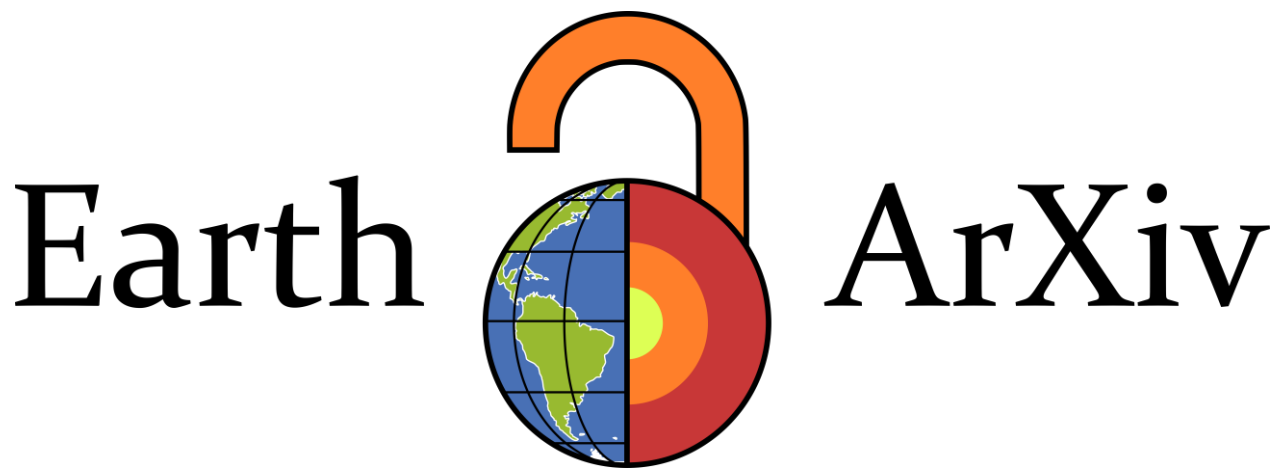




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**Resonant Platform Response and Vertical Velocity Biases in ADCP
Measurements from Quasi-Lagrangian Platforms**

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10 ABSTRACT: Autonomous surface and subsurface platforms equipped with acoustic Doppler
11 current profilers (ADCPs) are increasingly used to observe ocean velocities, but in the presence
12 of surface waves these measurements can be biased by orbital motion and wave-induced platform
13 tilting. Previous work quantified such biases for idealized platform responses that were in phase
14 with the wave forcing. Here we extend this framework to the general case of a partially resonant
15 platform response, in which the tilt amplitude is enhanced and phase-lagged relative to the waves, as
16 expected for real-world platforms. We derive analytical expressions for wave-induced ADCP biases
17 under arbitrary linear tilt response and show that phase-lagged platform motion generates biases
18 in vertical velocity in addition to previously reported horizontal biases. These vertical biases scale
19 with the imaginary part of the platform tilt transfer function and depend on wave properties, platform
20 depth, and measurement distance. Biases also depend on ADCP beam geometry, alignment with
21 wave propagation, and instrument orientation (upward or downward). Under certain conditions, a
22 five-beam vertical velocity reconstruction can be formed that is unbiased on average and generally
23 outperforms both the standard four-beam and the vertical-beam estimates. The theory is applied
24 to a Lagrangian float whose empirically determined tilt response suggests damped resonance at a
25 3-s wave period. Using realistic wind-wave spectra, we find typical biases of several centimeters
26 per second for horizontal velocities and several millimeters per second for vertical velocities under
27 open-ocean conditions. Because wave-induced biases depend strongly on platform configuration,
28 we provide an analytical framework and numerical tools to assess biases for individual platforms
29 and deployments.

30 SIGNIFICANCE STATEMENT: Autonomous ocean platforms commonly use acoustic Doppler
31 current profilers (ADCPs) to measure currents, but surface waves can induce platform motions
32 that affect these measurements. This study shows that partially resonant, phase-lagged platform
33 motion – expected for most real-world systems – can introduce systematic biases in velocity
34 estimates, including the biases in the vertical component that have not been previously identified.
35 We present a general analytical framework that accounts for platform dynamics, wave conditions,
36 and instrument geometry and enables deployment-specific bias assessment and mitigation. These
37 results are directly relevant to the interpretation of ADCP observations from autonomous platforms
38 and inform the design of future observing systems targeting weak vertical motions in the ocean.

39 1. Introduction

40 Compact autonomous marine vehicles, both surface and submersible, are now commonly used to
41 conduct observations of ocean velocities using acoustic Doppler current profilers (ADCPs). These
42 platforms enable persistent and spatially distributed measurements in regions or conditions that are
43 difficult to sample from ships, moorings, or bottom-mounted systems. When operated in a quasi-
44 Lagrangian mode, they also reduce the effect of dominant advection past the instrument, allowing
45 weaker relative currents, shear, vertical velocity, and turbulence-relevant velocity structure to be
46 resolved. However, in the inevitable presence of surface waves, ADCP measurements conducted
47 by these platforms are susceptible to biases stemming from wave-coherent orbital motion and
48 platform tilting. Our previous paper on this subject (Shcherbina and D’Asaro 2025, hereafter
49 SD25) derived analytical expressions and numerical estimates of wave-induced biases across a
50 range of scenarios. Among these, two limiting cases of platform tilt response to wave forcing
51 were examined: a hydrostatic response, in which the platform instantaneously aligns with the local
52 antigravity direction, and an inertial response, in which it aligns with the vertical material-line
53 vector.

54 Real-world sampling platforms can exhibit more complex behavior than either idealized limit. In
55 particular, some degree of resonant response may occur, producing both an increase in the platform
56 tilt amplitude and a phase shift relative to the wave forcing. It can be anticipated that the amplitudes
57 and vertical structure of wave-induced biases would be altered in this case. As will be shown in

58 this paper, *vertical* velocity bias can arise in addition to the horizontal velocity biases considered
59 in previous studies.

60 D’Asaro and Shcherbina (2026, hereafter DS26) investigated the actual response of an APL
61 Lagrangian Float (MLF) using data from multiple deployments under a range of wave conditions.
62 The float was found to exhibit partial resonance at a wave period of approximately 3 s. A full
63 response model for wave periods from 2 to 20 s was also developed. In this paper, we evaluate
64 the wave-induced biases associated with the specific response function of the MLF. Our analytical
65 framework, however, is formulated in terms of platform motion, tilt response, ADCP orientation,
66 beam geometry, range, and averaging, and is therefore applicable to other platforms with different
67 response functions and sampling configurations.

68 In section 2, we briefly revisit the analytical and numerical frameworks used in our analysis.
69 Section 3 then examines wave-induced platform tilt, including the phase-lagged resonant response,
70 and the resulting motion of the ADCP sampling volume. In Section 4, we derive analytical
71 expressions for wave-induced biases in the general partially resonant case and show how a phase-
72 lagged tilt response leads to vertical velocity bias. Section 5 explores alternative vertical-velocity
73 reconstruction methods that reduce the overall bias. In Section 6, we apply this framework to
74 quantify wave-induced biases for the measured response of a specific Lagrangian float, illustrating
75 how the general theory can be used for deployment-specific bias assessment. The main findings
76 are summarized and discussed in Section 7.

77 **2. Methods**

78 *a. Analytical framework*

79 We start with a short recap of the analytical framework developed in SD25; readers are referred
80 there for a more complete treatment. As before, we consider a quasi-Lagrangian platform conduct-
81 ing velocity measurements in the presence of surface gravity waves. For simplicity, we postulate a
82 monochromatic deep-water linear wave with the amplitude a , wavenumber k , and cyclic frequency
83 ω propagating in the x direction. Its surface elevation is given by

$$\eta = a \sin \phi, \tag{1}$$

where $\phi = kx - \omega t$ is the wave phase. The wave amplitude a is small compared to the wavelength, so that the wave steepness parameter $(ak) \ll 1$. We express coordinates in the vertical (x, z) plane using complex notation, $X = x + iz$. With this notation, orbital motion of a fluid particle can be expressed as

$$X = X_0 + X'_0 = X_0 + ae^{i(kx_0 - \omega t) + kz_0} = X_0 + ae^{i\phi_0 + kz_0}, \quad (2)$$

where $X_0 = x_0 + iz_0$ is the mean particle position. The corresponding wave velocity field, in complex notation, $U_0 = u_0 + iw_0$, is given by

$$U = -ia\omega e^{i(kx_0 - \omega t) + kz_0} = -ia\omega e^{i\phi_0 + kz_0}. \quad (3)$$

ADCP measurements in presence of waves must be averaged over wave phase to remove the oscillatory wave orbital velocities. In standard processing this averaging is performed at fixed range cell, so the mean is taken along the sampling-volume trajectory $X_m(t)$ rather than at a fixed spatial position. As shown in SD25, the velocity sampled along such an arbitrary trajectory can be linearized as

$$U_m \equiv U(X_m, t) \approx U_1 + \omega k X'_1 X'^*_m, \quad (4)$$

where $U_1 = U(X_1, t)$ is the “true” Eulerian velocity at the nominal measurement location X_1 , and $X'_m = X_m - X_1$ are the wave-induced perturbations of the sampling location. Although the leading-order wave orbital velocity averages to zero, the quadratic term does not generally vanish. When the sampling volume motion X'_m is partially coherent with the wave orbital motions X'_1 , phase averaging of $X'_1 X'^*_m$ leaves a nonzero mean component. This rectified component is the origin of the wave-induced velocity bias.

In section 3 we will discuss the measurement volume trajectories resulting from resonant wave-induced tilting of the float. In section 4 we will derive corresponding analytic expressions for the ensuing biases in horizontal and vertical velocity measurements.

b. Semi-analytical model

As in SD25, we will also use a semi-analytical model to validate and extend the analytical expressions for the wave-induced biases. Platform motion is simulated according to (2), and the platform tilt is modeled based on the wave properties and a specified platform response function

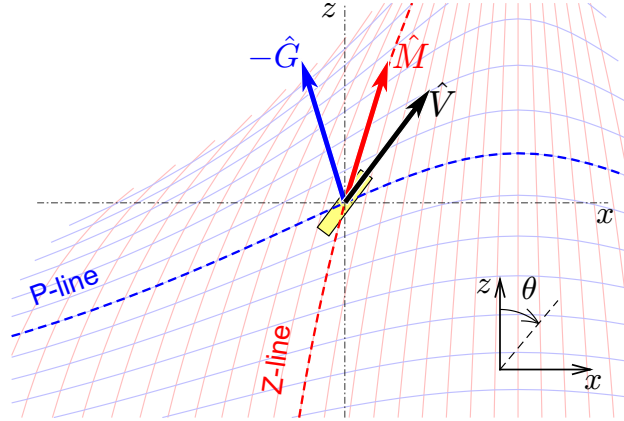


FIG. 1. Orientation of a sampling platform (yellow rectangle) in a wave-induced deformation field (the wave steepness is exaggerated). Material fluid P- and Z-lines are shown in blue and red, respectively, along with their orientation vectors $-\hat{G}$ (“anti-gravity”) and $\hat{M}$. Orientation of the platform “mast” vector $\hat{V}$ is determined by the platform dynamics and may not align with either $-\hat{G}$ or $\hat{M}$. The conventions for axes and tilt angles are illustrated in the bottom-right corner.

(see section 3). The velocity field is then sampled with multiple “beams” and “cells,” and processed as it would be with an ADCP.

This model is semi-analytical, in the sense that the platform trajectory and velocity sampling are computed analytically, while the subsequent averaging is numerical. Unlike the analytical framework, the model does not rely on a linearized expansion of the velocity field (4) and can therefore handle larger excursions of the sampling volume. Similarly, it does not require a small-angle approximation for platform tilt (see section 3) and can therefore handle arbitrary wave-induced variations in platform orientation.

3. Wave-induced platform tilt

a. Wave deformation field

In a monochromatic wave field, the deformation and tilt of fluid elements can be described using two complementary material lines (Fig. 1). The “P-line” is an isopotential line (actually, a surface in 3D) that would be horizontal in absence of waves. This line is tangent to the local instantaneous fluid velocity, and perpendicular to the local effective gravity vector G . In contrast, the “Z-line” is a material fluid line that would be vertical in absence of waves. As waves propagate, this line

128 deforms under the action of the velocity field. To characterize local tilting of the two lines, we
 129 will use the unit “anti-gravity” vector $-\hat{G}$ pointing normal to the P-line, and the vector $\hat{M}$ pointing
 130 upwards along the Z-line. In the absence of waves, the two vectors coincide with the upward unit
 131 vector $\hat{Z} = i$. Under wave motion, they oscillate symmetrically relative to the vertical, reflecting
 132 the tilting and deformation of the fluid elements.

133 From the wave kinematics equations, these directions are given, up to normalization, by

$$M = i + ake^{i\phi_0 + kz_0}, \quad (5)$$

$$-G = i - ake^{i\phi_0 + kz_0}. \quad (6)$$

134 Using the small-slope approximation ($ak \ll 1$), we can obtain corresponding unit vectors

$$\hat{M} \equiv \frac{M}{|M|} \approx i + \Re[ake^{i\phi_0 + kz_0}], \quad (7)$$

$$-\hat{G} \equiv -\frac{G}{|G|} \approx i + \Re[-ake^{i\phi_0 + kz_0}], \quad (8)$$

135 where $\Re[\cdot]$ is the real part operator (see SD25 for details). These vectors can also be expressed in
 136 terms of tilt angles,

$$\hat{M} = i + \theta_Z, \quad (9)$$

$$-\hat{G} = i + \theta_G, \quad (10)$$

137 where $\theta_Z = \Re[ake^{i\phi_0 + kz_0}]$ and $\theta_G = \Re[-ake^{i\phi_0 + kz_0}] = -\theta_Z$ are the material line tilt angles, as in
 138 DS26. We deliberately delay application of the real part operator to emphasize that $\pm ake^{i\phi_0 + kz_0}$
 139 is the analytic representation of the tilt angles – which would be useful in discussion of spectral
 140 transfer functions below.

141 *b. Platform response*

142 As in SD25 and DS26, we assume that, to first order, the platform follows the wave orbital
 143 motion of the surrounding water parcels. For neutrally-buoyant and profiling floats, observations
 144 support this approximation over a broad range of wave frequencies (D’Asaro 2003). The tilt of the
 145 platform is governed by a balance of hydrostatic and hydrodynamic torques, as discussed in detail

in DS26. Depending on the platform geometry and hydrostatic stability, the tilt response can be characterized by

$$\hat{V} = i + \theta = i + \Re[\gamma a k e^{i\phi_0 + k z_0}] \quad (11)$$

where γ is a complex frequency-dependent response factor. As discussed in SD25, $\gamma = -1$ corresponds to hydrostatic response mode ($\hat{V} = -\hat{G}$), and $\gamma = 1$ corresponds to inertial response ($\hat{V} = \hat{M}$). More generally, γ can be seen as a spectral transfer function between the platform tilt angle θ and the tilt of the Z-line θ_Z , $T_{\theta Z}(\omega)$ in DS26 notation. Note that since θ_z and θ_G are 180° out of phase,

$$\gamma = T_{\theta Z} = -T_{\theta G}. \quad (12)$$

It is important to note that the transfer function is applied to the analytic representation of the tilt angles *before* the real part is taken, allowing arbitrary phase shifts between the forcing and the response.

This complex response function $\gamma(\omega)$ is analogous to the response amplitude operators (RAOs) commonly used in naval architecture and offshore engineering to describe wave-induced motions of floating and submerged platforms (e.g., Faltinsen 1993; Journée and Massie 2001). Conventional RAOs are complex, frequency-dependent transfer functions relating incident wave amplitude to platform displacement, rotation, or acceleration in each degree of freedom. They can be interpreted analytically using damped harmonic oscillator models, in which hydrostatic restoring, added mass, and hydrodynamic damping set the resonant frequency, amplitude, and phase response for heave and rotational motions (e.g., Newman 2018; Orszaghova et al. 2019). More complex response may be predicted numerically from radiation–diffraction calculations, for example using boundary-element tools such as WAMIT, which compute excitation forces, added mass, radiation damping, and platform-motion RAOs for specified geometries (Lee and Newman 2006). RAOs may also be estimated empirically from scale-model tests or field measurements, with the response referenced to an independently measured or prescribed incident wave field (e.g., ITTC 2024; Jiao et al. 2019). The spectral response approach used in DS26 is analogous but not identical: instead of relying on external wave measurements, DS26 inferred the wave-induced forcing from the float measurements themselves, using the water-following approximation and the expected linear-wave consistency between wave-induced motion and tilt. The water-following assumption also effectively bypasses

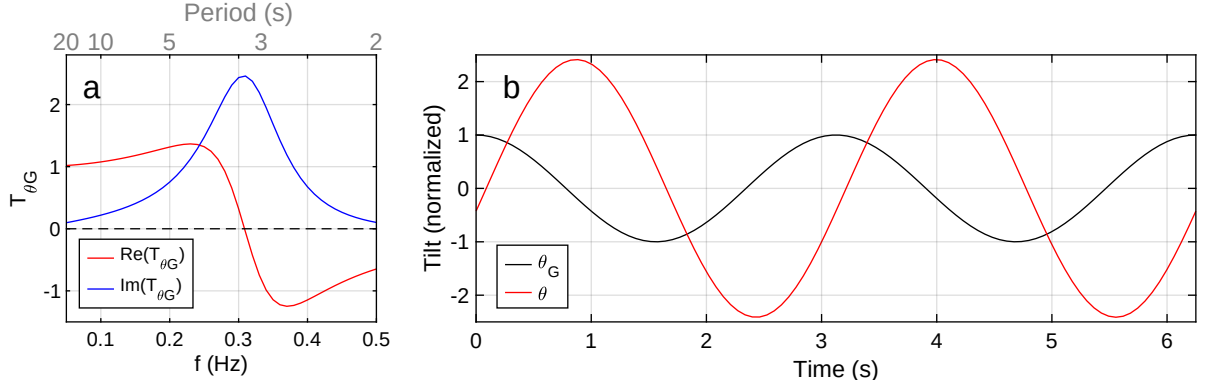


FIG. 2. Empirical model of Lagrangian Float tilt response based on DS26. a) Empirical spectral transfer function from the effective gravity tilt angle θ_G to the tilt of the float θ as a function of wave frequency. b) Time series of the forcing (θ_G , black) and the float response (θ , red) tilt angles at resonant frequency $f=0.32$ Hz; tilt angles are normalized by θ_G amplitude, ake^{kz_0} . Note the $\pi/2$ phase lag and significant amplification of the response, both hallmarks of resonance.

an explicit treatment of the translational RAOs and allows us to focus on the resonant tilt response that affects the ADCP sampling geometry.

DS26 established an empirical model of a Lagrangian Float response function

$$T_{\theta G}(\omega) \equiv -\gamma(\omega) = \frac{\sigma^2 - i\kappa\omega}{\sigma^2 - \omega^2 - iq\omega}, \quad (13)$$

where $\sigma = 2.01 \text{ rad s}^{-1}$ is the resonant angular frequency (corresponding to the cyclic resonant frequency $f = (2\pi)^{-1}\sigma = 0.32 \text{ Hz}$, $\kappa = 0.37 \text{ rad s}^{-1}$ is the shape eccentricity parameter, and $q = 0.88 \text{ rad s}^{-1}$ is the rotational frictional parameter. The MLF resonant frequency corresponds to the period $f^{-1} = 3.1 \text{ s}$ and wavelength of 16 m. Waves of this scale coincide with the peak of a fully developed sea spectrum generated by winds of about $U_{10} = 4 \text{ m s}^{-1}$ (Alves et al. 2003). Consequently, the MLF resonant response could be expected to be commonly excited during most deployments in typical ocean wave conditions.

The general shape of the spectral response function and the temporal behavior of the resonant tilt response are illustrated in Fig. 2. At low frequencies, the float response is in-phase with the forcing with $T_{\theta G}$ real and approaching 1 (corresponding to $\gamma \approx -1$). This behavior reflects the platform aligning with the effective gravity, as expected for any platform with strong hydrostatic

restoring moment (i.e., $\sigma \gg \omega$, $\sigma \gg \kappa$), which is the reason this response is called ‘hydrostatic’ (Longuet-Higgins 1986). Conversely, for a platform with a weak righting moment ($\sigma \ll \omega$, $\sigma \ll \kappa$), the response function would asymptote to $T_{\theta G}(0) = q^{-1}\kappa = -\lambda$, where λ is the Jeffery shape eccentricity parameter (Jeffery 1922; Bretherton 1962). This case corresponds to the classical Jeffery-type alignment with the principal strain direction – which can have characteristics of either inertial ($\gamma = \lambda > 0$) or hydrostatic ($\gamma = \lambda < 0$) response¹. Thus, the purely real-valued response considered in SD25 is applicable in the low-frequency limit, where either the hydrostatic (strong-restoring) or Jeffery (weak-restoring) asymptotes govern the tilt dynamics

At high frequencies, $T_{\theta G}$ is also real but negative (corresponding to $0 < \gamma < 1$), so the platform tilt is out of phase with the effective gravity tilt θ_G (but in-phase with $\theta_Z = -\theta_G$). Such behavior is universal among systems with finite inertia and corresponds to the attenuated inertial-response regime. SD25 formulae would be applicable in this regime as well.

At intermediate frequencies, particularly near the 0.32 Hz resonance, the response is amplified ($|T_{\theta G}|$ reaching 2.5) and increasingly lagging the forcing. The strong imaginary component of $T_{\theta G}$ observed over the broad range of frequencies requires us to extend the wave-induced bias analysis of SD25 to complex values of response parameter $\gamma = -T_{\theta G}$. As will be shown below, this extension is not as trivial as substituting a complex γ into the SD25 formulae.

c. Phase-lagged sampling volume trajectories

As described in detail in SD25, deriving the wave-induced bias formulae begins with determining the trajectories of the ADCP sampling volumes, $X_m(t)$. Wave-induced biases arise from the phase-locked relationships between these trajectories and the wave orbital velocities (see SD25, section 3a). We only need to reconsider the biases arising from the platform tilt – i.e., the “sweep” and frame rotation biases, plus the associated ADCP beam effects.

Consider a sampling volume at a nominal (vector) offset D from the platform, so that the nominal location of this volume is $X_1 = X_0 + D$. For a pair of symmetric ADCP beams,

$$D^{\pm} = r(i \pm \tan \beta) = ir(\cos \beta)^{-1} e^{\mp i \beta}, \quad (14)$$

¹Note that although a flat disc ($\lambda = -1$) exhibits a hydrostatic-like response ($T_{\theta G} \approx 1$), its alignment in this case would be governed by strain-induced Jeffery alignment rather than hydrostatic forces.

where r is the nominal vertical distance to the sampling volume (positive for upward-looking instrument), and β is the ADCP beam angle. For a hypothetical “vector” sampler that measures the full velocity vector at a remote location (as considered in SD25), the displacement vector D takes the simple form $D = ir$. The sampling volume trajectories around their nominal positions are the superposition of the platform’s orbital motion (X'_0) and the sweeping motion (X'_t) produced by time-varying platform tilt,

$$X'_m = X'_0 + X'_t. \quad (15)$$

As in SD25, the platform motion is described as

$$X'_0 = ae(i\phi_0 + kz_0), \quad (16)$$

and the sweeping term is given by

$$X'_t = -i\hat{V}D - D = -D(1 + i\hat{V}), \quad (17)$$

where $\hat{V}$ is the varying platform “mast” orientation unit vector. As expected, X'_t vanishes for an upright platform, $\hat{V} = i$. As discussed above, the platform tilt response can be described as

$$\hat{V} = i + \Re[\gamma a k e^{i\phi_0 + kz_0}] = i + a k e^{kz_0}(\gamma_r \cos \phi_0 - \gamma_i \sin \phi_0), \quad (18)$$

where $\gamma = \gamma_r + i\gamma_i$ is a complex frequency-dependent response factor (transfer function). This leads to the following linearized expression for the sweeping motion of the sampling volumes of the two ADCP beams:

$$X'^{\pm}_t = -D^{\pm}(1 + i\hat{V}) = a k r e^{kz_0}(\cos \beta)^{-1} e^{\mp i\beta}(\gamma_r \cos \phi_0 - \gamma_i \sin \phi_0). \quad (19)$$

Imaginary component of the response function, γ_i , produces a phase shift between the orbital and sweeping motions of the sampling volumes. As a result, the sampling volume trajectories are skewed and rotated (Fig. 3a) compared to the real-valued response parameter cases (Fig. 3b,c). There is also a notable difference between the two ADCP beams.

At this point we can already anticipate that, unlike the cases considered in SD25, the wave-induced biases arising from the resonant (phase-lagged) tilt response of the measuring platform would involve both the horizontal and the vertical components of measured velocities.

4. Wave-induced biases

Each of the two wave-induced time-varying terms in (15) gives rise to a bias when velocities are phase-averaged at fixed ADCP range cell. An additional frame-rotation bias arises if the instrument is not “aware” of its true wave-induced tilt and therefore conducts averaging in its own tilted frame of reference. To first order, these biases are additive and independent, allowing us to compute them separately. While these biases correspond directly to those considered in SD25, the resulting analytical expressions are critically different for the resonant tilt response considered here due to the imaginary component of the response function.

a. Platform motion bias

The platform motion bias arises from the $X'_0 = ae^{i\phi_0 + kz_0}$ component of sampling volume motion. This bias is independent of the platform tilt, therefore SD25 expression for the velocity bias still holds:

$$U_{wr} = U_{S0}e^{kz_0}(R_ue^{kz_1} - e^{kz_0}). \quad (20)$$

Here, $U_{S0} = a^2\omega k$ is the surface Stokes drift which serves as a common scaling factor, and

$$R_u = \frac{\sin(\beta + kr \tan \beta)}{\sin \beta} \quad (21)$$

is the horizontal velocity response function associated with ADCP measurements, which are derived from the radial velocities along two beams inclined at $\pm\beta$ (see SD25 for details). The associated vertical velocity response function

$$R_w = \frac{\cos(\beta + kr \tan \beta)}{\cos \beta} \quad (22)$$

does not enter this expression but will become important later.

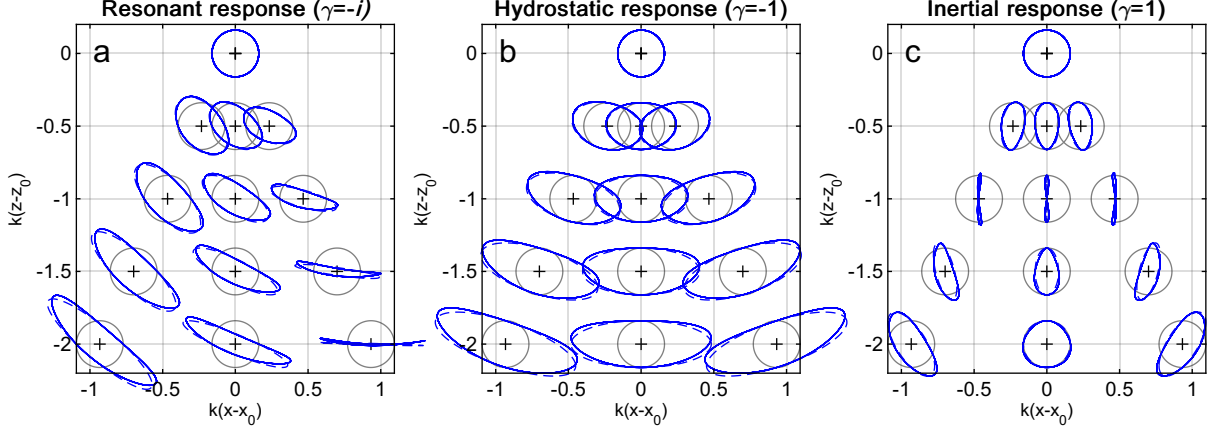


FIG. 3. (a) Wave-induced trajectories of ADCP sampling volumes for resonant platform tilt model ($\gamma = -i$), compared to (b) hydrostatic ($\gamma = -1$), and (c) inertial ($\gamma = 1$) responses. Trajectories obtained from a semianalytical model are shown in blue solid lines, with the linearized approximations shown with dashed lines. Trajectories in the absence of platform tilt are shown in grey for reference. Axes show nondimensional offsets from the nominal platform position. ADCP beam angle $\beta = 25^\circ$ assumed.

b. Sweeping bias

The sweeping bias arises from the X'_t component of sampling volume motion. Eulerian velocities along the sweeping trajectories are computed using the linearized expression

$$U_t^\pm = U_1^\pm + \omega k X'_1 X_t^{\pm*} = U_1^\pm + (a\omega k e^{i\phi_1^\pm + kz_1})r(\cos\beta)^{-1}e^{\pm i\beta}a k e^{kz_0}(\gamma_r \cos\phi_0 - \gamma_i \sin\phi_0) = U_1^\pm + U_{S0}e^{k(z_0+z_1)}kr(\cos\beta)^{-1}e^{i(\phi_1^\pm \pm \beta)}(\gamma_r \cos\phi_0 - \gamma_i \sin\phi_0) \quad (23)$$

Averaging over the wave phase eliminates harmonic terms, producing

$$\bar{U}_t^\pm = U_{S0}e^{k(z_0+z_1)}kr(\cos\beta)^{-1}e^{\mp\beta}\langle e^{i\phi_1^\pm}(\gamma_r \cos\phi_0 - \gamma_i \sin\phi_0) \rangle \quad (24)$$

The wave phase variation associated with the ADCP beam separation is

$$\phi_1^\pm = \phi_0 \pm kr \tan\beta = \phi_0 \pm \phi', \quad (25)$$

264 where $\phi' = kr \tan \beta$ is half the wave phase difference between the two beams. Therefore,

$$\begin{aligned} \bar{U}_t^\pm = U_{S0} e^{k(z_0+z_1)} kr (\cos \beta)^{-1} e^{\pm i(\beta+\phi')} \langle e^{i\phi_0} (\gamma_r \cos \phi_0 - \gamma_i \sin \phi_0) \rangle = \\ \frac{1}{2} (\gamma_r - i\gamma_i) U_{S0} e^{k(z_0+z_1)} kr (\cos \beta)^{-1} e^{\pm i(\beta+\phi')}. \end{aligned} \quad (26)$$

265 ADCP records the along-beam components of actual velocities,

$$\bar{B}^\pm = \Im[\bar{U}_t^\pm e^{\pm i\beta}], \quad (27)$$

266 where $\Im[\cdot]$ is the imaginary part operator. Expanding it, we obtain

$$\begin{aligned} \bar{B}^\pm = \frac{1}{2} U_{S0} e^{k(z_0+z_1)} kr (\cos \beta)^{-1} \Im[(\gamma_r - i\gamma_i) e^{\pm i(2\beta+\phi')}] \\ \frac{1}{2} U_{S0} e^{k(z_0+z_1)} kr (\cos \beta)^{-1} (\pm \gamma_r \sin(2\beta + \phi') - \gamma_i \cos(2\beta + \phi')), \end{aligned} \quad (28)$$

267 Following the standard ADCP processing procedure, we reconstruct the full velocity vector from
268 the beam velocities:

$$U_t = \frac{\bar{B}^+ - \bar{B}^-}{2 \sin \beta} + i \frac{\bar{B}^+ + \bar{B}^-}{2 \cos \beta} = \frac{1}{2} U_{S0} e^{k(z_0+z_1)} kr \left[\gamma_r \frac{\sin(2\beta + \phi')}{\sin 2\beta} - i \gamma_i \frac{\cos(2\beta + \phi')}{1 + \cos 2\beta} \right]. \quad (29)$$

269 Comparing this with the corresponding expression in SD25 (eq. 60), we see that the imaginary
270 part of the response function, γ_i , gives rise to a sweeping bias in the *vertical* velocity estimate, in
271 addition to the horizontal velocity bias examined previously. This expression can also be re-written
272 as

$$U_t = \frac{1}{2} (\gamma_r R_{ut} - i \gamma_i R_{wt}) kr U_{S0} e^{k(z_0+z_1)} \quad (30)$$

273 using modified ADCP sweeping response functions

$$R_{ut} = \frac{2 \sin(2\beta + \phi')}{\sin 2\beta} = \frac{2 \sin(2\beta + kr \tan \beta)}{\sin 2\beta}, \quad (31)$$

$$R_{wt} = \frac{2 \cos(2\beta + \phi')}{1 + \cos 2\beta} = \frac{2 \cos(2\beta + kr \tan \beta)}{1 + \cos 2\beta}. \quad (32)$$

274 *c. Frame rotation bias*

275 Next, we examine the frame rotation bias arising if the velocity averaging is carried out in the
 276 ADCP frame of reference. It should be reminded that this bias can, in principle, be eliminated by
 277 rotating each ping's velocity measurements to Earth coordinates prior to averaging, but doing so
 278 requires accurate attitude measurements. As discussed in DS26, accurately measuring platform
 279 attitude in the wave band is not an easy task even with the dedicated inertial sensors. Consequently,
 280 the frame rotation bias needs to be quantified. In the nominally-upright instrument frame of
 281 reference, measured relative velocities are recorded as

$$U_{1i} = (U_1 - U_0)(i\hat{V}^*), \quad (33)$$

282 where the multiplier $i\hat{V}^* = 1 + iake^{kz_0}(\gamma_r \cos \phi_0 - \gamma_i \sin \phi_0)$ represents the rotation from Earth to
 283 instrument frame. Applying phase averaging, we obtain

$$\begin{aligned} \overline{U_{1i}} = \langle (U_1 - U_0)i\hat{V}^* \rangle &= -ia\omega \langle (e^{kz_1+i\phi_0} - e^{kz_0+i\phi_0})(1 + iake^{kz_0}(\gamma_r \cos \phi_0 - \gamma_i \sin \phi_0)) \rangle = \\ &= \frac{1}{2}U_{S0}e^{kz_0} \left((\gamma_r - i\gamma_i)e^{kz_1} - (\gamma_r - i\gamma_i)e^{kz_0} \right). \end{aligned} \quad (34)$$

284 To obtain the final expression for the frame rotation bias, we need to apply the ADCP transfer
 285 functions to both components of the averaged ambient velocity $\langle U_1 i\hat{V}^* \rangle$ — but, importantly, not to
 286 the averaged platform motion $\langle U_0 i\hat{V}^* \rangle$, which is unaffected by ADCP measurement artifacts:

$$U_f = \frac{1}{2}U_{S0}e^{kz_0} \left((\gamma_r R_u - i\gamma_i R_w)e^{kz_1} - (\gamma_r - i\gamma_i)e^{kz_0} \right). \quad (35)$$

287 *d. Net wave-induced bias*

288 Net relative wave induced bias is obtained by adding the expressions for motion (20), sweeping
 289 (30), and frame-rotation biases (35). For insight into the structure of the resulting bias profiles, it
 290 is convenient to express them in terms of platform depth z_0 and measurement range $r = z_1 - z_0$.
 291

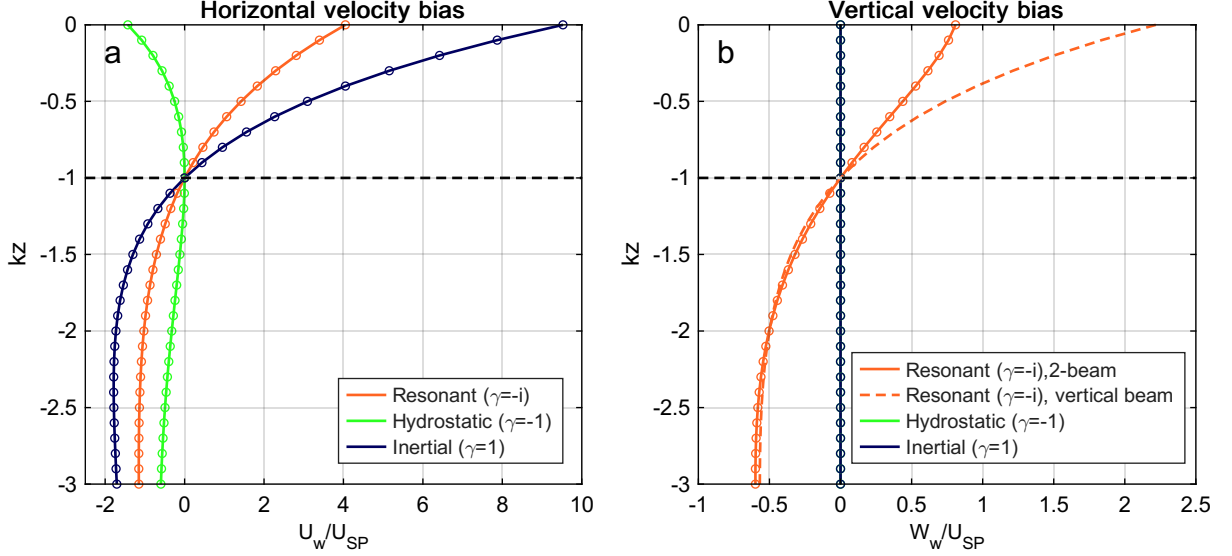


FIG. 4. Wave-induced tilt and motion biases in (a) relative horizontal and (b) vertical velocities measured by a up- and downward-looking ADCPs mounted on a subsurface quasi-Lagrangian platform with resonant ($\gamma = -i$), hydrostatic ($\gamma = -1$), and inertial ($\gamma = 1$) responses. Analytical estimates are shown in solid lines, overlaid with the semianalytical simulation results (circles). Dashed orange line in (b) shows the vertical beam-bias for the resonant case. Horizontal dashed line marks the platform depth (set to $z_0 = k^{-1}$ in this example). Velocity bias values are normalized by the Stokes drift at the level of the platform, U_{SP} ; the depth is normalized by the inverse wavenumber k^{-1} . Note the difference in the velocity axis scales.

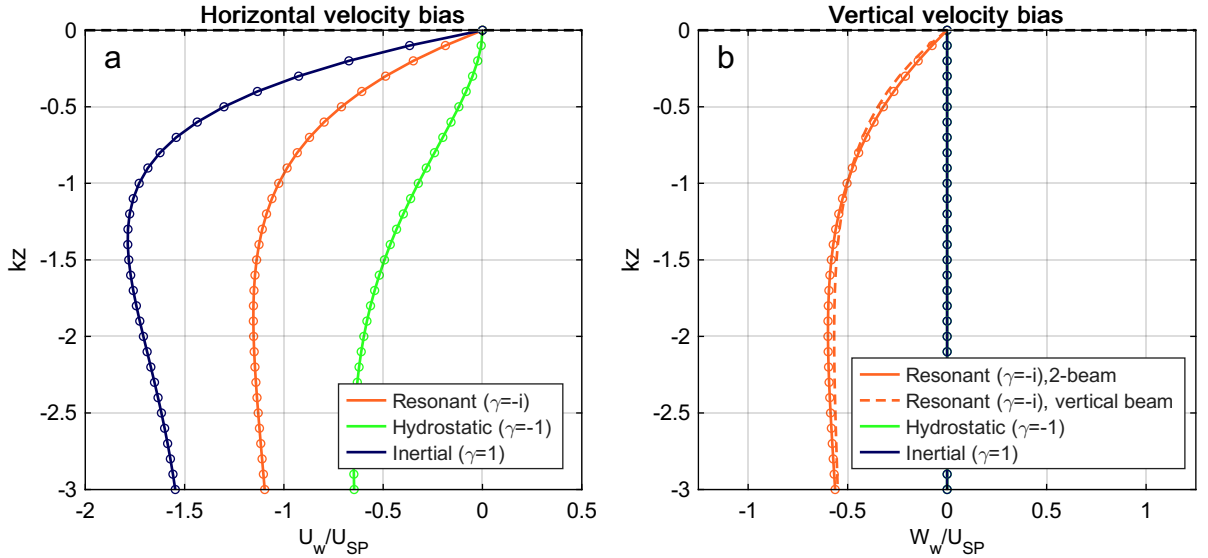


FIG. 5. Same as Fig. 4, but for a surface platform equipped with a downward ADCP.

We also present horizontal and vertical velocity biases separately for clarity:

$$u_w = \left((R_u + \frac{1}{2}\gamma_r(R_u + krR_{ut}))e^{kr} - (\frac{1}{2}\gamma_r + 1) \right) U_{SP}, \quad (36)$$

$$w_w = -\frac{1}{2}\gamma_i \left((R_w + krR_{wt})e^{kr} - 1 \right) U_{SP}, \quad (37)$$

where

$$U_{SP} = U_{S0}e^{2kz_0} = a^2\omega k e^{2kz_0} \quad (38)$$

is the Stokes drift at the platform depth that serves as the bias amplitude scaling parameter. In this form, it is easy to see that for a platform at fixed depth U_{SP} is also constant, and therefore bias profiles depend only on the scaled measurement range kr . Equations (36-37) imply that surface and sub-surface platforms share the same relative bias profiles, but the bias magnitudes for a subsurface platform are much smaller for the same range and wave forcing because of the rapid decay of the e^{2kz_0} factor with platform depth. Examples of normalized net wave-induced bias profiles for ADCPs mounted on a sub-surface and surface platforms with different response characteristics are shown in Fig. 4 and Fig. 5. Analytical expressions are in good agreement with the semi-analytical model results.

As expected, the resonant horizontal velocity bias ($\gamma = -i$) is intermediate between the hydrostatic ($\gamma = -1$) and inertial ($\gamma = 1$) response limits, reflecting the consistent effect of the real part of the response function on the wave-induced bias (36). In contrast, a resonant (and, more generally, any phase-lagged) response with $\gamma_i \neq 0$ introduces a substantial vertical velocity bias that is not present in hydrostatic or inertial response cases.

All biases vanish as $kr \rightarrow 0$, and most change sign between upward- and downward-looking orientations; the horizontal bias for the hydrostatic response case ($\gamma_r = -1$) is an exception, as it stays negative. For an upward-looking ADCP ($kr > 0$) the bias is dominated by the sweeping term, whose magnitude increases rapidly with range following the $\sim k r e^{kr}$ asymptotic. Presence of the periodic slant-beam transfer functions (Fig. 6), however, makes the biases oscillatory. Horizontal and vertical biases first change signs in the vicinity of the first zeros of R_{ut} and R_{wt} , occurring at $kr \approx 4.9$ and 1.5 , respectively. For a downward-looking ADCP, the biases are dominated by the frame rotation effects and the platform's own motion, to which the velocity measurements are

referenced. They approach finite limits as $kr \rightarrow -\infty$,

$$u_w^{-\infty} = -(\frac{1}{2}\gamma_r + 1)U_{SP}, \quad (39)$$

$$w_w^{-\infty} = -\frac{1}{2}\gamma_i U_{SP}. \quad (40)$$

As a result, downward-looking configurations typically exhibit a weak mid-range maximum in biases at a range of about $kr \sim 1 - 2$. Although the slant-beam transfer functions cause bias oscillations, their amplitudes decay rapidly and therefore not likely to be important.

Horizontal and vertical velocity biases generally scale with the real and imaginary parts of the response factor γ , respectively. Near the resonance frequency, both the real and (especially) the imaginary components of the response function can exceed unity in magnitude (e.g., see Fig. 2). In such cases, the wave-induced biases will be correspondingly amplified.

e. ADCP beam alignment

In the preceding derivations, we assumed that the ADCP beam pair was aligned with the wave propagation direction, an assumption that is seldom satisfied in real life observations. At first glance, one might expect the wave-induced biases to be a simple projection of the down-wave biases onto the plane of the ADCP beam pair. In practice, the situation is considerably more complex.

When the ADCP beam-pair axis is rotated by an angle α relative to the wave propagation direction, several aspects of the wave-beam interaction are modified simultaneously. The most direct effect is a reduction of the horizontal component of the wave orbital velocity "seen" by the beams, scaling as $\cos \alpha$. Additionally, the projection of the beam spread onto the (x, z) plane is reduced, producing a smaller effective beam angle $\beta_x = \tan^{-1}(\cos \alpha \tan \beta)$. This reduction decreases the wave phase difference between the two slanted ADCP beams, which, in turn can be expected to reduce the effects of the beam geometry. Moreover, the wave-induced sweeping trajectories of the sampling volumes are affected by misalignment because these also depend on the projected beam angle. Together, these effects modify wave-induced biases in a non-trivial manner. Platform motion:

$$U_{wr}^\alpha = U_{S0} e^{kz_0} (R_u^\alpha e^{kz_1} - \cos \alpha e^{kz_0}). \quad (41)$$

345 Sweeping:

$$U_t^\alpha = \frac{1}{2}krU_{S0}e^{k(z_0+z_1)}(\gamma_r R_{ut}^\alpha - i\gamma_i R_{wt}^\alpha). \quad (42)$$

346 Frame rotation:

$$U_f^\alpha = \frac{1}{2}U_{S0}e^{kz_0}((\gamma_r R_u^\alpha - i\gamma_i R_w^\alpha)e^{kz_1} - (\gamma_r \cos \alpha - i\gamma_i)e^{kz_0}). \quad (43)$$

347 Most of the alignment effects are captured by the modified ADCP beam response functions:

$$R_u^\alpha = \frac{\cos \beta \sin(\phi'_x) + \cos \alpha \sin \beta \cos(\phi'_x)}{\sin \beta}, \quad (44)$$

$$R_w^\alpha = \frac{\cos \beta \cos(\phi'_x) - \cos \alpha \sin \beta \sin(\phi'_x)}{\cos \beta}, \quad (45)$$

348 where $\phi'_x = kr \tan \beta_x = kr \cos \alpha \tan \beta$. Similarly, the sweeping response functions become

$$R_{ut}^\alpha = \frac{\cos \beta \sin(\beta_x + \phi'_x) + \cos \alpha \sin \beta \cos(\beta_x + \phi'_x)}{\sin \beta \cos \beta_x}, \quad (46)$$

$$R_{wt}^\alpha = \frac{\cos \beta \cos(\beta_x + \phi'_x) - \cos \alpha \sin \beta \sin(\beta_x + \phi'_x)}{\cos \beta \cos \beta_x}. \quad (47)$$

352 Derivation of these expressions is given in the Appendix A. For the aligned ADCP case ($\alpha = 0$,
 353 $\beta_x = \beta$), these reduce to (21)-(22) and (31)-(32). In general, both the horizontal and vertical velocity
 354 response functions vary with the misalignment angle α as shown in Fig. 6. All response functions
 355 are periodic in kr with the period $2\pi/\tan \beta_x$. As α increases, the oscillation period increases
 356 accordingly. In the limiting case of the beam pair oriented perpendicular to the wave propagation
 357 direction ($\alpha = \frac{\pi}{2}$), the response functions reduce to constants: $R_u = R_{ut} = 0$ and $R_w = R_{wt} = 1$.
 358 Thus, the horizontal velocity biases vanish while the vertical velocity biases remain finite (as long
 359 as $\gamma_i \neq 0$) but unaffected by the beam spread.

360 *f. Vertical beam considerations*

361 Several modern ADCPs, such as the Nortek Signature series and Teledyne RDI Sentinel V, can
 362 be equipped with a vertical (fifth) beam in addition to the standard slanted beams. In principle,
 363 this vertical beam enables direct line-of-sight measurements of vertical velocity because it does
 364 not rely on the multi-beam geometric reconstruction used by conventional ADCP configurations

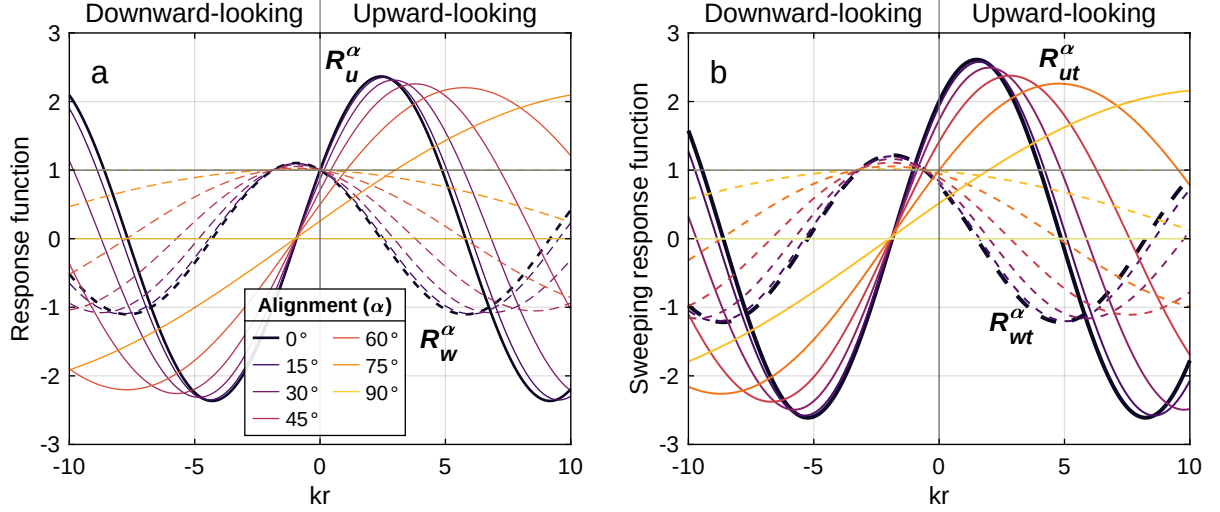


FIG. 6. (a) ADCP response functions and (b) sweeping response functions for horizontal (solid lines) and vertical (dashed lines) velocities as a function of ADCP alignment angle α . Bold lines correspond to aligned ADCP case ($\alpha = 0$), as considered in SD25. ADCP beam angle of 25° is used.

(Shcherbina et al. 2018; Comby et al. 2022). In other words, the vertical velocity response function for the vertical beam is always unity, $R_w^V = R_{wt}^V = 1$.

When it comes to wave-induced vertical velocity biases, the advantages of the dedicated vertical beam are less straightforward. Using (30) and (35), and assuming unity ADCP response functions, the biases in the vertical-beam estimates of w can be obtained as

$$w_{Vt} = -\frac{1}{2}\gamma_i kr U_{S0} e^{k(z_0+z_1)}, \quad (48)$$

$$w_{Vf} = -\frac{1}{2}\gamma_i U_{S0} e^{kz_0} (e^{kz_1} - e^{kz_0}). \quad (49)$$

For a downward-looking configuration, the vertical beam exhibits nearly the same bias as the conventional two-beam reconstruction of vertical velocity (Fig. 5b). This occurs because the two-beam ADCP response functions R_w and R_{wt} are already near unity across the relevant downward-looking ranges (dashed lines in Fig. 6). In a 25° ADCP shown, the departure of the response function from unity remains small $|R_{wt} - 1| < 0.25$ for $-4 \lesssim kr < 0$ (Fig. 6a). Moreover, near $kr \approx -1$, the opposing effects of the ADCP response functions (R_w and R_{wt}) on tilt and sweeping biases nearly cancel, so that the net vertical velocity bias is effectively equivalent to that of the vertical-beam velocities. For an upward-looking subsurface platform, such as the Lagrangian float,

the wave-induced vertical velocity bias in vertical-beam measurements is substantially larger than that of the conventional two-beam reconstruction Fig. 4). As discussed in SD25, $|R_w| < 1$ for relatively short upward ranges ($0 < kr \lesssim 5$, and so is $|R_{wt}|$ (cf. Fig. 6). Therefore, the ADCP beam reconstruction effect is beneficial: it reduces both the sweeping and frame-rotation biases in measured vertical velocities (in stark contrast to the horizontal biases, see Fig. 12–13 in SD25).

For slanted-beam velocity reconstructions, the frame-rotation biases U_f can be removed by rotating each ping’s velocity measurements to Earth coordinates prior to averaging, provided that accurate attitude data are available. For vertical-beam velocity measurements, however, this approach is not viable: A single beam does not provide the cross-beam velocity components required for such a transformation, and reconstructing them from additional beams would remove the benefits of a single-beam measurement. Frame-rotation bias is therefore appears to be unavoidable in vertical-beam measurements.

5. Optimal unbiased vertical velocity reconstruction

A five-beam ADCP provides three independent vertical velocity estimates: two reconstructed from the opposing slanted-beam pairs (w_1, w_2) , and one measured directly by the vertical beam (w_3) . Even more estimates can be obtained by linear combination of these three (e.g., an estimate using all four slanted beams). From the preceding discussion it can be seen that each of these estimates experiences biases due to wave-induced tilt of the platform, but to a different degree.

The biases associated with the slanted-beam reconstructions depend on the alignment of each beam-pair and follow (41)-(43) with two orthogonal angles, α and $\alpha + \frac{\pi}{2}$. In general, both would be affected by the ADCP beam response functions to some extent — except for a beam pair perpendicular to the wave propagation direction. The vertical-beam biases, given by (48) and (49), do not depend on the ADCP alignment with the waves. It can be seen that the vertical-beam biases match those of the vertical velocity reconstructed from the cross-wave beam pair ($\alpha = \pi/2$).

It is reasonable to ask whether we can determine which of the three vertical velocity estimates is “better”, i.e. least affected by the tilt biases. The three velocity estimates and their biases can be expressed as

$$w_i = w + R_i w_w, \quad i = 1 \dots 3, \quad (50)$$

where w is the “true” vertical velocity. The bias is expressed as the product of the common bias amplitude w_w and the geometric factors R_i that capture how this bias projects onto each of the three velocity reconstructions. Thus w_w contains all dependence on the wave field and platform response, while the coefficients R_i encode the purely geometric differences among the three ADCP-based reconstructions. As will become apparent later, it is convenient to take the vertical-beam bias as the bias amplitude, i.e., for the combination of sweeping and frame rotation biases, we set

$$w_w = w_{Vt} + w_{Vf} = -\frac{1}{2}\gamma_i \left((1 + kr)e^{kr} - 1 \right) U_{SP}. \quad (51)$$

Then the geometric factors for the two slanted-beam reconstructions are

$$R_i = \frac{(R_w^{\alpha_i} + R_{wt}^{\alpha_i} kr)e^{kr} - 1}{(1 + kr)e^{kr} - 1}, \quad i = 1, 2, \quad (52)$$

where α_i is the alignment angle of the two beam-pairs, $\alpha_2 = \alpha_1 + \frac{\pi}{2}$. As discussed in the previous section, the vertical-beam estimate does not depend on the alignment, so corresponding geometric factor is unity ($R_3 = 1$). Geometric factors can be calculated analytically using the expressions for R_w^α and R_{wt}^α derived earlier. Despite the presence of exponentials, R_i remain $O(1)$ for both the positive and negative values of kr . Singularity at $kr = 0$ is avoided by continuity extension $R_i|_{kr=0} = 1 - \cos^2 \alpha \tan^2 \beta$.

If the ADCP orientation angles were known, identifying the “best” estimate would be as trivial as finding the smallest bias scaling factor, $\min |R_i|$ at each range cell. Note that we could make this choice without knowing wave amplitude or platform response characteristics. In a more realistic case where the wave field is poorly known it is impossible to unambiguously determine which of the three biases is smaller. No single estimate is universally superior.

We can, however, exploit the fact that the three reconstructions provide redundant but differently biased estimates of the same quantity, and seek an appropriately weighted recombination

$$\tilde{w} = \sum_{i=1}^3 c_i w_i \quad (53)$$

that systematically reduces the biases for an arbitrary ADCP alignment². The choice of the recombination weights c_i depends on the adopted optimality criterion, and the combination can be expected to vary with range. One obvious choice is to seek a linear combination that is unbiased on average over a uniform distribution of unknown alignment angles α from 0 to 2π :

$$\langle \tilde{w} - w \rangle_\alpha = 0. \quad (54)$$

Here, angle brackets represent averaging over all possible α . This condition leads to the constraints

$$\sum c_i = 1, \quad (55)$$

$$\sum c_i \bar{R}_i = 0. \quad (56)$$

The two slanted-beam reconstructions differ only by a $\pi/2$ rotation, which implies that $\langle R_1 \rangle = \langle R_2 \rangle = \bar{R}_1$. The average geometric factor $\bar{R}_1$ serves as a measure of the mean (expected value) of the bias in either of the slant-beam reconstructions, normalized by the corresponding bias of the vertical-beam estimate. Its analytical expression can be obtained from the expressions for R_w^α and R_{wt}^α derived earlier (see Appendix B for details). For the direct vertical-beam estimate, $\langle R_3 \rangle = R_3 = 1$. Because the two slant-beam pairs are symmetric and their orientation relative to the wave direction is unknown, there is no reason to weigh the two slant-beam estimates differently; we therefore set $c_1 = c_2$. Solving the two constraints then gives

$$c_1 = c_2 = \frac{1}{2}(1 - \bar{R}_1)^{-1}, \quad (57)$$

$$c_3 = -\bar{R}_1(1 - \bar{R}_1)^{-1}. \quad (58)$$

Therefore, the optimal recombination is uniquely determined by the average geometric factor $\bar{R}_1$ which, given a specific ADCP configuration, depends only on the scaled range kr . This dependence for an up-/downward-looking ADCP subject to both sweeping and frame-rotation biases is illustrated in Fig. 7a, along with the corresponding optimal recombination weights (Fig. 7b). Note that the slant-beam contribution to the recombination, $c_1 w_1 + c_2 w_2 = c_1(w_1 + w_2)$, is

²Another way to see this recombination as an optimal five-beam vertical velocity reconstruction.

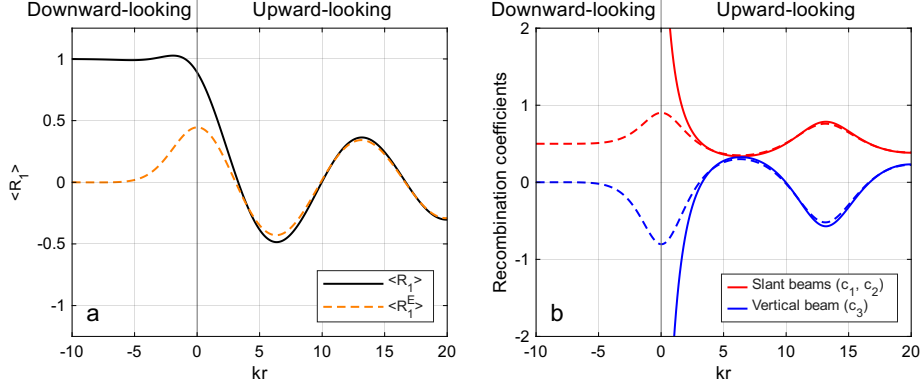


FIG. 7. (a) Mean geometric factor $\bar{R}_1$ describing the relative scaling of vertical-velocity bias in slant-beam reconstruction. (b) Optimal recombination weights for the slant-beam ($c_1 = c_2$, red) and vertical-beam (c_3 , blue) estimates. In both panels, solid curves correspond to the full-bias case (sweeping + frame rotation). Dashed curves show the Earth-frame geometric factor and weights applicable when slant-beam estimates are first rotated into the Earth frame prior to averaging. ADCP beam angle of 25° assumed.

equivalent to $2c_1w_4$, i.e. the standard four-beam estimate $w_4 = \frac{1}{2}(w_1 + w_2)$ taken with the weight $2c_1$.

For upward-looking ADCP ($kr > 0$), $\bar{R}_1$ oscillates with the same period as R_w and R_{wt} ($2\pi/\tan\beta$). The oscillation amplitude decays slowly with increasing kr , and $\bar{R}_1$ generally remaining in the $[-0.5, 0.5]$ range. Corresponding optimal recombination weights also oscillate with normalized range and remain $O(1)$. Net-zero bias recombination is therefore feasible in this regime.

Fig. 8 illustrates several examples of unbiased recombination at different normalized ranges. It can be seen that even though the optimal recombination is constrained only to be unbiased *on average*, it tends to have smaller bias than any of the individual estimates at most (though not all!) misalignment angles. Variability of the bias with α is also reduced compared to that of the slant-beam estimates (R_1 and R_2). As expected, the optimal five-beam reconstruction generally outperforms the standard four-beam vertical-velocity estimate, which corresponds to $c_1 = c_2 = 0.5$, $c_3 = 0$ and the bias factor $R_4 = \frac{1}{2}(R_1 + R_2)$ (dashed line).

For downward-looking measurements ($kr < 0$), the behavior is markedly different: $\bar{R}_1$ quickly approaches 1 and then remains nearly constant. This situation ($\bar{R}_1 \approx \bar{R}_3 = 1$) implies that all three reconstructions carry essentially the same bias. This behavior can be understood by examining the structure of the tilt-induced bias: for a downward-looking ADCP ($kr < 0$), the bias is dominated

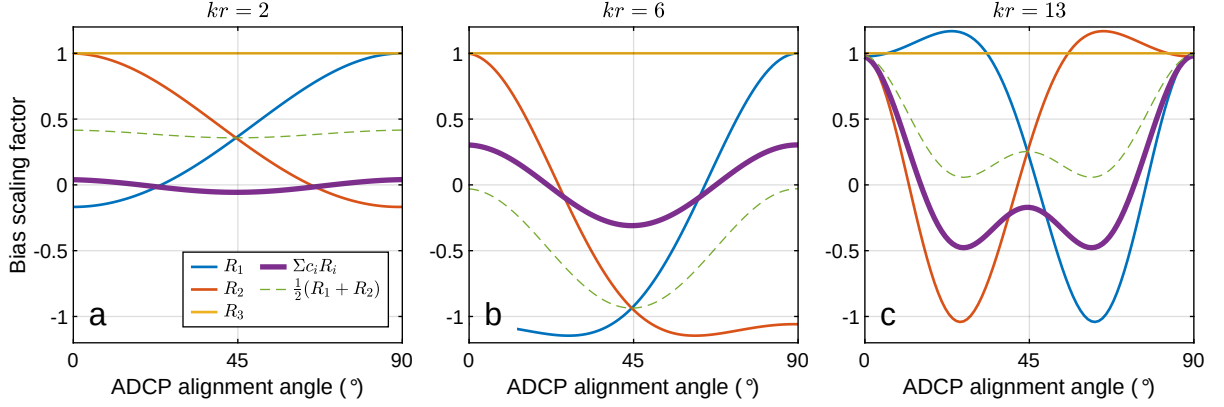


FIG. 8. Examples of optimal unbiased recombination of ADCP vertical velocity estimates for (a) $kr = 2$, (b) $kr = 6$, and (c) $kr = 13$. Bias scaling factors for the two slant-beam reconstructions (R_1 , R_2) and the vertical-beam estimate (R_3) are shown, along with their optimal recombination ($\sum c_i R_i$, thick line). The standard four-beam vertical-velocity estimate bias factor, $R_4 = \frac{1}{2}(R_1 + R_2)$, is included for reference (dashed line).

by the frame-rotation distortion of the platform motion, to which the measured beam velocities are referenced. As discussed in section 4c, this component is unaffected by the velocity reconstruction artifacts because it impacts all the beams equally. Therefore, all three vertical velocity reconstructions experience roughly the same bias. Although an optimal solution (57)-(58) formally exists for any $\bar{R}_1 \neq 1$, the corresponding values of the recombination weight become very large when $\bar{R}_1$ is close to unity. Using such weights would drastically amplify noise in the reconstructed velocity.

As discussed earlier, the frame-rotation bias can be removed from the slant-beam reconstructions (but, importantly, not from the vertical-beam estimate) by rotating each ping's velocity measurements to Earth coordinates prior to averaging. Doing so would produce a simpler “Earth-frame” geometric factor

$$\bar{R}_1^E = \frac{R_{wt}^{\bar{\alpha}} k r e^{kr}}{(1 + kr) e^{kr} - 1}. \quad (59)$$

Analytical expression for this factor is given in the Appendix B, and its dependence on kr is illustrated in Fig. 7a. Unlike $\bar{R}_1$, the Earth-frame factor $\bar{R}_1^E$ approaches zero for large negative ranges ($kr \ll -1$). In this regime, the optimal recombination is obtained with weights $c_1 = c_2 = 0.5$, $c_3 = 0$ – i.e., the vertical-beam velocity estimate w_3 is ignored, while the two slant-beam reconstructions are averaged. This solution is intuitive: the estimate affected by the dominant frame-rotation bias should be excluded from the recombination.

Net-zero bias is not the only optimality criterion worth considering. One may instead prefer to minimize RMS bias over all possible wave directions, $J = \langle |\tilde{w} - w|^2 \rangle_\alpha = |w_w|^2 \langle |\sum R_i|^2 \rangle_\alpha$. Such minimization process would produce a different set of optimal recombination weights, which, however, behave similarly to those derived above (not shown). Yet another approach would be to minimize the average (or worst-case) bias over a particular range of misalignment angles instead of the full $[0, 2\pi]$ interval. For now, we will leave these options as opportunities for future research.

6. Wave-induced biases for the Lagrangian Float

With the general expressions for the wave-induced biases derived in section 4, we can quantify these biases in ADCP velocity measurements obtained from a Lagrangian float with a particular response model derived in DS26 (see section 3b). Unlike the generic analysis of the prior sections, all the results shown below apply only to a specific platform (Lagrangian float) and its response model. Our goal is twofold: estimate the biases inherent in our past and future Lagrangian float observations as well as provide a road map for similar analysis for other platforms.

Generally speaking, the wave-induced biases depend on three primary variables: the wave field, the float depth, and the measurement depth (or range). For a realistic representation of the wave field, we choose the fully developed wind-sea surface elevation spectrum of Alves et al. (2003), which provides a compact one-parameter representation similar to the classic Pierson-Moskowitz spectrum but with an f^{-4} high-frequency tail. We further assume a unidirectional wave field aligned with the ADCP beam pair. This idealized geometry provides a useful reference case because it isolates the frequency, depth, and platform-response dependencies of the bias. It is nevertheless a substantial simplification: observed wind seas and swell often exhibit considerable directional spreading (e.g., Smit and Janssen 2019), so wave energy is distributed over a range of beam-alignment angles. Here we keep the aligned unidirectional configuration as a simple reference case, allowing the interpretation to remain focused on the primary frequency–depth structure of the bias without introducing additional sea-state-dependent parameters. With these assumptions, the wave field is uniquely parameterized by the 10-m wind speed, U_{10} .

The ADCP geometry used in the calculations is representative of a five-beam Nortek Signature1000, with a 25° slant-beam angle. Some of the measurement ranges considered below may

exceed the practical range of that instrument, but they are included to show the depth dependence of the bias more clearly.

To extend the monochromatic bias analysis to a broadband wave field, we examine how individual frequency bands contribute to the total bias. The monochromatic expressions for the net bias derived above, e.g., (36)–(37), scale with the squared wave amplitude, a^2 . For a broadband sea state described by the one-sided surface-elevation spectrum $S(f)$, the variance contained in a narrow frequency band $[f, f + df]$ is $S(f) df$. A monochromatic component representing that band therefore has equivalent squared amplitude $a_f^2 = 2S(f) df$. Substituting a_f^2 into the monochromatic bias expressions and dividing by df gives the corresponding bias contribution per unit frequency. With this replacement, the monochromatic Stokes drift scaling parameter (38) becomes

$$\tilde{U}_{SP}(f) = 2S(f)\omega k e^{2kz_0}, \quad (60)$$

which represents the spectral contribution to the float-depth Stokes drift per unit frequency. Similarly to (36)–(37), this amplitude modulates the relative bias profiles determined by the ADCP beam geometry to produce frequency-resolved bias density $\tilde{U}_w(f) = \tilde{u}_w + i\tilde{w}_w$, with units of velocity per unit frequency. The net broadband bias is then obtained by integration,

$$U_w = \int_0^\infty \tilde{U}_w(f) df. \quad (61)$$

The frequency-resolved bias density $\tilde{U}_w(f)$ provides insight into which wave frequencies contribute most strongly to the broadband bias. These contributions depend not only on the platform response, but also on the spectral distribution of wave energy. The fully developed wind-sea spectrum of Alves et al. (2003), normalized by wind speed, can be viewed as an f^{-4} high-frequency tail modified by a wind-dependent low-frequency rolloff (Fig. 9a). We therefore first examine the bias density obtained using the $S(f) \sim f^{-4}$ spectral weighting alone. The absolute spectral level is unimportant here because we are interested in the relative frequency–depth structure of the bias contribution.

The maps in Fig. 9c–d give an example of the frequency–depth structure of the bias density for a float at nominal depth 20 m. They illustrate the combined effects of the f^{-4} spectral shape, the exponential decay of wave orbital motion with depth, the resonant platform response, and

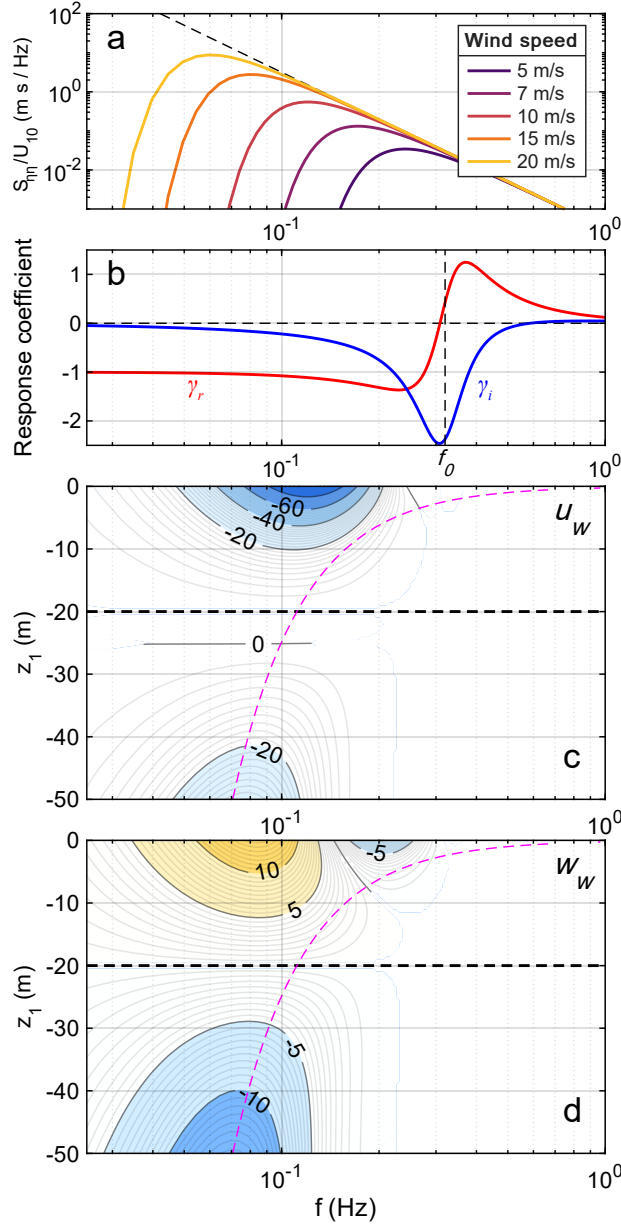


FIG. 9. Frequency-depth structure of wave-induced biases for a Lagrangian float at 20 m. (a) Alves et al. (2003) surface-elevation spectra, normalized by wind speed U_{10} , for different wind speeds. The normalized high-frequency asymptote, $S(f)/U_{10} \sim f^{-4}$, is shown by the black dashed line. (b) Real (red) and imaginary (blue) components of the float tilt response coefficient γ based on DS26; f_0 is the float resonant frequency. Normalized frequency-resolved bias contribution for (c) horizontal and (d) vertical velocities obtained from slant-beam ADCP reconstruction using an f^{-4} equilibrium-range spectral weighting. All bias values are relative, normalized by maximum absolute value of u_w and expressed in percent. Magenta dashed lines in (c–d) show the wave length scale k^{-1} for reference.

the ADCP beam geometry. For a given wind speed, the full spectrum-dependent bias density would be obtained by applying the corresponding wind-dependent low-frequency rolloff to this equilibrium-range response and then integrating over frequency.

The main feature of these maps is that the largest bias contributions tend to arise from waves whose length scale is comparable to the depth of the float, i.e. from the band where $kz_0 \sim 1$, provided those waves are present in the sea state. For a float at 20 m depth, this corresponds to $f \sim 0.1$ Hz, the waves excited for wind speeds $U_{10} \gtrsim 10 \text{ m s}^{-1}$. Thus, the dominant contribution at this depth is expected from the wind-sea peak band under moderate winds (Fig. 9a), and the resulting bias magnitude depends strongly on the detailed spectral shape in that band. Over these dominant frequencies, the float tilt response remains close to the hydrostatic limit ($\gamma \approx -1$). The horizontal velocity bias is therefore generally negative throughout the range of observations, as in the hydrostatic monochromatic case shown in Fig. 4a, corresponding to a bias directed opposite to wave propagation.

The float resonant frequency $f_0 = 0.32$ Hz lies at substantially higher frequency. At 20 m depth, wave orbital motion in this band is strongly attenuated, so the resonant response peak has only a weak effect on the velocity bias. Higher frequencies, including the resonant band, can contribute more strongly for shallower float observations (not shown); however, their net contribution remains limited by the f^{-4} decay of the wave-elevation spectrum. Lower-frequency waves have a progressively weaker effect on the biases despite their larger amplitudes, because their vertical scales k^{-1} are large compared with the ADCP sampling range r leading to $kr \approx 0$.

Finally, to characterize the operationally relevant total bias magnitudes, we integrate $\widetilde{U}_w$ over the actual wind-wave spectrum. To summarize the parameter space dependence, we show the biases for a given float depth and varying wind speed, and a given wind speed and varying float depth. We also include the alternative velocity reconstructions discussed earlier – the Earth-frame processing (4c), vertical-beam measurements (4f), and optimal recombination (5).

Fig. 10 and Fig. 11 show estimates of horizontal velocity biases. As expected, the biases increase with increasing wind speed and typically reach the maximum of up to $6 - 8 \text{ cm s}^{-1}$ near the surface for 20 m s^{-1} winds. Earth-frame averaging improves the upward-looking biases slightly, while making the downward-looking biases much worse. This behavior could be anticipated from

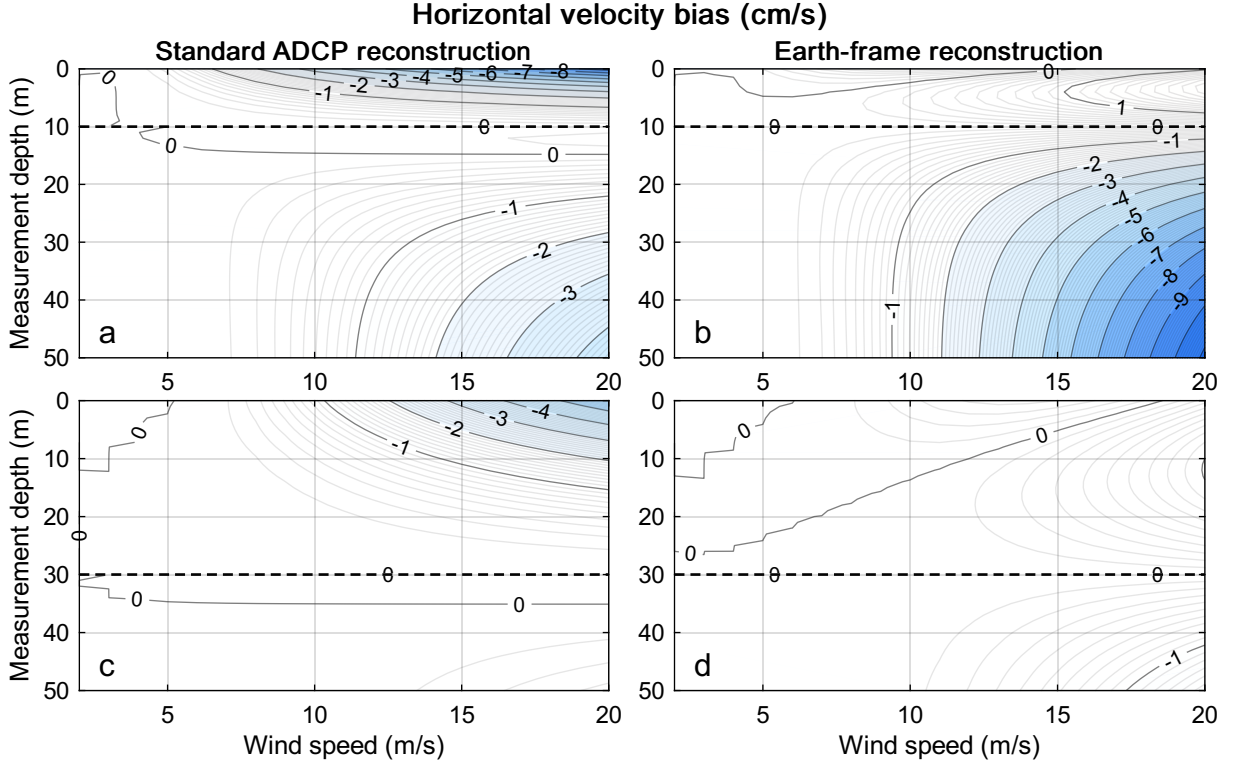


FIG. 10. Wave-induced biases in horizontal velocity measurements from a Lagrangian float situated at 10 m (top row) and 30 m (bottom row) depth as a function of measurement depth and wind speed. The left column shows the full bias of the standard ADCP reconstruction; the right column shows the Earth-frame bias (excluding the frame rotation bias contribution). Velocity bias contour labels are in cm s^{-1} .

the asymptotic relationship (39) that shows the frame-rotation bias partially offsetting the motion bias for $\gamma_r < 0$.

Vertical velocity biases are shown in Fig. 12 and Fig. 13. They are typically an order of magnitude smaller than horizontal biases, on the order of a few millimeters per second. As discussed in section 4f, the two-beam vertical velocity bias is substantially smaller than the vertical-beam bias but has a more complex pattern with subsurface maxima of both signs. Optimal beam recombination (section 5) efficiently reduces the vertical velocity bias by another order of magnitude, as could be anticipated from the examples in Fig. 8.

These quantitative bias estimates should be interpreted as reference values for an idealized aligned-wave geometry, corresponding to a near-worst-case configuration for the wave-induced horizontal velocity bias. In this configuration, the wave orbital motions lie in the ADCP beam-pair

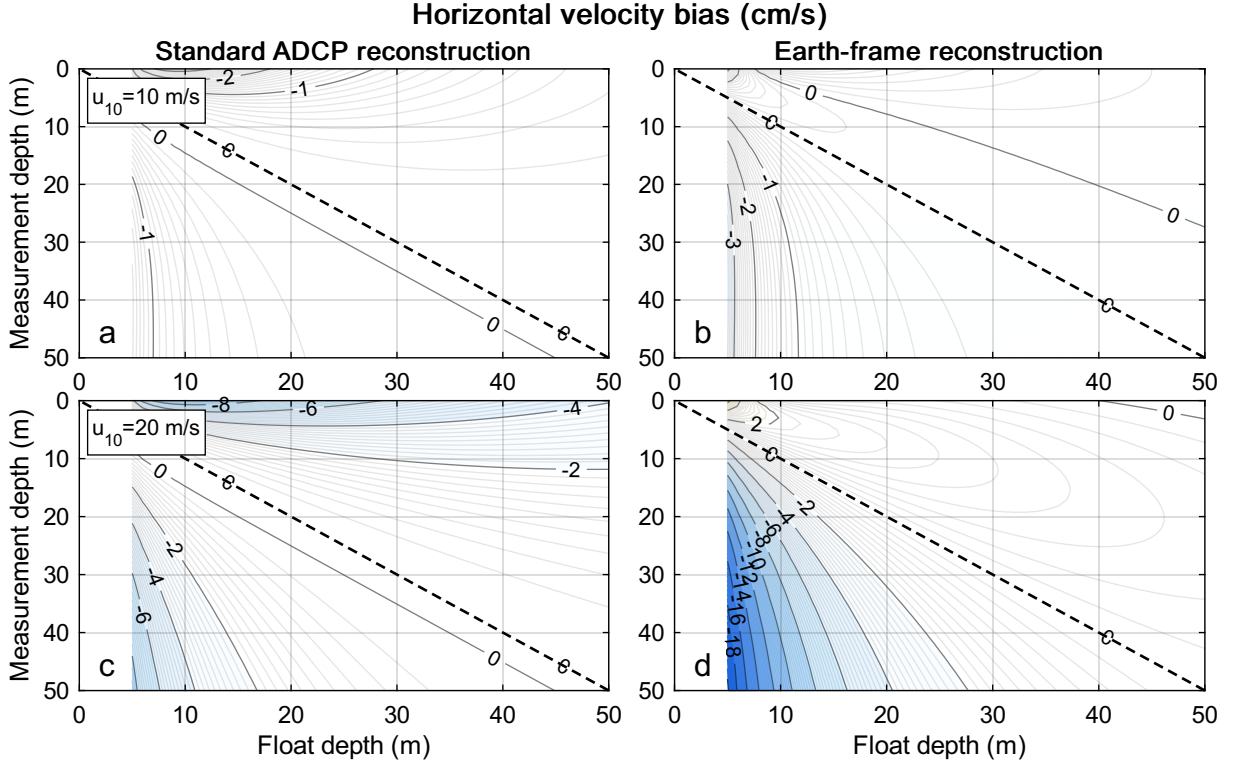


FIG. 11. Same as Fig. 10, but as a function of measurement depth and float depth for moderate (top row) and strong (bottom row) winds.

plane, producing a strong coherent projection onto the reconstructed horizontal velocity. In an actual wave field, directional spreading and oblique wave components would generally reduce this projection, although vertical velocity biases and biases in other beam combinations need not vanish. A fully directional estimate would require integrating the directional wave spectrum against the alignment-dependent response functions as described in section 4e. We do not present this calculation here because it would introduce additional sea-state-dependent parameters and obscure the primary frequency–depth structure emphasized in this reference case.

7. Discussion and conclusions

As discussed in SD25, all wave-induced biases considered here arise from the same fundamental mechanism: the superposition of wave orbital motion and the motion of the platform (and hence of the ADCP sampling volume). Because these motions are at least partially coherent, their nonlinear coupling produces wave-averaged biases in the measured velocities. For monochromatic

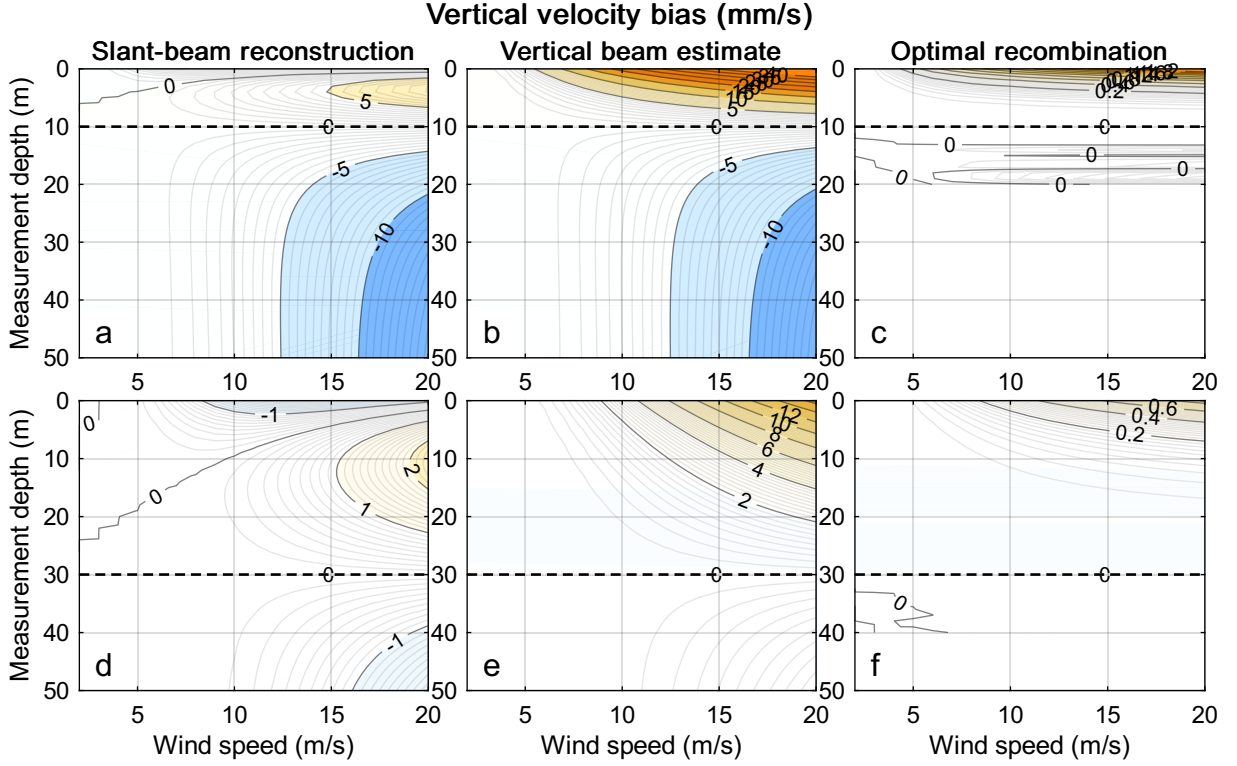


FIG. 12. Wave-induced biases in vertical velocity measurements from a Lagrangian float at 10 m (top row) and 30 m (bottom row) as a function of measurement depth and wind speed. Columns show the biases of the standard slant-beam reconstruction, the direct vertical-beam estimate, and the optimal beam recombination, from left to right (see section 5 for details). Velocity bias contour labels are in mm s^{-1} .

wave forcing (or an individual spectral component), the resulting biases generally scale with the geometric mean of the Stokes drift velocities evaluated at the nominal depths of the platform and the measurement. The detailed bias structure, however, depends critically on the trajectory of the sampling volume through wave phase space, which in turn is controlled by the platform's wave-induced motion and tilt response as well as by the ADCP beam geometry.

Real autonomous platforms can generally be expected to exhibit partially resonant response, leading to both amplification and phase lag of platform's tilt relative to the wave forcing. These effects alter the sampling-volume trajectory through the wave space and therefore modify the coupling between wave orbital motion and the sampling. The resulting phase-averaged biases affect both horizontal and vertical velocity estimates, as described by (36-37). In general, tilt-induced biases in horizontal velocity scale with the real part of the platform tilt transfer function,

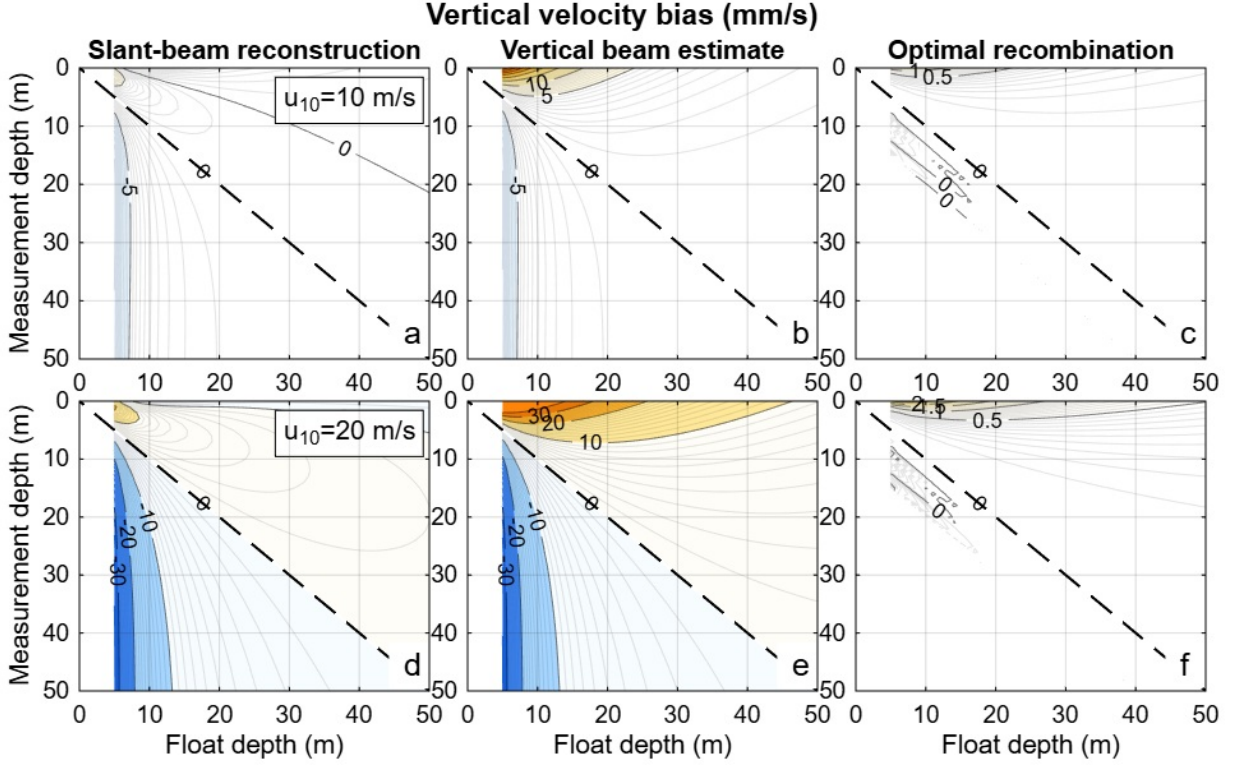


FIG. 13. Same as Fig. 12, but as a function of measurement depth and float depth for moderate (top row) and strong (bottom row) winds.

while biases in vertical velocity scale with its imaginary part. The latter mechanism, absent in the in-phase response considered in SD25 and other studies, provides a direct pathway for wave-induced vertical velocity bias.

The magnitude and sign of the biases further depend on the relative alignment of the ADCP beams with the direction of wave propagation. Biases are typically largest when a slanted beam pair is aligned with the wave direction (alignment angle $\alpha = 0$), although bias cancellation can occur for particularly favorable ADCP beam spread for a given wavelength (see Fig. 6 and Section e). When the beam pair is orthogonal to the wave direction ($\alpha = 90^\circ$), the horizontal bias vanishes, while the vertical bias remains finite. This directional dependence underscores the importance of beam geometry in interpreting wave-contaminated ADCP observations.

Five-beam ADCPs, which include a vertical beam in addition to the standard slanted beams, offer additional flexibility for mitigating wave-induced vertical velocity bias by providing three independent estimates: two from the slanted beam pairs and one from the vertical beam. Although the

vertical-beam estimate alone is typically more strongly biased than the slant-beam reconstruction, it is theoretically possible to form an optimal five-beam vertical velocity estimate that is unbiased on average for arbitrary ADCP–wave alignment. This reconstruction must be performed in Fourier space and can be poorly conditioned in certain parameter regimes, leading to noise amplification and potential instability. Whether such unbiased reconstructions are beneficial in practice therefore remains an open question.

Present study focuses on the resonant tilt response and does not explicitly include resonant or phase-lagged translational motion, such as heave or surge. Both here and in SD25, the platform is assumed to be fully Lagrangian, following wave orbital motion without phase lag, attenuation, or resonance (with the notable exception of self-propelled platforms considered in SD25). Under this assumption, the unity motion response function is implicitly assumed in (20). For Lagrangian floats, this approximation has been shown to be valid at spatial scales larger than the float itself ($\gtrsim 1\text{ m}$; D’Asaro 2003, 2015). However, it need not hold for all platforms or deployment scenarios. Surface platforms, in particular, may exhibit resonant heave response because their buoyancy provides a strong hydrostatic restoring force. If the translational platform response is frequency dependent or phase lagged, the appropriate motion RAOs should be included explicitly in (20); this would introduce additional contributions to the horizontal and vertical velocity biases beyond the tilt-induced terms considered here.

The biases discussed here arise from averaging over wave-correlated sampling-volume trajectories, so the averaging procedure is part of the bias mechanism. As in SD25, the present analysis assumes averaging within fixed ADCP range cells. If individual velocity measurements could instead be mapped to an absolute vertical coordinate before averaging, some of the wave-induced biases derived here could be reduced or eliminated. Such fixed- z averaging is difficult to implement, particularly for submerged platforms that infer their depth from hydrostatic pressure, which is insensitive to wave-induced heave. This obstacle could potentially be overcome with sufficiently accurate estimates of wave-band platform motion, for example from inertial, GNSS-aided, or bottom-referenced acoustic tracking measurements. Existing AHRS measurements may already contain enough information to estimate wave-band heave using appropriate wave-specific sensor-fusion algorithms, although the accuracy of such estimates for submerged quasi-Lagrangian floats

663 remains to be assessed. If such motion correction is achieved, the wave-induced bias problem
664 should be revisited in the corresponding fixed- z averaging framework.

665 Although the analytical expressions derived here fully characterize wave-induced biases, we
666 remain skeptical that such biases can be reliably removed from observational data in most practical
667 situations, because the wave field parameters, platform orientation, and the platform response
668 functions may be insufficiently constrained. Instead, we suggest that wave-induced biases be
669 treated as an inherent source of uncertainty in autonomous and moored ADCP measurements.
670 These biases can be quantified using the expressions developed here³, and experimental design
671 measures should be considered to minimize their impact. While wave orbital motions are obviously
672 beyond control, platform configuration, positioning, and dynamic response can sometimes be
673 modified. As suggested in SD25 and confirmed here, "an upward-looking ADCP mounted on a
674 subsurface quasi-Lagrangian platform [...] can be expected to have weaker wave-induced biases
675 when observing velocities at a given depth than other configurations". In addition, shifting
676 the platform tilt resonance toward higher frequencies (i.e., increasing hydrostatic stability) and
677 increasing the quality factor Q to sharpen the response may further reduce wave-induced biases,
678 particularly in vertical velocity estimates. It remains to be seen whether such modifications are
679 feasible from the engineering and operational perspective.

³A MATLAB implementation of the analytical expressions is provided with the semi-analytical model at <https://github.com/shcher2018/wave-bias>.

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687 our own, though.

688 *Data availability statement.* No datasets were generated or analyzed during the current
689 study. MATLAB implementation of the semi-analytical model is freely available at
690 <https://github.com/shcher2018/wave-bias> under MIT license.

691 APPENDIX A

692 Derivation of misaligned ADCP response functions

693 Consider an ADCP beam-pair oriented at an angle α to the wave propagation direction. The motion
694 bias response function derivation is modified as follows:

$$\begin{aligned} \overline{U_m^\pm} &= a^2 \omega k e^{k(z_1+z_0)} \langle e^{i(\phi_1^\pm - \phi_0)} \rangle = a^2 \omega k e^{k(z_1+z_0)} e^{\pm i k r \tan \beta_x} = \\ & a^2 \omega k e^{k(z_1+z_0)} (\cos(kr \tan \beta_x) \pm i \sin(kr \tan \beta_x)) \quad (A1) \end{aligned}$$

$$\begin{aligned} B^\pm &= \pm u \cos \alpha \sin \beta + w \cos \beta = \\ & \pm a^2 \omega k e^{k(z_1+z_0)} (\cos(kr \tan \beta_x) \cos \alpha \sin \beta + \sin(kr \tan \beta_x) \cos \beta) \quad (A2) \end{aligned}$$

$$u_{ADCP} = \frac{B^+ - B^-}{2 \sin \beta} = a^2 \omega k e^{k(z_1+z_0)} \frac{\cos(kr \tan \beta_x) \cos \alpha \sin \beta + \sin(kr \tan \beta_x) \cos \beta}{\sin \beta} \quad (A3)$$

695 Thus

$$R_u = \frac{\cos(kr \tan \beta_x) \cos \alpha \sin \beta + \sin(kr \tan \beta_x) \cos \beta}{\sin \beta}. \quad (\text{A4})$$

696 For $\alpha = 0$, this simplifies to the original expression. For $\alpha = \pi/2$, $R_u = 0$.

697 Let's consider the sweeping bias:

$$\begin{aligned} \bar{U}_t^\pm = U_{S0} e^{k(z_0+z_1)} kr (\cos \beta_x)^{-1} e^{\pm i(\beta_x + kr \tan \beta_x)} \langle e^{i\phi_0} (\gamma_r \cos \phi_0 - \gamma_i \sin \phi_0) \rangle = \\ \frac{1}{2} (\gamma_r - i\gamma_i) U_{S0} e^{k(z_0+z_1)} kr (\cos \beta_x)^{-1} e^{\pm i(\beta_x + kr \tan \beta_x)}, \quad (\text{A5}) \end{aligned}$$

$$\begin{aligned} B_t^\pm = \pm u \cos \alpha \sin \beta + w \cos \beta = \\ \frac{1}{2} U_{S0} e^{k(z_0+z_1)} kr (\cos \beta_x)^{-1} [\pm \cos \alpha \sin \beta (\gamma_r \cos (\beta_x + kr \tan \beta_x) \pm \gamma_i \sin (\beta_x + kr \tan \beta_x)) + \\ \cos \beta (\pm \gamma_r \sin (\beta_x + kr \tan \beta_x) - \gamma_i \cos (\beta_x + kr \tan \beta_x))] = \\ \frac{1}{2} U_{S0} e^{k(z_0+z_1)} kr (\cos \beta_x)^{-1} [\pm \gamma_r (\cos \alpha \sin \beta \cos (\beta_x + kr \tan \beta_x) + \cos \beta \sin (\beta_x + kr \tan \beta_x)) - \\ \gamma_i (\cos \beta \cos (\beta_x + kr \tan \beta_x) - \cos \alpha \sin \beta \sin (\beta_x + kr \tan \beta_x))]. \quad (\text{A6}) \end{aligned}$$

698 From this, we get

$$R_{ut} = \frac{\cos \alpha \sin \beta \cos (\beta_x + kr \tan \beta_x) + \cos \beta \sin (\beta_x + kr \tan \beta_x)}{\sin \beta \cos \beta_x}, \quad (\text{A7})$$

$$R_{wt} = \frac{\cos \beta \cos (\beta_x + kr \tan \beta_x) - \cos \alpha \sin \beta \sin (\beta_x + kr \tan \beta_x)}{\cos \beta \cos \beta_x}. \quad (\text{A8})$$

699 APPENDIX B

700 Analytic expression for mean response functions

701 We wish to average the ADCP beam geometric factor over all possible values of alignment angle

702 α ,

$$\bar{R}_1 = \frac{1}{2\pi} \int_0^{2\pi} R_1(\alpha) d\alpha, \quad (\text{B1})$$

703 where

$$R_1 = \frac{(R_w^\alpha + R_{wt}^\alpha kr)e^{kr} - 1}{(1 + kr)e^{kr} - 1}. \quad (\text{B2})$$

704 First, let's obtain the angle averages of the two ADCP response functions:

$$\bar{R}_w^\alpha = \frac{1}{2\pi} \int_0^{2\pi} R_w^\alpha d\alpha = \frac{1}{2\pi} \int_0^{2\pi} [\cos \phi'_x - \cos \alpha \tan \beta \sin \phi'_x] d\alpha, \quad (\text{B3})$$

705 where $\phi'_x = \phi' \cos \alpha = kr \tan \beta \cos \alpha$. The integrals can be expressed in terms of the Bessel functions
706 of the first kind,

$$\frac{1}{2\pi} \int_0^{2\pi} \cos \phi'_x d\alpha = \frac{1}{2\pi} \int_0^{2\pi} \cos(\phi' \cos \alpha) d\alpha = J_0(\phi'), \quad (\text{B4})$$

$$\frac{1}{2\pi} \int_0^{2\pi} \cos \alpha \sin \phi'_x d\alpha = \frac{1}{2\pi} \int_0^{2\pi} \cos \alpha \sin(\phi' \cos \alpha) d\alpha = J_1(\phi'), \quad (\text{B5})$$

707 therefore

$$\bar{R}_w^\alpha = J_0(\phi') - J_1(\phi') \tan \beta. \quad (\text{B6})$$

708 Next,

$$\bar{R}_{wt}^\alpha = \frac{1}{2\pi} \int_0^{2\pi} R_{wt}^\alpha d\alpha = \frac{1}{2\pi} \int_0^{2\pi} \left[\frac{\cos(\beta_x + \phi'_x)}{\cos \beta_x} - \cos \alpha \tan \beta \frac{\sin(\beta_x + \phi'_x)}{\cos \beta_x} \right] d\alpha. \quad (\text{B7})$$

709 Transforming the integrand to eliminate β_x :

$$\frac{\cos(\beta_x + \phi'_x)}{\cos \beta_x} = \frac{\cos \beta_x \cos \phi'_x - \sin \beta_x \sin \phi'_x}{\cos \beta_x} = \cos \phi'_x - \cos \alpha \tan \beta \sin \phi'_x, \quad (\text{B8})$$

$$\begin{aligned} \cos \alpha \tan \beta \frac{\sin(\beta_x + \phi'_x)}{\cos \beta_x} &= \cos \alpha \tan \beta \frac{\sin \beta_x \cos \phi'_x + \cos \beta_x \sin \phi'_x}{\cos \beta_x} = \\ &\cos \alpha \tan \beta \sin \phi'_x + \cos^2 \alpha \tan^2 \beta \cos \phi'_x, \end{aligned} \quad (\text{B9})$$

710 and using

$$\frac{1}{2\pi} \int_0^{2\pi} \cos^2 \alpha \cos \phi'_x d\alpha = \frac{1}{2\pi} \int_0^{2\pi} \cos^2 \alpha \cos(\phi' \cos \alpha) d\alpha = \frac{1}{2} (J_0(\phi') - J_2(\phi')) \quad (\text{B10})$$

711 We obtain

$$\bar{R}_{wt}^{\alpha} = J_0(\phi') - 2J_1(\phi') \tan \beta - \frac{1}{2}(J_0(\phi') - J_2(\phi')) \tan^2 \beta. \quad (\text{B11})$$

712 Using the recurrence relation $J_2(x) = 2x^{-1}J_1(x) - J_0(x)$, this can be simplified as

$$\bar{R}_{wt}^{\alpha} = J_0(\phi')(1 - \tan^2 \beta) + J_1(\phi')\left(\frac{1}{kr} - 2\right) \tan \beta. \quad (\text{B12})$$

713 Finally,

$$\bar{R}_1 = \frac{[J_0(\phi') - J_1(\phi') \tan \beta + (J_0(\phi')(1 - \tan^2 \beta) + J_1(\phi')(\frac{1}{kr} - 2) \tan \beta)kr]e^{kr} - 1}{(1 + kr)e^{kr} - 1} = \frac{[J_0(\phi')(1 + kr(1 - \tan^2 \beta)) - 2J_1(\phi') \tan \beta]e^{kr} - 1}{(1 + kr)e^{kr} - 1}. \quad (\text{B13})$$

714 Similarly, the Earth-frame geometric factor can be calculated as

$$\bar{R}_1^E = \frac{[J_0(\phi')kr(1 - \tan^2 \beta) + J_1(\phi')(1 - 2kr) \tan \beta]e^{kr}}{(1 + kr)e^{kr} - 1}. \quad (\text{B14})$$

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