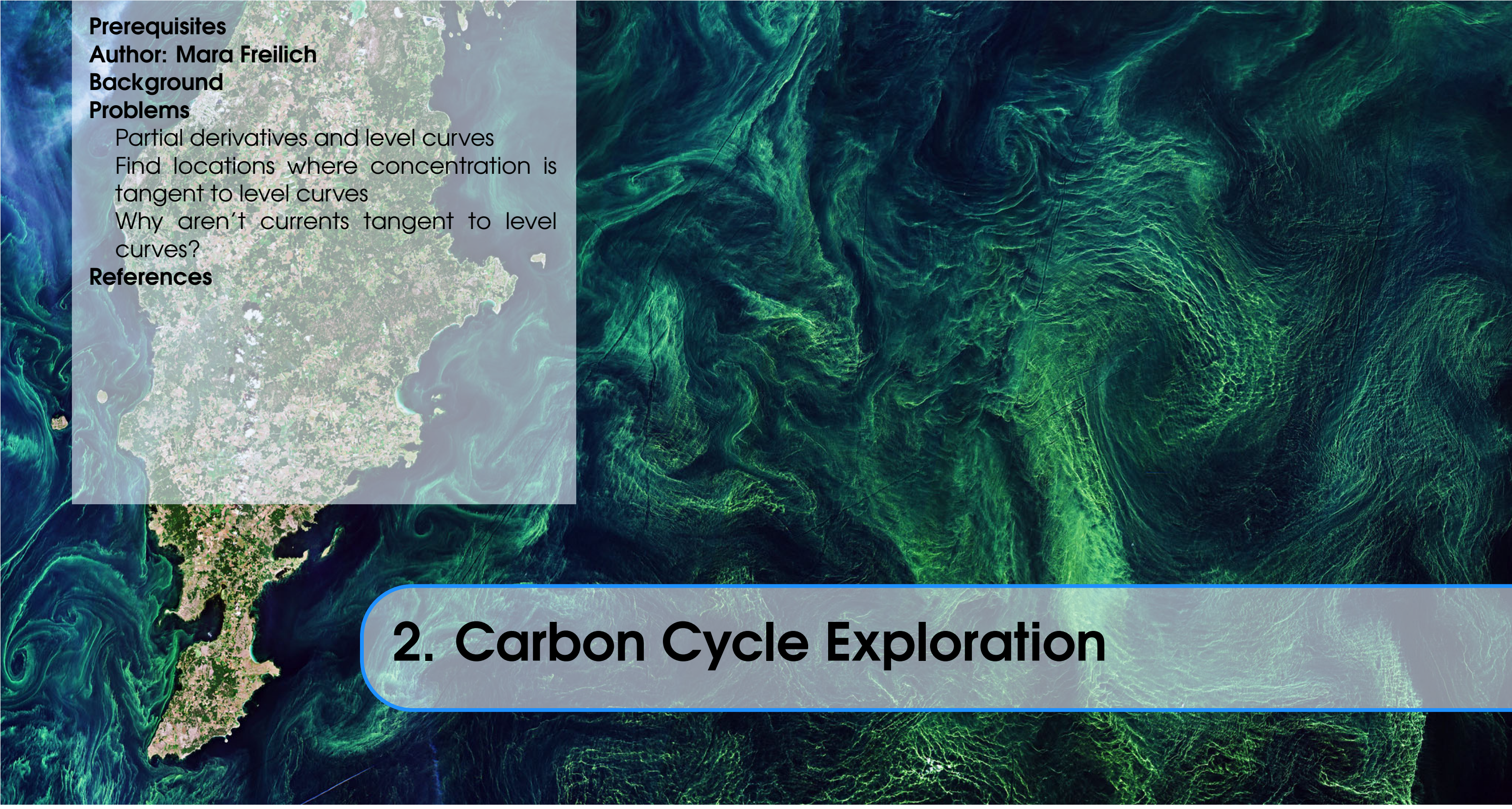


2. Carbon Cycle Exploration

2.1 Prerequisites

- Level curves / contour plots
- Partial derivatives
- Critical points

2.2 Author: Mara Freilich

Mara Freilich completed her PhD in 2021 in the MIT-Woods Hole Oceanographic Institution joint program. She is currently a postdoc fellow at Scripps Institution of Oceanography. Her research focuses on physical and biological oceanography, and especially on the ways that ocean currents affect plankton communities and the carbon cycle. Mara studied applied math as an undergraduate. She decided to study oceanography because it is intellectually engaging, combining math, physics, ecology, and chemistry, and socially impactful, providing expertise to engage in many pressing environmental and social issues. Her research combines mathematical modeling with data and observations. As an oceanographer, she has built a global community, doing research on four continents. Mara also organizes alongside community organizations to work towards social and environmental justice.

2.3 Background

The Earth's oceans have taken in more than one quarter of the human emitted carbon dioxide. However, climate change is affecting how effectively the oceans are absorbing carbon dioxide to mitigate the greenhouse effect, which could further accelerate global warming. (See for example Southern Ocean Climate Sink.)

Ocean currents redistribute heat and carbon dioxide in the oceans. To understand ocean currents to better model cycles of carbon dioxide absorption, oceanographers use satellite data. The image below was generated from two types of satellite data – sea surface height contours are obtained from altimeters, and chlorophyll concentrations are obtained through photographs (the missing photograph is due to cloud interference). The concentration of chlorophyll is used as a proxy for the primary production of oxygen from carbon dioxide through photosynthesis.

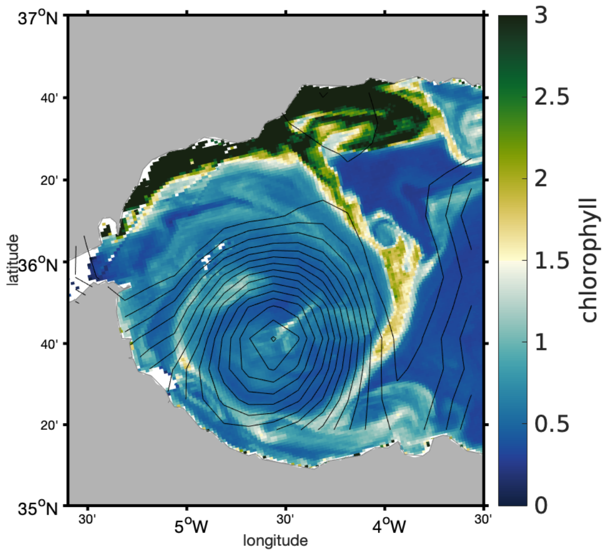


Figure 2.1: Sea surface height (contours) obtained from AVISO altimeters (radar) and chlorophyll concentration acquired from photos from NASA MODIS-Aqua. Data obtained and image created by Mara Freilich March 11, 2020 in the Mediterranean Sea. Used with permission.

2.4 Problems

2.4.1 Partial derivatives and level curves

Answer the following questions for the function $h(x,y)$, the sea level height measured in meters depicted in Figure 2.1. Use the fact that the highest sea level point can be observed at $(4°30'W, 35°40'N)$ in the image above. Note that the plot is centered on the point $(4°30'W, 36°N)$.

1. Is the partial derivative h_x positive, negative, or zero at $(4°30'W, 36°N)$?
2. Is the partial derivative h_y positive, negative, or zero at $(4°30'W, 36°N)$?
3. Which has larger magnitude (absolute value)?

2.4.2 Find locations where concentration is tangent to level curves

Ocean currents, responsible for the flow of heat and carbon, are created by the interaction of surface winds and the rotation of the Earth, which gives rise to the Coriolis effect. These forces create a vertical churning of waters, bringing carbon rich deep sea waters up to the surface, which then emit carbon into the atmosphere. Oceanographers are interested, for this reason among others, in tracking and monitoring evidence of high concentrations of carbon rising to the surface to provide more accurate models of ocean behavior and better predict the complicated dance of carbon emission and absorption.

Consider the concentrations of chlorophyll obtained from satellite photographs in the image below. The dark green shows highest concentrations, light yellow and white lower concentrations, and the blue regions of very little chlorophyll. We can consider high concentrations of chlorophyll to be evidence of high amounts of carbon absorption or regions of vertical currents bringing deep sea water rich in carbon up to the surface.

We expect ocean current to move along level curves. You can see evidence of these currents in the paths of chlorophyll through the ocean.

Identify locations on the image where the chlorophyll concentrations move along (tangent to) level curves.

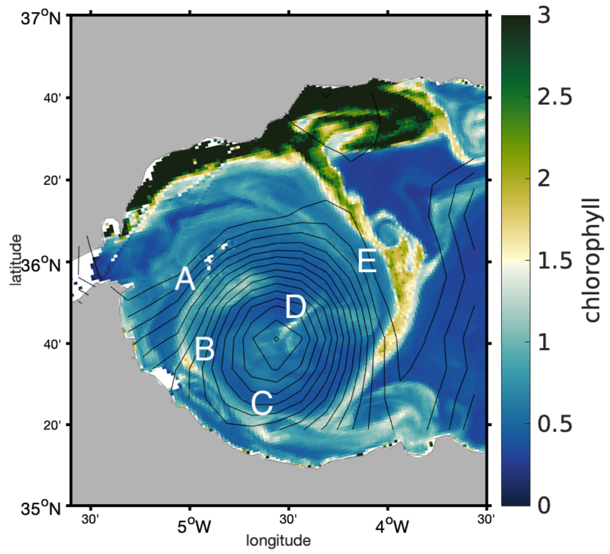


Figure 2.2: Sea surface height (contours) obtained from AVISO altimeters (radar) and chlorophyll concentration acquired from photos from NASA MODIS-Aqua. Data obtained and image created by Mara Freilich March 11, 2020 in the Mediterranean Sea. Used with permission.

2.4.3 Why aren't currents tangent to level curves?

Which of the following may be reasons that the chlorophyll is not moving directly along the level curves (as seen in Figure 2.2)?

(This problem is here to get you thinking, but we do not expect you to know the answer! This question is central to modern oceanography research.)

- The currents are 3-dimensional, so surface level curves are only telling part of the story
- The sea surface height level curves are taken as average from radar data collected, thus there is an error in the curves drawn
- There are concentrations of carbon that lead to algae bloom unrelated to direct ocean currents and are related to deep undersea events
- Algae have dynamics of motion that allow them to move against currents

Discuss why you selected the reasons you did.

2.5 References

- Watson, A. J., et al. (2020). Revised estimates of ocean-atmosphere CO_2 flux are consistent with ocean carbon inventory. Nature Communications, 11(1), 1-6.
- Southern Ocean Climate Sink: `https://knowablemagazine.org/article/physical-world/2020/southern-ocean-carbon-sink`

3. Humidity and Temperature

3.1 Prerequisites

- Linear approximation
- Partial derivatives
- Level curves
- Chain rule

3.2 Author: Tristan Abbott

Tristan Abbott received his PhD in 2021 from the MIT Program in Atmospheres, Oceans and Climate and was an instructor for Weather and Climate at MIT (12.307). He is currently a postdoc at the NOAA Geophysical Fluid Dynamics Laboratory. His research focuses on tropical weather and climate, with particular emphasis on understanding how human activity (including global warming) affects patterns of clouds and rainfall. Most of his work relies on a combination of theory and numerical modeling. He chose to study climate science because it allows him to combine interests in the natural world and in computational science to address urgent social and environmental challenges.

3.3 Background

In hot weather, we regulate our body temperatures by sweating. When sweat evaporates from our skin, evaporative cooling allows us to feel "apparent temperatures" that are significantly lower than the "actual air temperature". By "actual air temperature", we are referring to dry-bulb temperature, which is the temperature measured by a dry thermometer. The apparent temperature we feel can increase because of increases in the actual temperature and because of increases in humidity, the latter because sweat evaporates less quickly when there is more moisture in the air around us.

One simple measure of "apparent temperature" is the wet-bulb temperature, or the temperature measured if the thermometer's bulb were covered in water-soaked fabric, and is always less than or equal to the dry-bulb temperature. The wet bulb and dry bulb temperature are equal when the relative humidity is 100%. Humidity is expressed as either mass of water vapor per volume of moist air, or as mass of water vapor per mass of dry air (usually in grams per kilogram). Relative humidity is defined as the ratio of the absolute humidity to the maximum humidity possible given the temperature. By definition, the relative humidity is between 0 and 100%.

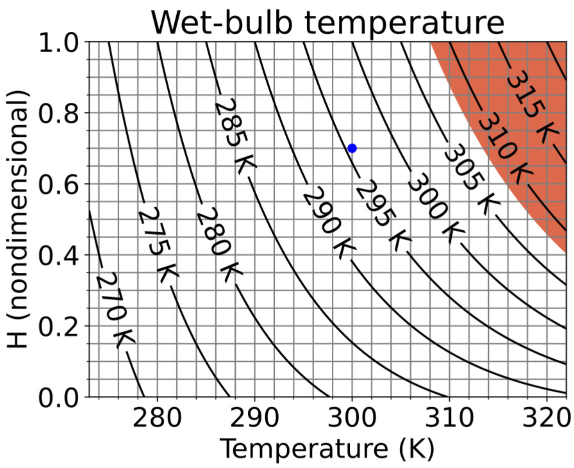
Wet-bulb temperature W is a function of both dry-bulb temperature T and relative humidity H (i.e., $W = W(T,H)$). It is always lower than the dry-bulb temperature, because evaporating water into air will cool the air while increasing humidity, and the wet-bulb temperature increases with increasing dry-bulb temperature and increasing humidity. It is also a useful measure of heat stress: humans cannot survive exposure to wet-bulb temperatures above 95°F for more than a few hours. As global climate changes, it is important to understand how the heat stress of a region can change to make different regions more or less habitable. While wet bulb temperature is important for human impacts of climate, output from climate model projections usually includes dry-bulb temperature and humidity but not wet-bulb temperature. Therefore, it is necessary to compute wet-bulb temperature from dry-bulb temperature and humidity.

Wet-bulb temperature is related to the dry-bulb temperature and the relative humidity by the following implicitly defined function:

$$T - W = a\left(e^{-b/W} - He^{-b/T}\right) \tag{3.1}$$

The equation above is a statement about energy balance: it says that the energy consumed to evaporate water (right-hand side) is equal to the energy provided by cooling the air (left-hand side). The independent variables are T (temperature) and H (relative humidity). The dependent variable is $W = W(T,H)$, and $a = 4.2 \times 10^9$ Kelvins and $b = 5400$ Kelvins are constants.

The equation above cannot be solved for W expressed in terms of elementary functions. Instead, the wet-bulb temperature is typically calculated numerically as a function of T and H. The figure below shows level curves of W for T between 273 K and 322 K (32°F to 120°F) and H between 0 and 1 (0% to 100%). The region where the wet-bulb temperature is above 95°F (308 K), or above the limit humans can tolerate, is shaded in red. A warm summertime day in Boston, with relative humidity around 70% (H = 0.7), dry-bulb temperature around 85 °F (T = 300 K), and a wet-bulb temperature of 295 K (71° F) is marked with a blue dot.



Although solving explicitly for wet-bulb temperature requires numerical calculations, linear approximation is a useful tool for quantifying the sensitivity of wet-bulb temperature to changes in dry-bulb temperature and humidity. In the following problems, we will use linear approximation to examine how heat stress during a warm summertime day in Boston changes when the weather becomes hotter and more humid.

Note Relative humidity itself is typically determined from physical measurements of the wet-bulb and dry-bulb temperatures. That is, $H = H(T,W)$. However, in the series of questions that follows, we consider H and T to be independently measured quantities.

3.4 Problems

3.4.1 Identify independent and dependent variables

We are interested in understanding the wet-bulb temperature.

1. Identify the independent variables in the equation for the wet-bulb temperature.
2. Identify the constants determined by material and physical properties.

3.4.2 Compute the differential

Using the implicit function

$$T - W = a\left(e^{-b/W} - He^{-b/T}\right), \tag{3.2}$$

find an expression for the differential dW in terms of the variables H, T and W, the differentials dT and dH and the constants a and b.

(Hint: To take the differential of an implicitly defined function, we take the differential of each side independently.)

3.4.3 Sensitivity to temperature and humidity

A warm summertime day in Boston has relative humidity around 70% (H = 0.7), dry-bulb temperature around 85 °F (T = 300 K), and a wet-bulb temperature of W = 295 K (71° F). We are interested in determining how sensitive is the wet-bulb temperature to increases in the dry-bulb temperature and to increases in relative humidity. Recall that $a = 4.2 \times 10^9$ Kelvins and $b = 5400$ Kelvins.

Compute W_T and W_H on a day described above.

3.4.4 Which creates a larger heat stress?

Recall that a warm summertime day in Boston has relative humidity around 70% (H = 0.7), dry-bulb temperature around 85 °F (T = 300 K), and a wet-bulb temperature of W = 295 K (71° F).

Use linear approximation to determine which would produce a larger increase in heat stress as measured by wet-bulb temperature:

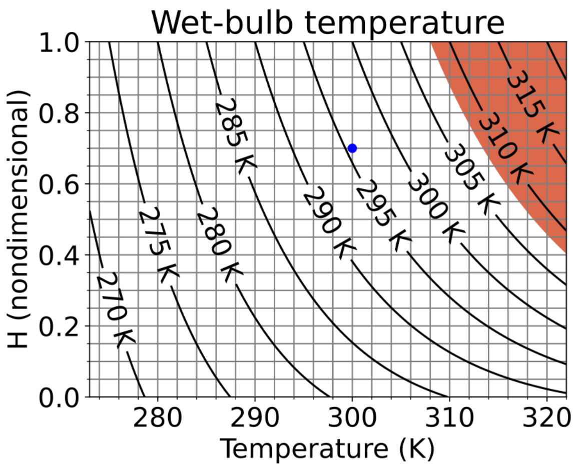
- a 5° F (2.8 K) increase in dry-bulb temperature
- a 0.2 increase (20%) in relative humidity
- they increase heat stress by the same amount

3.4.5 Increasing actual temperature

Recall that a warm summertime day in Boston has relative humidity around 70% (H = 0.7), dry-bulb temperature around 85 °F (T = 300 K), and a wet-bulb temperature of W = 295 K (71° F).

1. Approximate, using linear approximation, how large of an increase in dry-bulb temperature would be required to exceed the maximum wet-bulb temperature (308 K) that humans can tolerate.
2. Compare the approximation to the dry-bulb temperature increase seen in the level curves image. According to the image, how large of an increase in dry-bulb temperature would be

required to exceed the maximum wet-bulb temperature (308 K) that humans can tolerate? (Enter to the nearest degree.)



3.4.6 Increasing relative humidity

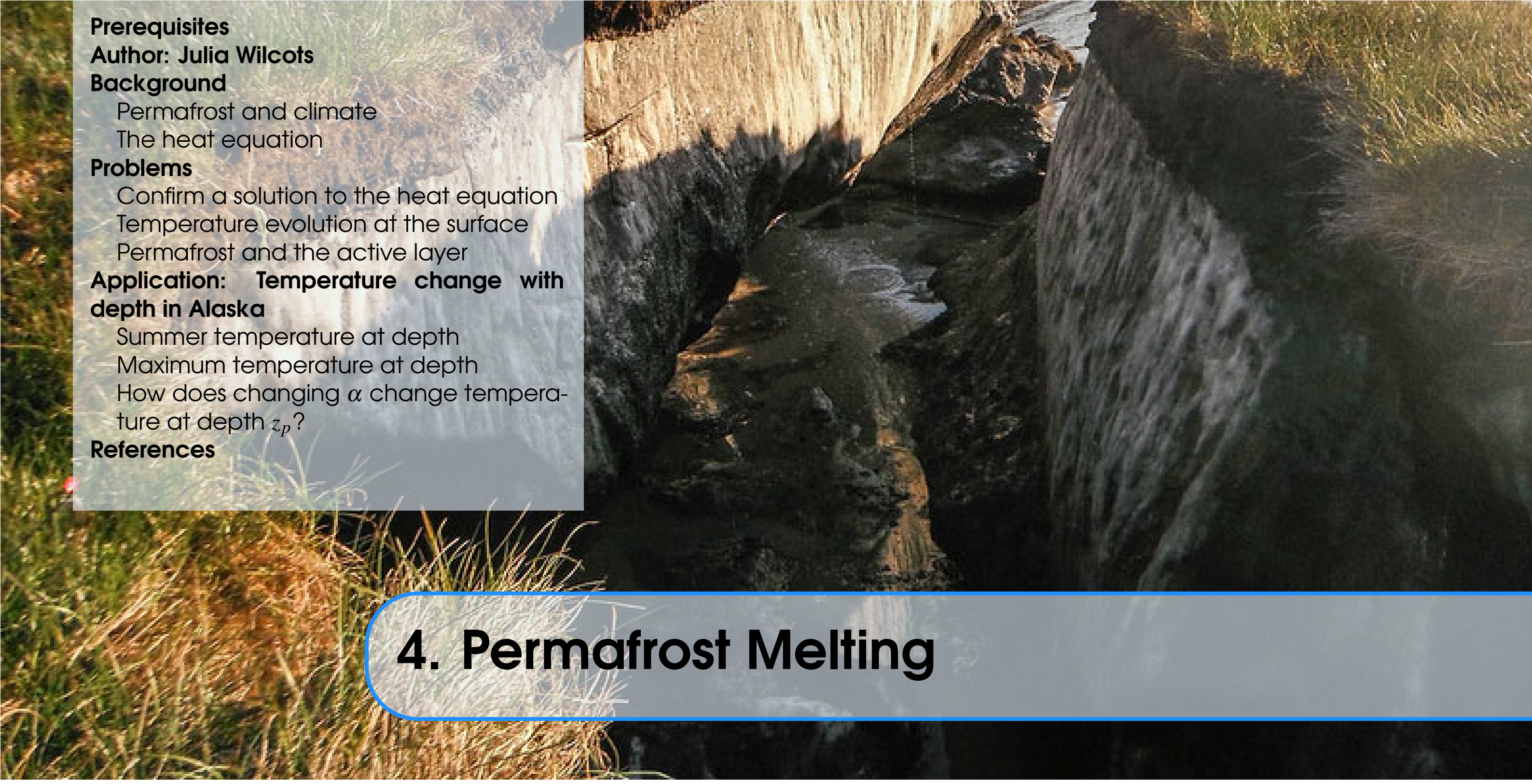
Recall that a warm summertime day in Boston has relative humidity around 70% (H = 0.7), dry-bulb temperature around 85 °F (T = 300 K), and a wet-bulb temperature of W = 295 K (71° F).

1. Approximate the maximum wet-bulb temperature experienced by increasing the relative humidity to 1.
2. Note that when the relative humidity is equal to 1, we expect the wet-bulb and dry-bulb temperatures to agree. What is the absolute value of the percent error in your approximation above?

3.5 References

For more information on psychrometrics, the study of the thermodynamic properties of air-water vapor mixtures:

- Wikipedia: Psychrometric charts
- Psychrometric chart app from Ashrae



4. Permafrost Melting

4.1 Prerequisites

- Partial derivatives
- Level curves
- Chain rule

4.2 Author: Julia Wilcots

Julia Wilcots is a 5th year geology PhD student in MIT's Department of Earth, Atmospheric and Planetary Science. In her research, she uses sedimentary rocks deposited in ancient oceans to understand the climate and environments of Earth hundreds of millions of years ago, before animals or land plants had evolved! As an undergraduate at Princeton, Julia studied Civil and Environmental Engineering, a major which allowed her to develop a solid quantitative foundation. Now, as a sedimentary geologist, she pieces together geological puzzles using field observations, geochemical analyses, microscopy, and computational methods. Her work mapping and collecting rocks has taken her many places around the world, including the Arctic archipelago Svalbard, a trip which inspired her math question about permafrost melting.

4.3 Background

4.3.1 Permafrost and climate

Permafrost, defined as ground that remains frozen for at least two consecutive years, is found across the world in high latitude and high alpine regions. Importantly for climate, permafrost soils store large amounts of carbon in the form of dead, frozen organic matter. As long as permafrost soils remain perennially frozen, this carbon stays put and cannot be consumed and respired by microbes and released as CO_2 or CH_4, greenhouse gasses. Worryingly, however, human-induced climate change is disproportionately warming the Arctic and long-frozen permafrost soils are melting during the warm summer months. As permafrost melts, the organic carbon it stores is consumed by microbes, respired, and released as CO_2 into the atmosphere. Permafrost melting thus provides a positive feedback loop on global warming: as the planet warms, more permafrost melts, releasing more carbon, which via the greenhouse effect acts to further warm the planet. Permafrost melting is not just a concern because of global warming. Millions of people live in

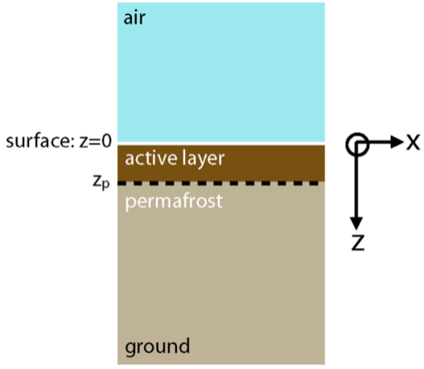


Figure 4.1: Depth profile through arctic soil. The active layer freezes and thaws seasonally, while permafrost stays frozen year-round.

the Arctic on permafrost and their infrastructure and the local ecological balances are threatened by melting permafrost. Here, we will explore how changes in air temperature propagate to warm permafrost below the surface.

4.3.2 The heat equation

The time (t) evolution of heat through a material with thickness z and thermal diffusivity α is governed by the heat equation

$$\frac{\partial T(x,y,z,t)}{\partial t} = \alpha \nabla^2 T(x,y,z,t) \tag{4.1}$$

$$= \alpha \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right). \tag{4.2}$$

In our study of permafrost melting, we are concerned with temperature variations with depth (z); we will assume the ground is infinite and homogeneous in composition in the x and y directions (see Figure 4.1), which simplifies the heat equation to one spatial dimension:

$$\frac{\partial T(z,t)}{\partial t} = \alpha \frac{\partial^2 T}{\partial z^2}. \tag{4.3}$$

where α, the thermal diffusivity, is a constant property of the material (in our case, frozen ground).

4.4 Problems

4.4.1 Confirm a solution to the heat equation

While solving the heat equation 4.3 is outside the scope of this class, we can confirm that solutions exist. Please show that

$$T(z,t) = Ae^{\left(\frac{-z}{z_w}\right)} \sin\left(\omega t - \frac{z}{z_w} + C_1\right) + C_2 \tag{4.4}$$

is a solution to the heat equation 4.3 by calculating the derivatives $\frac{\partial T}{\partial t}$ and $\frac{\partial^2 T}{\partial z^2}$. Then, write an expression for the thermal diffusivity (α) in terms of other variables; this term describes how quickly temperature diffuses through a material. (Do the units of α make sense?) The "skin depth" of the material is given by z_w and describes the depth at which temperature has attenuated by a factor of $1/e$. In our coordinate system, z_w is always positive (see Figure 4.1).

4.4.2 Temperature evolution at the surface

In locations on Earth that experience four seasons, annual surface air temperature can be approximated by a sinusoidal function with a period of one year (3.15576×10^7 s) where temperatures rise from midwinter through midsummer and fall from midsummer through midwinter. We define our coordinate system so that the surface (the interface between air and ground) is at depth $z = 0$ and set $C_1 = \frac{3\pi}{2}$ so that temperatures are at a minimum at $t = 0$. Write an expression for the temperature at the air-ground interface, $z = 0$.

4.4.3 Permafrost and the active layer

We will now define the top of the permafrost to be the depth z_p below which $T(z,t)$ for all t is less than 0°C. This is ground that will remain frozen all year. Soils between z_p and $z = 0$ are known as the "active layer" and freeze/thaw seasonally: as temperatures fall from summer through winter, the active layer freezes, but will thaw again down to depth z_p as temperatures rise the following year.

Abrupt, seasonal, freezing and thawing in the active layer drives many physical processes that affect the landscape and communities that live there, including ground collapse, rapid erosion, and the formation of short-lived lakes[1]. Determining z_p – and predicting how it will change with our warming climate – is crucial for understanding permafrost melting and its impact on carbon emissions, as well as for local communities' building and planning.

Determining the thickness of the active layer (i.e. the depth to permafrost) is difficult to do analytically, so scientists typically rely on field temperature measurements taken in a borehole (a core through the ground) to identify the active layer. However, we can still use equation 4.4 to explore interesting behavior of temperature at depth.

We want to write an expression for the thickness of the active layer (i.e. depth to permafrost), z_p, at a given time, t_0. (Find $z = z_p(t)$ such that $T(z,t_0) < 0$).

One way to do this is to consider the following:

1. Find all z_p such that $T(z = z_p, t = t_0) < 0$.
2. Find the shallowest (smallest) depth, $z_p(t_0)$, at which $T(z = z_p, t = t_0) < 0$.

Note A will always be positive for our problem set up (check out the Alaskan examples below for real-world values of A). ω (frequency) is equal to one cycle of yearly temperatures ($\omega = 2\pi/t$) and t is one year of time.

4.5 Application: Temperature change with depth in Alaska

Based on monthly average temperature data [2], we can approximate the annual (air) temperature curve for two locations in Alaska, Utqiagvik (~ 71°N) and Bettles (~ 66°N), shown in Figure 4.2:

$$T_{71^\circ}(t) = 16 \times \sin\left(\omega t + \frac{3\pi}{2}\right) - 10 \tag{4.5}$$

$$T_{66^\circ}(t) = 19 \times \sin\left(\omega t + \frac{3\pi}{2}\right) - 3 \tag{4.6}$$

$$\omega = \frac{2\pi}{3.15576 \times 10^7} \mathrm{s}^{-1}. \tag{4.7}$$

We'll now use these real-world examples to explore how temperature propagates with depth.

[1] See, for example Schuur et al. 2015

[2] Source: http://climate.gi.alaska.edu/

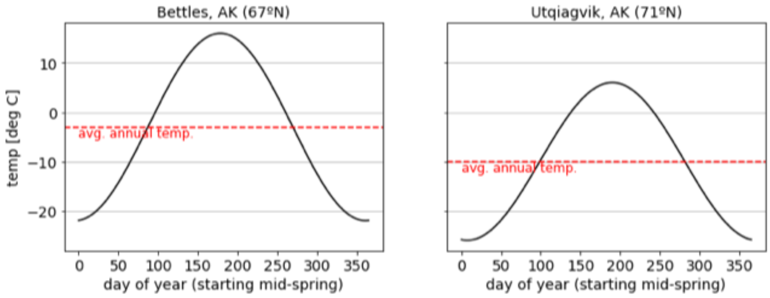


Figure 4.2: Approximation of annual (mean monthly) temperature variation for two Alaskan localities. See equations 4.5 and 4.6.

4.5.1 Summer temperature at depth

We have defined C_1 such that "January 1" occurs at $t = 0$ and halfway through the year ($t =$ year$/2$, "midsummer") surface temperatures are at a maximum.

1. What is the temperature at 1m depth at midsummer at Bettles?
2. How does it compare to the surface temperature at the same time of year? Use $\alpha = 0.2 \times 10^{-6}$ m^{2}s^{-1}.

4.5.2 Maximum temperature at depth

Now let's explore the general case of what we just saw with Bettles' midsummer temperatures. Imagine you are standing on the surface in Bettles at midsummer.

1. How many days will elapse (δt) before the ground 1 m below your feet ($z = 1$ m) reaches its maximum annual temperature?
2. For each meter of depth, how much later in the year does the maximum temperature occur?

4.5.3 How does changing α change temperature at depth z_p?

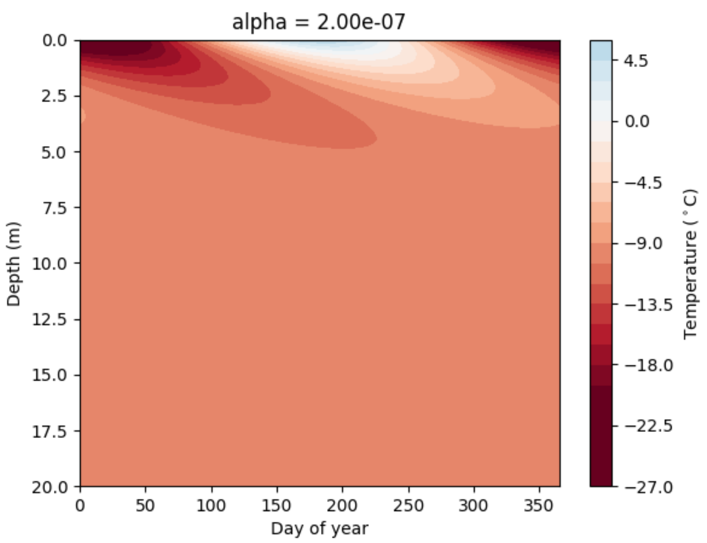


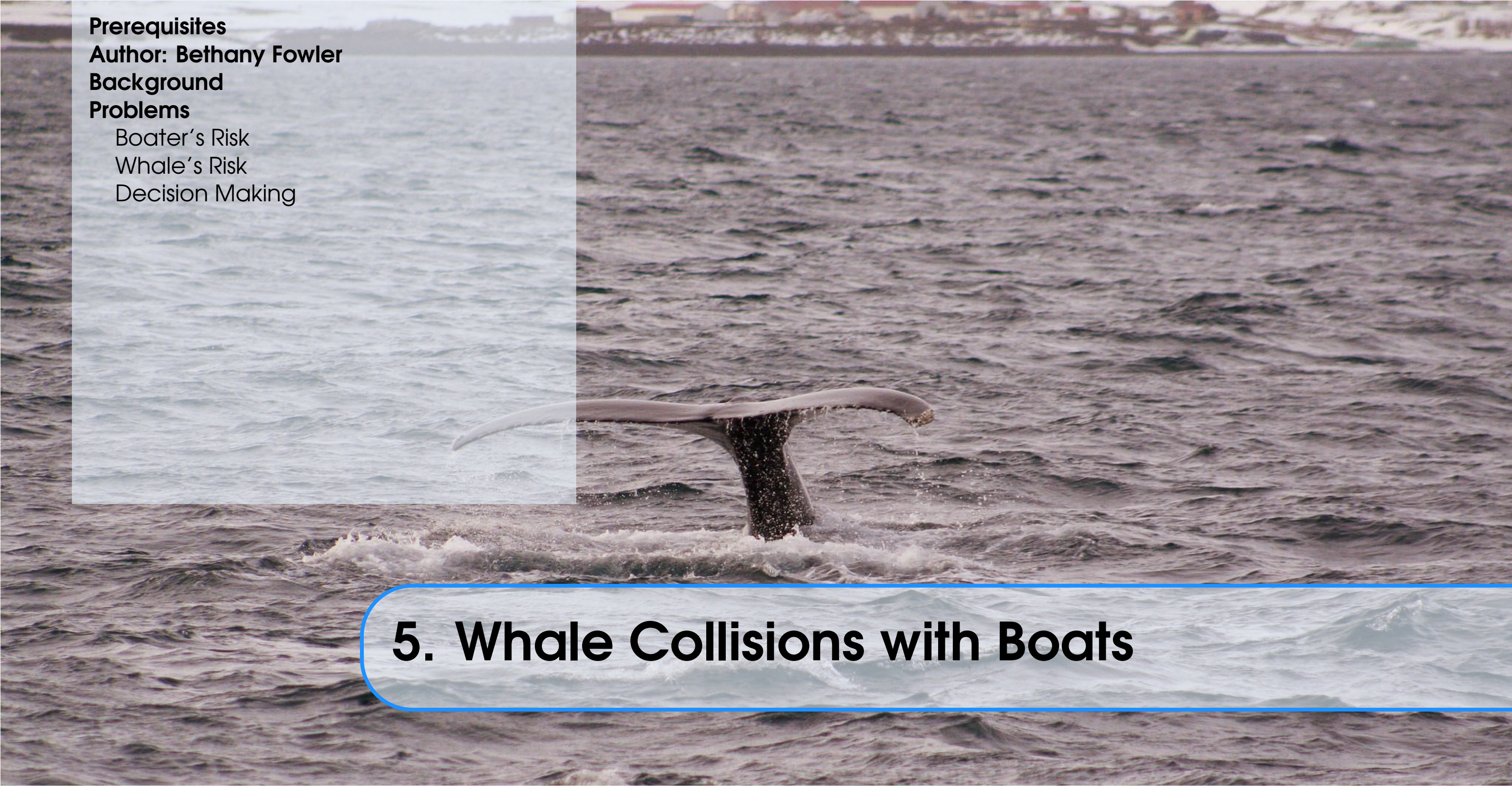
Figure 4.3: Temperature with time and depth for $\alpha = 0.2 \times 10^{-6}$ m^{2}s^{-1}.

In colder climates, where more of the ground is ice, α approaches 1×10^{-6} m^{2}s^{-1}.

1. Find δt if $\alpha = 1 \times 10^{-6}$ m^{2}s^{-1}.
2. How would Figure 4.3 change for this value of α?

4.6 References

- Schuur, E. A. G., and Coauthors, 2015: Climate change and the permafrost carbon feedback. Nature, 520, 171-179, https://doi.org/10.1038/nature14338.
- Learn more and find data on Alaskan climate available from Alaska Climate Research Center(https://akclimate.org/)
- Impacts of permafrost thaw on Arctic communities from Arctic Council.
- As the Arctic thaws, Indigenous Alaskans demand a voice in climate change research, https://dx.doi.org/10.1126/science.abe7149



5. Whale Collisions with Boats

5.1 Prerequisites

- Arc length
- Line Integrals

5.2 Author: Bethany Fowler

Bethany is a graduate student working towards a PhD in Biological Oceanography in the MIT-Woods Hole Oceanographic Institution joint program. As an undergraduate, she studied mathematics at Rice University and learned that math could be used to model, understand, and even protect natural ecosystems. She now studies the ecology of marine plankton with a combination of mathematical models and observations collected at sea. She loves gardening, beekeeping and spending any time by the water or in the sun.

5.3 Background

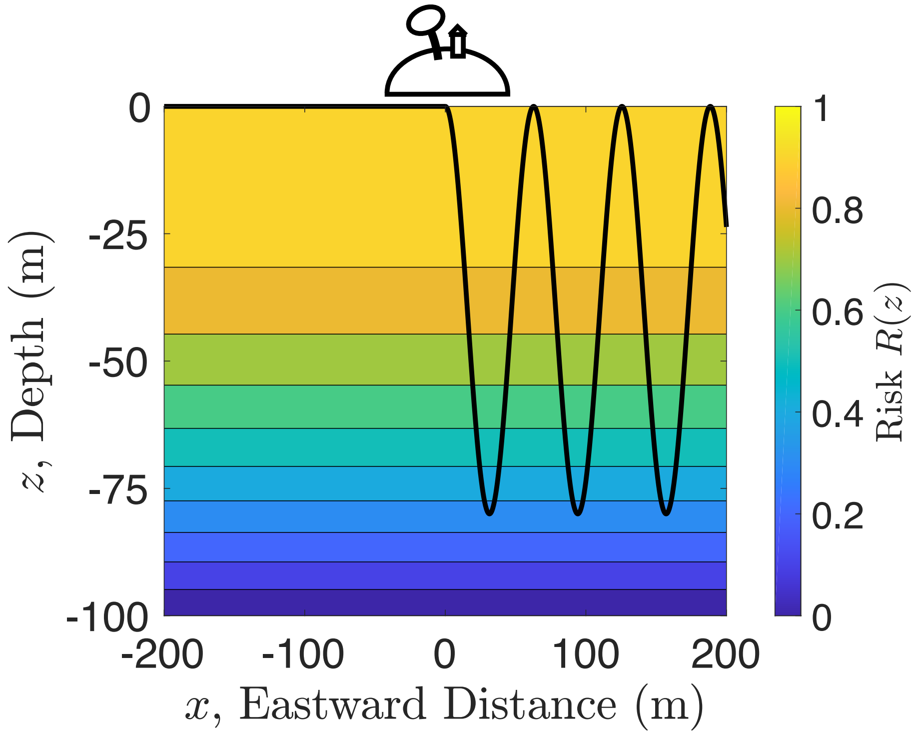
An island's government is going to limit boat traffic to protect a local species of whales. They plan to close one side of the island to traffic in order to reduce the risk of interactions between boats and whales (e.g. collisions and entanglements). Accounting for both the likelihood and severity of the interaction, the risk value, R, is a decreasing function of a whale's depth, z, in meters below the surface: $R(z) = 1 - (z/100)^2$.

By tagging and tracking the whales, biologists have found that the waters to the West of the island are used primarily for feeding at the surface and those to the East for deeper foraging dives. The depth of a whale at any location, x, relative to the island can be modeled by $z = C(x)$ where

$$C(x) = \begin{cases} 0 & \text{if } -200 < x < 0, \\ 40(\cos(\frac{x}{10}) - 1) & \text{if } 0 < x < 200. \end{cases} \tag{5.1}$$

The figure below shows the complete model. The whale behavior, indicated by the curve, C, is overlaid on the risk landscape shown by the color gradient.

Assume that the whales are evenly distributed across the path, C, at any point in time and that individual whales travel along this path at constant speed. Also assume that no whales travel beyond 200 meters of the island, or that this is beyond the government's region of interest.



5.4 Problems

5.4.1 Boater's Risk

1. Consider the risk experienced by a boater traveling along the surface of the water. Given that x is the east-west distance from the island, z is depth, and s is the distance measured along the curve, which integral expression bests describes the risk to the boater?
   (a) $\int_C R(z)\,dz$
   (b) $\int_C R(z)\,dx$
   (c) $\int_C R(z)\,ds$
   (d) $\iint_C R(z)C(x)\,dx\,dz$
   (e) None of these
2. How would you estimate the total risk experienced by an individual boat traveling directly to the island from the West at constant speed? Write your answer as an integral (from $x = -200$ to 0).

3. Write the corresponding integral that estimates the risk experienced by a boater traveling to the island from the East.

4. Which of the two values is greater?

5.4.2 Whale's Risk

Now consider the risk experienced by an individual whale traveling along the path C.

1. Which integral best describes the risk to the whale?:

   (a) $\int_C R(z)\,dz$
   (b) $\int_C R(z)\,dx$

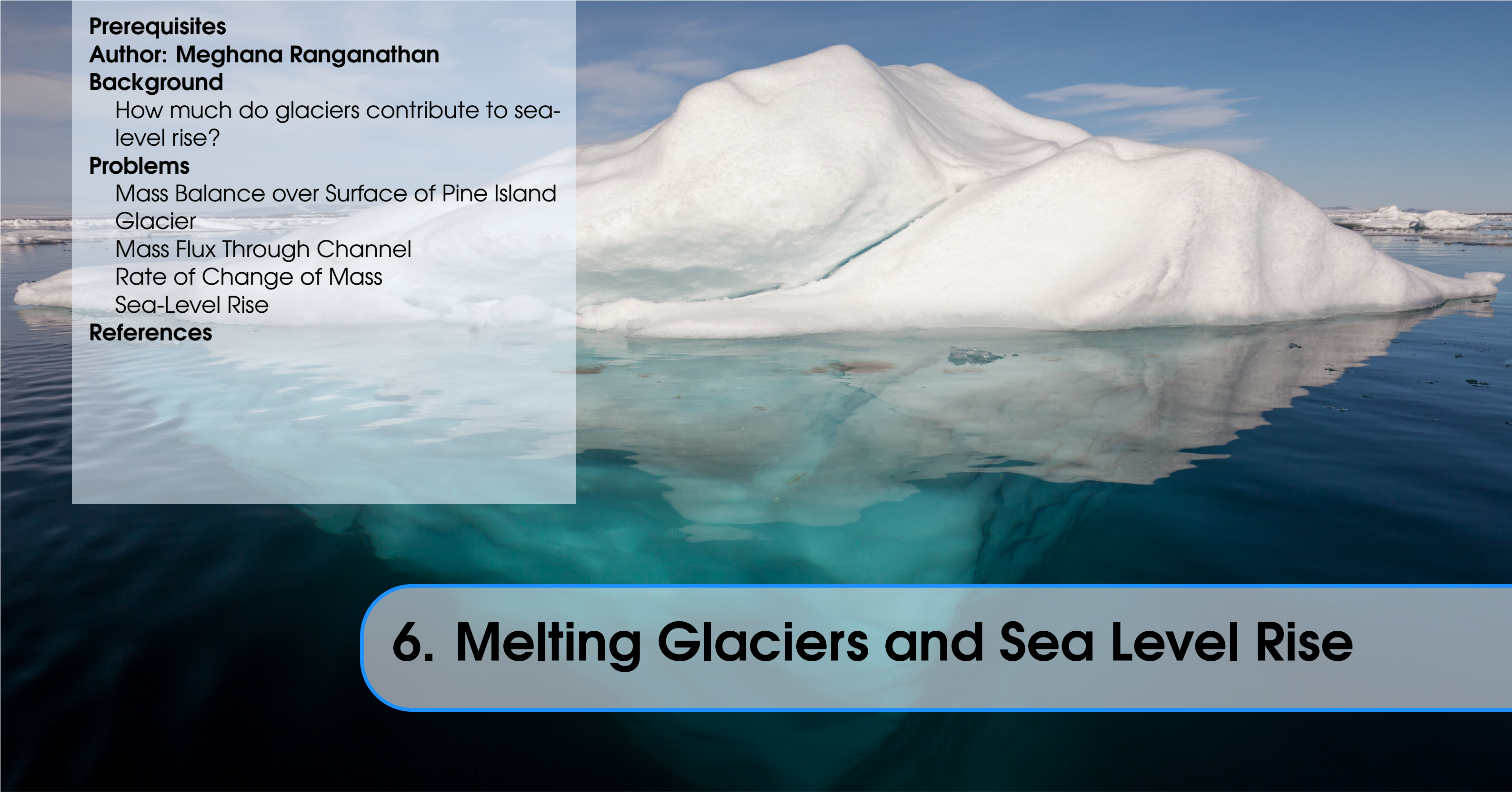
   (c) $\int_C R(z)\,ds$
   (d) $\iint_C R(z)C(x)\,dx\,dz$
   (e) None of these
2. Write the definite integral that describes the risk experienced by a whale traveling on the West side of the island. Evaluate this integral.
3. Write the definite integral that describes the risk experienced by a whale traveling on the East side of the island. Estimate this value graphically or with a computer.

5.4.3 Decision Making

1. Based on Question 1: Boater's Risk, which side of the island should be closed to traffic in order to minimize the risk experienced by the boaters?
2. Based on Question 2: Whale's Risk, which side of the island should be closed to traffic in order to minimize the risk experienced by the whales?
3. What are some of assumptions of this model which may not be realistic and might affect the final policy decision?
4. Which side of the island would you recommend closing to boating traffic and why?

Note

Key Takeaway: This problem was written to highlight the difference between the line integrals, $\int_C R(z)\,dx$ and $\int_C R(z)\,ds$. Note how the variable of integration changes how you both interpret and evaluate the expressions.



6. Melting Glaciers and Sea Level Rise

6.1 Prerequisites

- Flux

6.2 Author: Meghana Ranganathan

Meghana Ranganathan received a Ph.D in Climate Science from the Massachusetts Institute of Technology and is currently a postdoctoral fellow at the Georgia Institute of Technology, studying the Antarctic Ice Sheet. As a high schooler, Meghana had no idea what she wanted to do - except she knew she hated math and wanted to help the environment. When she went to college, she (somewhat reluctantly) took one math class and it changed the way she thought about math. Rather than being a series of problems with no goal, math became a tool to better understand our world. She ended up majoring in math and choosing to use math to understand how our climate changes. In her research now, she works towards understanding how glaciers and ice sheets respond to climate changes and how large ice sheets (such as Antarctica!) will change in the next few centuries. She uses a ton of math and computer modeling to explain how glaciers are accelerating and fracturing into icebergs, and she dreams of one day going to Antarctica. She also hopes to work towards creating community collaborations, working with community leaders, policymakers, and activists towards a more sustainable and just future.

6.3 Background

6.3.1 How much do glaciers contribute to sea-level rise?

Global sea-level rise is of significant concern as our climate changes. Sea-level rise is already resulting in coastal erosion, increased flooding and destruction of coastal cities and communities, and forced migration. Climate change often most impact the most vulnerable populations, with low-income communities and communities of color disproportionately impacted. Due to segregation, economic disinvestment, and environmental injustices that include increased proximity to toxic sites that are vulnerable to flooding, sea level rise is one dimension of the increased vulnerability of low-income and communities of color to sea level rise.

Much of the current concern surrounding global sea-level rise comes from the two ice sheets - the Greenland Ice Sheet and the Antarctic Ice Sheet. In particular, the Antarctic Ice Sheet has

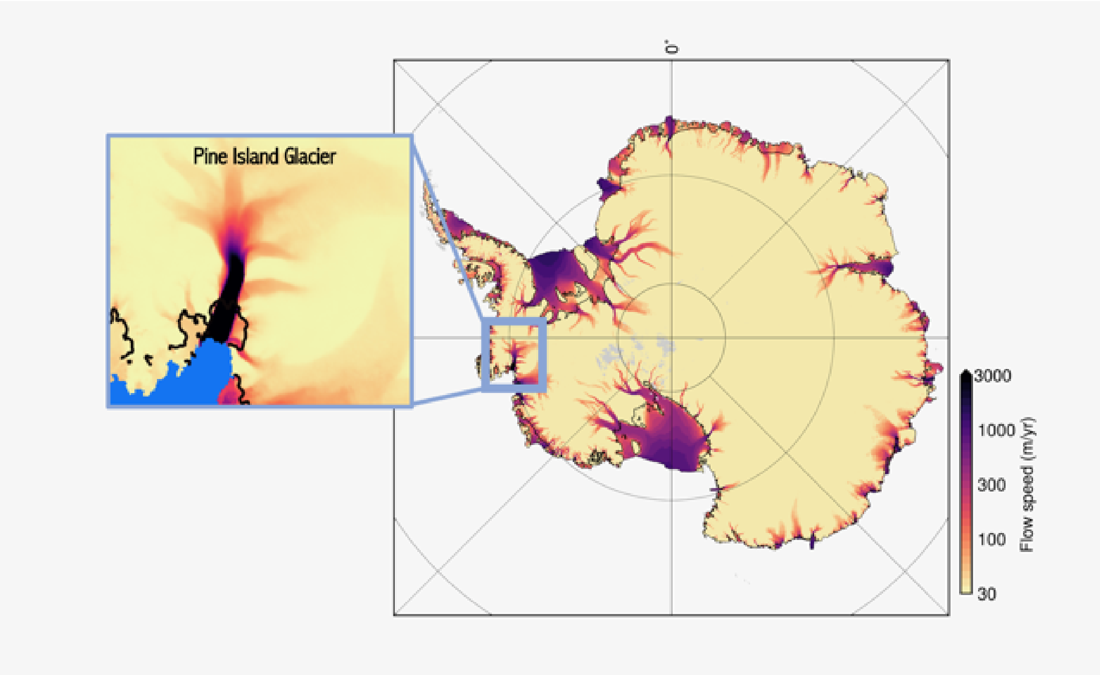


Figure 6.1: The Antarctic Ice Sheet, showing how fast the ice is moving in meters per year. The inset shows Pine Island Glacier.

the potential to add significantly to global sea-level rise over the next 100-500 years, making it a significant area of attention for scientific research. **Here, you will be estimating how much one Antarctic glacier may contribute to sea-level rise in the next 100-500 years.**

Glaciers are functionally rivers of ice that "flow" from "upstream" (near the center of the ice sheet) to "downstream", where they eventually flow onto the ocean and the ice breaks off into icebergs. The icebergs float away and melt. This is how Antarctica loses most of its mass - by glaciers flowing to the ocean and then breaking off into the ocean. The glacier we're focusing on here is called Pine Island Glacier, which is one of the fastest-flowing glaciers in Antarctica and considered to be the possible site of collapse of half of the Antarctic Ice Sheet due to how much mass it loses. Figure 6.1 shows the whole Antarctic ice sheet (with the colormap showing how fast the ice is moving) and the inset shows Pine Island Glacier from "upstream" to "downstream" (at the ocean).

Here, you'll compute how much mass is lost from Pine Island Glacier each year and then how much sea-level rise that mass loss creates. We will assume that Pine Island Glacier is a simple rectangular prism sitting on top of rock of height $H = 1,000$ m, length $L = 100,000$ m, and width $w = 40,000$ m (Figure 6.2). To compute the total mass loss, we will have to estimate how much ice mass is added every year (Problem 1.1), then how much mass flows through the channel and is lost (Problem 1.2). We will then compute total mass loss (Problem 1.3) and estimate how much sea-level rise equivalent this is (Problem 1.4).

6.4 Problems

6.4.1 Mass Balance over Surface of Pine Island Glacier

Mass is added to a glacier by snow falling on the surface of the glacier and, over time, compacting into ice. How much mass is added to the surface at each point is called the *surface mass balance* of the glacier. Surface mass balance (in meters per year) is generally found from regional climate

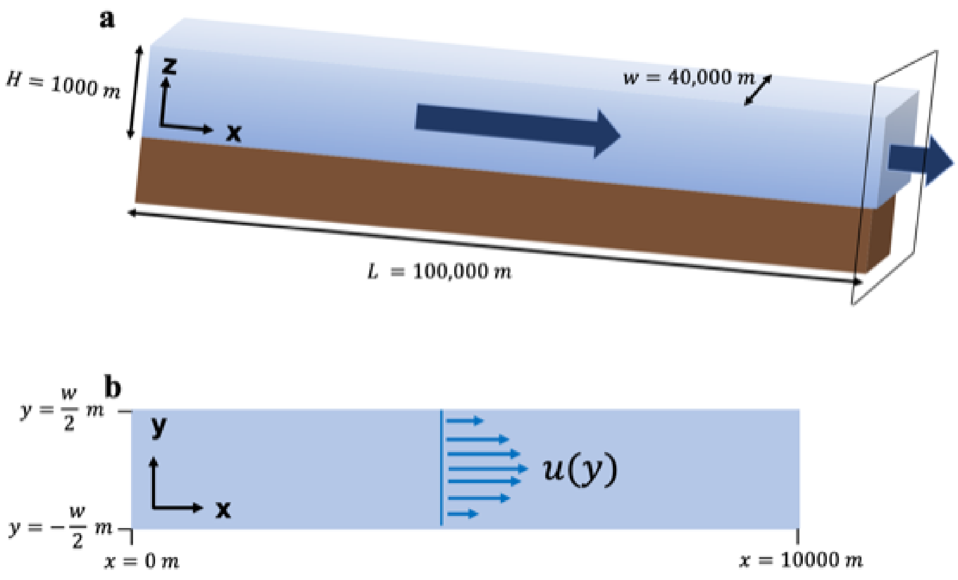


Figure 6.2: The problem set-up. (a) A 3-dimensional view of the glacier. The glacier can be approximated as a rectangular prism, of height $H = 1,000$ m, length $L = 100,000$ m, and width $w = 40,000$ m, flowing uniformly with depth. (b) A top-down (birds' eye) view of the glacier. The glacier does not flow uniformly in the y direction, however. The glacier flows fastest in the centerline and the flow speed tapers off towards the edges of the glacier.

models showing how much snowfall occurs in Antarctica along with an estimate of how long it takes for snowfall to compact into ice. Figure 6.3a shows the surface mass balance computed from one of these regional climate models (called RACMO) over Pine Island Glacier, in which surface mass balance increases slightly "downstream", then decreases, and is generally higher on the southern part of the glacier (the top of the box in Figure 6.3a), likely due to mountains increasing elevation. We will approximate this surface mass balance by the following equation:

$$\dot{b}_i = 0.2\sin\left(\frac{\pi}{L}x\right) + 0.6\left[1 + \frac{y + \frac{w}{2}}{20w}\right] \tag{6.1}$$

where $\dot{b}_i$ has units of m yr^{-1}. Compute the mass added to the surface of all of Pine Island Glacier each year, noting that $\rho_i = 917$ kg m^{-3} is the density of ice.

6.4.2 Mass Flux Through Channel

This glacier is moving towards the ocean in such a way that the flow does not change with depth (Figure 2a). However, because of the unique properties of ice, the flow does change across the glacier (Figure 2b). The ice moves fastest in the centerline and slowest in the margins. The flow of ice can be approximated by the following equation:

$$u(y) = C\left(\frac{w}{2}\right)^{n+1} - C\left(\frac{w}{2} - y\right)^{n+1} \tag{6.2}$$

where $C = 8 \times 10^{-14}$ yr^{-1} m^{-3} is a constant representing the viscosity of ice and the stress the ice is experiencing and $n = 3$ is a constant that dictates how fast ice flows due to an applied stress. Compute the total mass flux of ice out of the Pine Island Glacier channel per year.

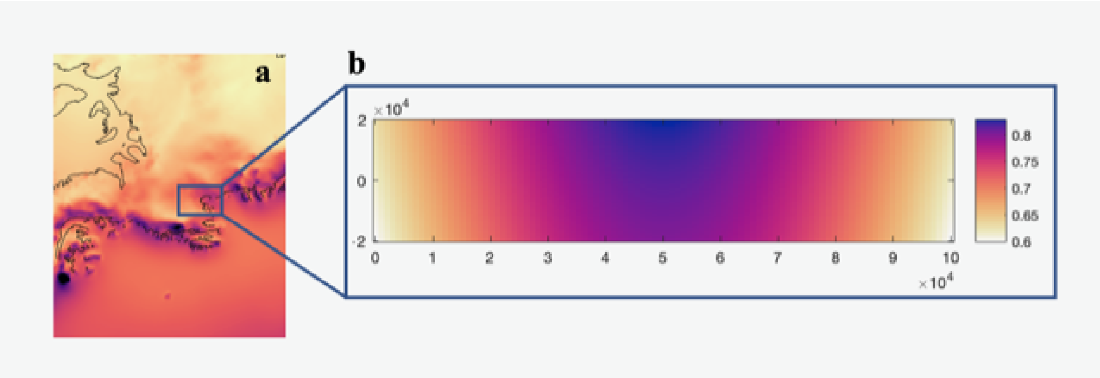


Figure 6.3: Surface Mass Balance: (a) Output from a regional climate model showing surface mass balance in West Antarctica, with the box drawn around Pine Island Glacier. (b) An approximation of surface mass balance found using Equation 6.1.

Important note + hint: The model in Equation 6.2 is only valid from $y = 0$ to $y = \frac{w}{2}$, which is from the centerline of the glacier to one of the margins. However, there is symmetry with respect to the centerline $y = 0$.

6.4.3 Rate of Change of Mass

The total rate of change of mass on Pine Island Glacier is found by subtracting the mass flux out of Pine Island Glacier from the mass added to Pine Island Glacier. Estimate the total rate of change of mass on Pine Island Glacier per year in Gigatonnes per year (noting that 1 Gt $= 1 \times 10^{12}$ kg).

6.4.4 Sea-Level Rise

We can estimate the contributions to global sea-level rise by assuming that global sea levels change based on the change in total volume of the ocean. Volume is added to the ocean through mass flux into the ocean from glacier flow, and volume is subtracted from the ocean by the ocean evaporating and precipitating as snow on top of the glacier (defined by glacier mass balance).

However, sea-level rise is generally described as a change in ocean height, rather than ocean volume. Once we have the change in the volume of the ocean, we note that the change in the volume of the ocean is equal to the surface area of the ocean multiplied by the change in the height of the ocean:

$$\text{Surface Area of Ocean} \times \Delta H_o = \Delta V \tag{6.3}$$

Estimate how much Pine Island Glacier is contributing to global sea-level rise each year. How much total sea-level rise will Pine Island Glacier produce over the next 100 years? How about 500 years? Assume that the radius of the Earth is $R = 6.378 \times 10^6$ m and assume that 70% of the surface of the Earth is ocean.

6.5 References

- Bick, I. A., and co-authors (2021). Rising seas, rising inequity? Communities at risk in the San Francisco Bay Area and implications for adaptation policy. Earth's Future, 9(7), e2020EF001963.

- Toxic Tides: Sea Level Rise, Hazardous Sites, and Environmental Justice in California `https://sites.google.com/berkeley.edu/toxictides`
- Pine Island Glacier from antarcticglaciers.org

7. Flux Through a River

7.1 Prerequisites

- Flux

7.2 Author: Adrian Mikhail P. Garcia

Adrian Mikhail P. Garcia is a PhD candidate in the MIT-WHOI Joint Program in Applied Ocean Science & Engineering studying Environmental Fluid Mechanics. In his research, he applies numerical modeling and observational studies to understand hydrodynamics and transport processes in the environment, with the goal of contributing towards the sustainable development of the coastal zone. Furthermore, Adrian is passionate about improving STEM representation among historically excluded groups through mentorship and educational outreach.

7.3 Background

Environmental engineers are often concerned with the transport of contaminants in the environment. For example, they might work on quantifying how much of a contaminant might be flowing in a river downstream from a chemical spill and how quickly it will might reach a certain threshold concentration. To do this, environmental engineers take discrete measurements of water velocity and contaminant concentrations to calculate fluxes through a river cross-section.

The total volume flowrate through a river cross-section can be defined as

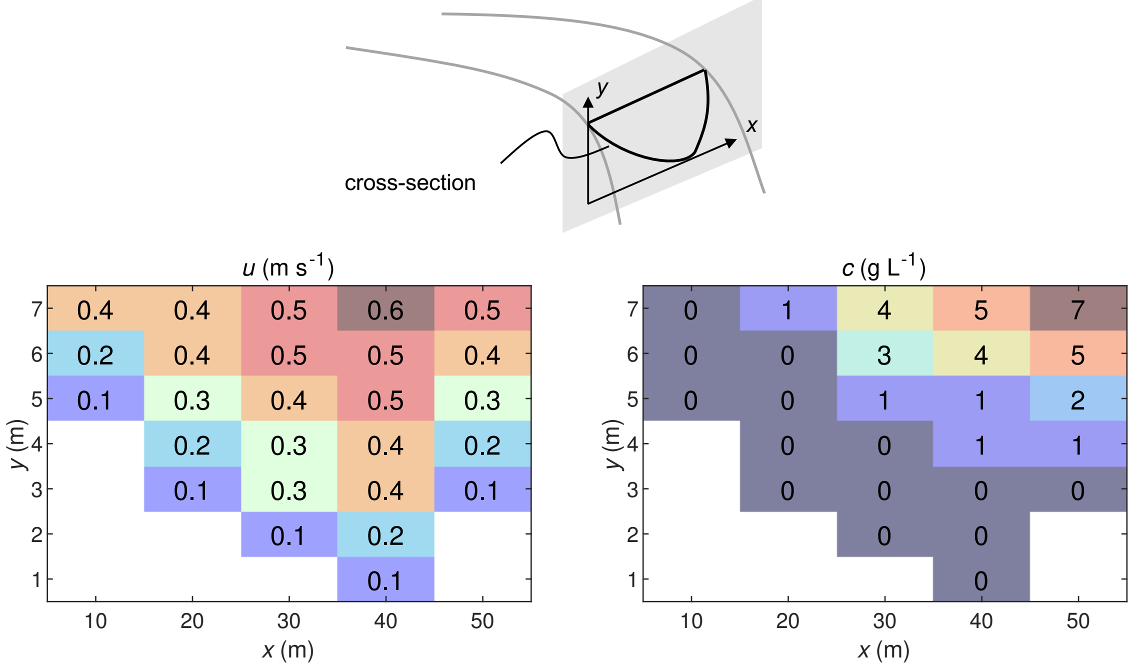
$$Q = \iint u\,dx\,dy, \tag{7.1}$$

where Q is the volume flowrate and $u(x,y)$ is the velocity normal to the cross-section, which is defined on the x,y plane. For a contaminant with concentration $c(x,y)$, the total flux is defined as

$$F = \iint uc\,dx\,dy. \tag{7.2}$$

7.4 Problems

In this problem, we will practice computing fluxes given discrete points which define an arbitrary function that varies over 2D space (x,y). We will consider a set of evenly-spaced velocity and



concentration measurements across a river, as shown in the schematic and figure below.

7.4.1 Volume flowrate

Recall that the integral of a continuous function $f(x)$ over an interval $[a,b]$ is defined by dividing the interval into n subintervals of width Δx and taking the limit as $n \to \infty$, i.e.,

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x. \tag{7.3}$$

For discrete measurements, n is a finite number – however, if it is sufficiently large, then the errors are not too significant. Write a discrete version of the integral given by (7.1), then compute the volume flowrate Q, with units of $\mathrm{m^3\,s^{-1}}$. Note: missing values can be treated as zeros.

7.4.2 Flux

Now, write a discrete version of the integral given by (7.2) and then compute the contaminant flux F, with units of $\mathrm{kg\,s^{-1}}$. Note: $1\,\mathrm{g\,L^{-1}} = 1\,\mathrm{kg\,m^{-3}}$.

8. Plastic Accumulation in the Oceans

8.1 Prerequisites

- Flux
- Divergence
- Green's Theorem

8.2 Author: Arianna Krinos

Arianna Krinos is a PhD candidate in Biological Oceanography in the MIT-WHOI Joint Program. She is an aquatic microbial ecologist with special focus on protists. Primarily, she uses bioinformatics and experimental culture work to infer trends in the population and community ecology of eukaryotic microbes. Using bioinformatics, we can sequence the genetic code of organisms to create a string of data that corresponds to the physical reality of the instruction manual of each cell. Processing these instructions systematically with computer programs can help decode the family tree of each type of algae and the decisions that it makes when faced with a new environment. She is excited about teaching, especially with respect to computer science and data analytics education. Arianna grew up excited about freshwater and marine science, but gradually became interested in mathematics and computation. Finding the intersection between the two in the form of environmental bioinformatics has inspired Arianna to explore new ways that quantitative tools can be used to solve ecological problems.

8.3 Background

Plastic pollution is becoming a major problem, with over 300 million tons of plastic waste being generated every year. Eight million tons of this waste ultimately makes it to the ocean, and is a significant environmental hazard for wildlife. What's worse, sites of plastic accumulation are growing. One site is the Great Pacific Garbage Patch, a millions-of-square-kilometer swatch of ocean with a swirling mass of more than 70,000 metric tons of plastics. (See references for more information.)

Plastic *floats* on the surface of the ocean, moving along with the surface velocity. We assume the plastic floats with the same velocity as the surface flow. Thus, whereas the vertical depth component is important in the movement of ocean water, we can model changes and accumulation in ocean plastic by focusing on horizontal convergence or divergence. Thus, we can write the equation below, with solely u and v components.

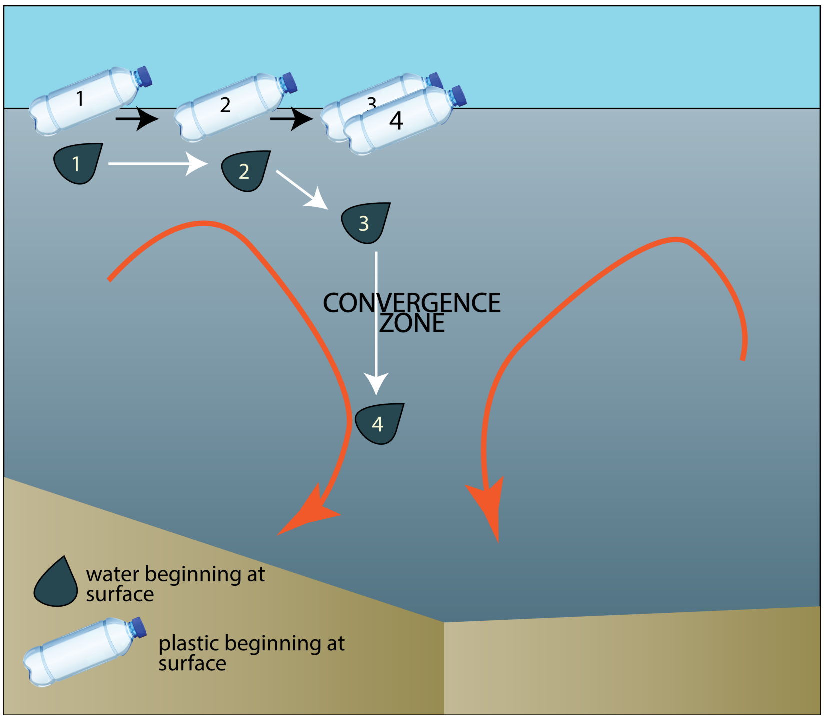


Figure 8.1: Why plastics can accumulate even when surface water does not.

8.4 Problems

8.4.1 Surface velocity

Let's say we have a model for flow as, where d is some constant and $\mathbf{u}$ is the velocity vector for the surface flow:

$$\mathbf{u} = -\text{curl}(\hat{\mathbf{k}}\psi(x,y)) - d\nabla\psi(x,y)$$

Show that, in *Cartesian* coordinates, this expression for $\mathbf{u}$ is equal to:

$$(u,v) = \hat{\mathbf{k}} \times \nabla\psi - d\nabla\psi$$

8.4.2 Conditions for accumulation

Using the result from 8.4.1, what value of $\mathbf{u}$ causes plastic to potentially accumulate?

8.4.3 Accumulation and divergence of ψ.

The sign of the expression $\nabla \cdot \mathbf{u}$ determines whether the plastic is moving closer to or further from a given point.

Given what we have established for the velocities, how is the expression for the divergence related to the function $\psi(x,y)$?

8.5 Exploration of a special case

Let's say we are given a domain $-2L \leq x \leq 0$ and $-\frac{L}{2} \leq y \leq \frac{L}{2}$ for the function ψ, given as:

$$\psi(x,y) = -\frac{UL}{\pi}\frac{x}{L}\cos\left(\frac{\pi y}{L}\right) + \frac{VL}{\pi}\sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{\pi y}{L}\right)$$

and can assume $U < V\pi$.

8.5.1 Locations of accumulation

Write an expression for locations in x and y where there may be plastic accumulation, given our equation for the flow and the function ψ as defined above.

8.5.2 Divergence of the surface velocity

What is the divergence of $\mathbf{u}$?

8.5.3 Deep dive into divergence expression

In 8.4.2, you identified the location(s) where the flow is stationary. Assuming that $V >> U$, what are the signs of the divergence expression from 8.5.2 at these locations of potential plastic accumulation?

8.5.4 Deep dive into divergence expression

Based on your results from the previous question 8.5.3, at which of the stationary points from 8.4.2 do you expect plastic to accumulate?

Note

While we use an example of a water bottle floating on the sea surface to illustrate convergence, the majority of the plastic in the ocean is microplastics. These tiny fragments can be seen in the image in the problem header, which shows microplastics (multicolored objects) mixed in with other objects that are naturally found at the sea surface, including a fish and seaweed, that were collected by dragging a net through the water from a ship.

8.6 References

- *Beat Plastic Pollution: Our planet is drowning in plastic pollution?it's time for change!* UN Environmental Program.https://www.unep.org/interactive/beat-plastic-pollution/
- Lebreton et. al. (2018). *Evidence that the Great Pacific Garbage Patch is rapidly accumulating plastic*. Scientific Reports. 8(1), 1–15.
- *Marine Plastics*. (2018). International Union for the Conservation of Nature. https://www.iucn.org/resources/issues-briefs/marine-plastics