Introducing HYRISK R package

HYRISK: An R package for hybrid uncertainty analysis using probability, imprecise probability and possibility distributions

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Abstract

Uncertainty analysis is an unavoidable risk assessment task (for instance for natural hazards, or for environmental issues). In situations where data are scarce, incomplete or imprecise, the systematic and only use of probabilities can be debatable. Over the last years, several alternative mathematical representation methods have been developed to handle in a more flexible manner the lack of knowledge related to input parameters of risk assessment models. This article presents an R package **HYRISK** dedicated to jointly handling different mathematical representation tools, namely probabilities, possibility distributions and probability functions with imprecise parameters, for the different stages of uncertainty treatment in risk assessments (i.e. uncertainty representation, propagation, sensitivity analysis and decision-making). We support the description using the case study of a dike stability analysis. The package is available at: https://cran.r-project.org/web/packages/HYRISK/index.html.

Keywords: Uncertainty quantification, sensitivity Analysis, probability, possibility, p-box, R.

1. Introduction

Uncertainty analysis is an unavoidable task for a proper risk assessment, like for instance for natural hazards (like volcanoes, landslides, floods, etc.), or for environmental issues (like

groundwater or soil contamination), or for engineered structures. When dealing with uncertainties, two facets should be considered as outlined by several authors. See for instance for seismic risk: Rohmer and Baudrit (2011), for volcano risks: Marzocchi, Sandri, Gasparini, Newhall, and Boschi (2004), for environmental risk: Baudrit, Couso, and Dubois (2007). The first facet corresponds to aleatory uncertainty (also named randomness or intrinsic variability). It arises from natural variability owing to either heterogeneity or to the random character of natural processes (i.e. stochasticity). A common example of aleatory uncertainty is the variability in weather. The second facet corresponds to epistemic uncertainty and arises when dealing with partial ignorance i.e. when facing vague, incomplete or imprecise information such as limited databases and observations or imperfect modelling.

For representing aleatory uncertainty, there is a large consensus in the community about the use of probabilities under the frequentist perspective: when many observations are available, probability distributions can be inferred. In geotechnics, it is the fit of power-law to the relationship on frequency-volumes of cliff rockfalls (Dewez, Rohmer, Regard, Cnudde *et al.* 2013). However, for representing epistemic uncertainty, several options exist.

The systematic use of probabilities as a tool to represent epistemic uncertainties has been criticized by several authors in situations where the available data are imprecise, scarce, incomplete, vague, qualitative, etc. (Ferson 1996; Helton, Johnson, and Oberkampf 2004; Baudrit 2005; Baudrit and Dubois 2006; Baudrit *et al.* 2007; Dubois and Guyonnet 2011; Beer, Ferson, and Kreinovich 2013). In such highly uncertain situations, the challenge is to formulate appropriate mathematical tools and models in a quantitative manner, on the one hand, accounting for all data and pieces of information, but, on the other hand, without introducing unwarranted assumptions (Beer *et al.* 2013). Therefore, to overcome the shortcomings of the classical probabilistic setting, several alternative mathematical representation methods have been developed: e.g., probability boxes, possibility distributions, Dempster-Shafer structures, etc. (see an overview by Dubois and Guyonnet (2011)).

To date and to the authors′ best knowledge, only a few open-source solutions exist to jointly handle different uncertainty mathematical tools (e.g., probabilities and intervals; or probabilities and fuzzy numbers) in uncertainty analysis of risk assessments (Table 1). Under the Matlab platform (The MathWorks, Inc 2012), the **DSI**-toolbox exists (Auer, Luther, Rebner, and Limbourg 2010) but it is restricted to Dempster-Shafer structures. More recently the more complete **OpenCossan** software has been developed (Patelli 2016) and handles probabilities, probability-boxes fuzzy variables and intervals at different stages of uncertainty treatment (representation, propagation, sensitivity analysis, reliability analysis, optimisation, etc.). Under the R platform (R Core Team 2017), no such computational framework exists. Available packages mainly focus on uncertainty representation, as for example the **imprProbEst** (Hable 2008) or **dst** R packages (Boivin 2015), or on data mining; e.g., clustering (**evclust** R package) and classification (**evclass** R package).

The purpose of this communication is to present the R package **HYRISK**. The paper is organised as follows. In a first section, we summarise the main choices for the package design in order to fill the gap identified among the available open-source software solutions for uncertainty quantification with mixed uncertainty representation tools. Then, we describe the case study of a dike stability analysis (adapted from (Ferson and Tucker 2006)), which is used to describe and illustrate the HYRISK functionalities for the uncertainty quantification procedure in the subsequent sections.

Package	Platform	Description
DSI toolbox	Matlab	Collection of methods for uncertainty quantification and propagation using Dempster-Shafer Theory and imprecise probabilities
COSSAN	Matlab	Algorithms and methods for uncertainty quantification and management: numerical analysis, reliability analysis, simulation, sensitivity, optimization, robust design
imprProbEst	R	Minimum distance estimator for an imprecise probability model
dst	R	Basic probability assignments on a set of possibilities (events) and combine these events with Dempster's rule of combination
evclust	R	Clustering algorithms that produce a credal partition
evclass	R	Evidential distance-based classifier

Table 1: Main statistical open-source software solutions handling a combination of uncertainty representation tools

2. Package functionalities

To fill the gap identified in the introduction, the **HYRISK** package provides functions at each step of the uncertainty quantification procedure (Table 2).

The first step focuses on uncertainty representation. It aims at selecting the most appropriate mathematical tool to represent uncertainty on the considered parameter. The available options are: interval, possibility distribution (trapezoidal or triangular, see e.g. Baudrit and Dubois (2006)), probability distribution (normal, lognormal, beta, triangle, uniform, Gumbel or user-defined), a probability distribution with imprecise parameters, i.e. a family of parametric probability distributions represented by a p-box (Ferson, Kreinovich, Ginzburg, Myers, and Sentz 2002). For the sake of clarity, we use the generic term imprecise probability to designate such a uncertainty representation tool. The procedure in **HYRISK** first uses the `CREATE_INPUT` function to define the input variables (imprecise, random or fixed); for instance by setting the values of the bounds of the interval, the mean and the standard deviation of a normal probability distribution, etc. Second, the `CREATE_DISTR` function assigns the corresponding distribution (probability or possibility) to each uncertain input previously defined. A visualisation function `PLOT_INPUT` has also been implemented to handle the different representation forms. This is further described in Section 4.

The second step aims at conducting uncertainty propagation, i.e. evaluating the impact of the uncertainty pervading the input on the outcome of the risk assessment model. To do so, the main function is `PROPAG`, which implements the Monte-Carlo-based algorithm of (Baudrit *et al.* 2007) for jointly handling possibility and probability distributions and the algorithm of (Baudrit, Dubois, and Perrot 2008) for jointly handling possibility, probability distributions and p-boxes. Different options for summarizing and visualizing the results of this step are available, which are fully described in Section 5.

The last step focuses on sensitivity analysis, i.e. on the study of how uncertainty in the output

Step	Function	Description
Uncertainty Representation	CREATE_INPUT	Defines the input variables (imprecise, random or fixed). Five types are accounted for: Probability distribution ; Possibility distribution (either trapezoidal or triangular); Intervals; Fixed scalar value; Imprecise probability distributions (p-boxes)
Uncertainty Representation	CREATE_DISTR	Assigns the corresponding distribution (probability or possibility) to each uncertain input based on the input definition using CREATE_INPUT
Uncertainty Representation	PLOT_INPUT	Plots the input variables
Uncertainty Propagation	PROPAG	Conducts the joint propagation of probability, imprecise probability and possibility distributions (or intervals)
Uncertainty Propagation	PLOT_CDF	Creates a plot summarizing the uncertainty propagation in the form of a pair of lower and upper Cumulative Distribution Function CDFs
Uncertainty Propagation	SUMMARY_1CDF	Summarizes the uncertainty propagation results in the form of a single CDF via the weighting average approach of (Dubois and Guyonnet 2011)
Uncertainty Propagation	QUAN_INTERVAL	Summarizes the uncertainty propagation results in the form of an interval of quantiles at a given level
Uncertainty Propagation	UNCERTAINTY	Summarizes the uncertainty propagation results in the form of a global indicator corresponding to the area between the upper and lower CDFs
Sensitivity Analysis	PINCHING_fun	Pinches an imprecise variable to a fixed value following (Ferson and Tucker 2006)
Sensitivity Analysis	SENSI_PINCHING	Conducts a sensitivity analysis using the pinching approach of (Ferson and Tucker 2006)

Table 2: Overview of the main functions implemented in the HYRISK package at each step of the uncertainty treatment procedure.

of a model (numerical or otherwise) can be apportioned to different sources of uncertainty in the model input (Saltelli, Ratto, Andres, Campolongo, Cariboni, Gatelli, Saisana, and Tarantola 2008). The approach, based on the pinching method of Ferson and Tucker (2006), is implemented using the PINCHING_fun and SENSI_PINCHING functions.
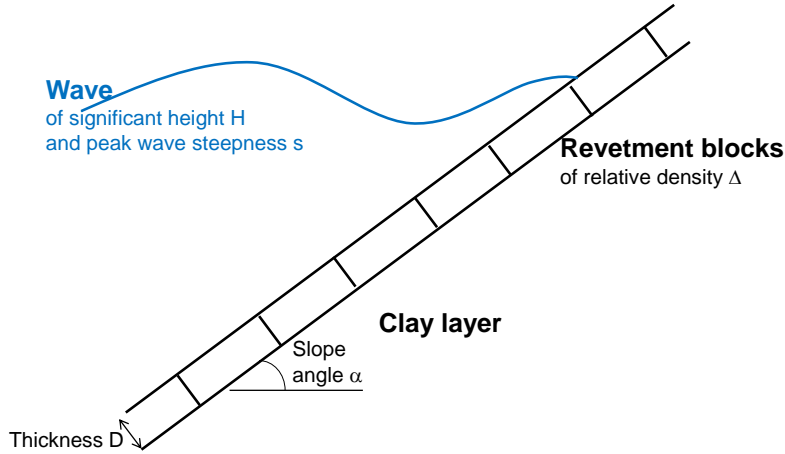
3. Case study

Figure 1: Schematic representation (plane view) of the dike (of slope angle α) made of masonry revetment blocks (with characteristics D and Δ) subject to wave actions (with characteristics H and s). Adapted from (Ferson and Tucker 2006).

In this section, we present the study used to describe and illustrate the functionalities of the **HYRISK** package. This case study is focused on the stability analysis of a dike as described by Ferson and Tucker (2006). The dike has revetments made of masonry blocks subject to wave action as depicted schematically in Fig. 1.

The stability is estimated as the difference between the dike strength minus the stress acting on it as follows:

$$Z = strength - stress = \Delta.D - \frac{H.\tan(\alpha)}{\cos(\alpha).M.s^{0.5}} \tag{1}$$

where Δ is the relative density of the revetment blocks, D is their thickness, α is the slope of the revetment. The wave characteristics are the significant wave height H, and the offshore peak wave steepness s. The factor M reflects the risk analyst's vision on the uncertainty related to the model itself, i.e. its ability to reproduce reality.

If $Z \geq 0$, the dike is stable (the strength is greater than the stress); unstable otherwise. The study is focused on the estimate of the probability for Z to become negative, which is considered a measure of the dike reliability.

4. Step 1: Uncertainty representation

As described in the introduction, depending on the available data and information and on the nature of the model parameter, the most appropriate mathematical tools and procedures

should be selected (Dubois and Guyonnet 2011). The **HYRISK** package provides a broad range of different mathematical representation tools to handle different uncertainty situations.

4.1. Representing imprecision

The first situation corresponds to an imprecise model parameter, i.e. a parameter whose value is ill- known, meaning that the uncertainty is purely epistemic and if we had complete information, we could expect this imprecision to drop to zero. The simplest tool to mathematically represent this imprecision is the interval, which is defined by a lower and an upper bound. In the dike example, this is the case for the revetment density Δ and the thickness D, which are respectively represented by $[1.60, 1.65]$ and $[0.68, 0.72]$ meters.

Yet, the risk analyst may provide in some cases, more information by expressing preferences inside this interval, i.e. the interval representation can be refined, see e.g., (Baudrit *et al.* 2007; Beer *et al.* 2013). This is the case for the slope angle α. The risk analyst is certain that the value for α is located within the interval $[0.309, 0.328]$ radians. However, according to a few measurements and his/her own experience, the analyst may be able to judge that the value 0.318 is the most likely. Using both pieces of information, the imprecision on α can be represented by a triangular possibility distribution. Such a distribution can be seen as a consonant set of intervals each with a level of confidence (Dubois and Prade 1988), which represents the analyst's preference and ranges between confidence one (assigned to the certain interval $[0.309, 0.328]$ referred to as the *support* such that values located outside this interval are considered impossible) and confidence zero (assigned to the most likely value 0.318 referred to as the core of the possibility distribution).

The afore-described situation is also valid for factor M since this factor reflects the confidence of the analyst on the model uncertainty (i.e. its ability to reproduce real situations), and it is subjectively defined. Yet, instead of using a unique value for defining the core (as for slope angle α), the risk analyst prefers here to provide an interval so that the imprecision is in this case represented by a trapezoidal possibility distribution. Figure 2 provides some illustrations of the triangle and trapezoidal possibility distributions. The representation of the afore-described imprecision situations is achieved using the following R code.

```
R> input<-vector(mode="list", length=8)
R> input[[1]]=CREATE_INPUT(
                        name="delta",
                        type="possi",
                        distr="interval",
                        param=c(1.6,1.65)
                        )
R> input[[2]]=CREATE_INPUT(
                        name="D",
                        type="possi",
                        distr="interval",
                        param=c(0.68,0.72)
                        )
R> input[[3]]=CREATE_INPUT(
                        name="alpha",
                        type="possi",
```

```
                          distr="triangle",
                          param=c(0.309,0.318,0.328)
                          )
R> input[[4]]=CREATE_INPUT(
                          name="M",
                          type="possi",
                          distr="trapeze",
                          param=c(3,4,5,5.2)
                          )
```

4.2. Representing randomness

To represent the random nature of model parameters (stochasticity), probabilities are the most appropriate representation tools; for instance by inferring the underlying statistical law. In the dike example, the wave characteristics (s and H) are random variables by nature since they reflect the stochastic nature of the processes generating the waves (related to weather, seasonality and the chance of storm events, etc.). These are respectively represented by a Normal and a Weibull probability distribution. In the **HYRISK** package a series of different probability distribution laws are available (Normal, log-Normal, Triangular, Uniform, Beta, Gumbel) but the user can also easily define a new model (by specifying the quantile and the random sampling function).

Yet, when too few observations are available, the inference of a statistical law may be unfeasible in which case the parameters of an appropriate probability distribution may be difficult to estimate and those parameters are associated with imprecision. This is the case for the Weibull distribution of H used in this example, whose shape and scale parameter (k, λ) values cannot be precisely known due to observation scarcity. This means that there is not a unique Weibull distribution to represent the randomness of H, but a whole set of laws, which may be applied given the imprecision on the afore-described parameters. Two intervals for k and λ, $[10, 12]$ and $[1.2, 1.5]$, are used to represent this imprecision. The ill-known Weibull probability distribution is then represented by a family probability distributions termed p-box (Ferson *et al.* 2002), which is bounded by a lower and an upper distribution determined by the intervals on the shape and scale parameters. The corresponding p-box is illustrated in Fig. 2.

The representation of the afore-described randomness and ill-known randomness situations is conducted using the following R code.

```
R> input[[5]]=CREATE_INPUT(
                          name="H",
                          type="impr proba",
                          distr="user",
                          param=c(7,8),
                          quser=qweibull,
                          ruser=rweibull
                          )
R> input[[6]]=CREATE_INPUT(
                          name="s",
```
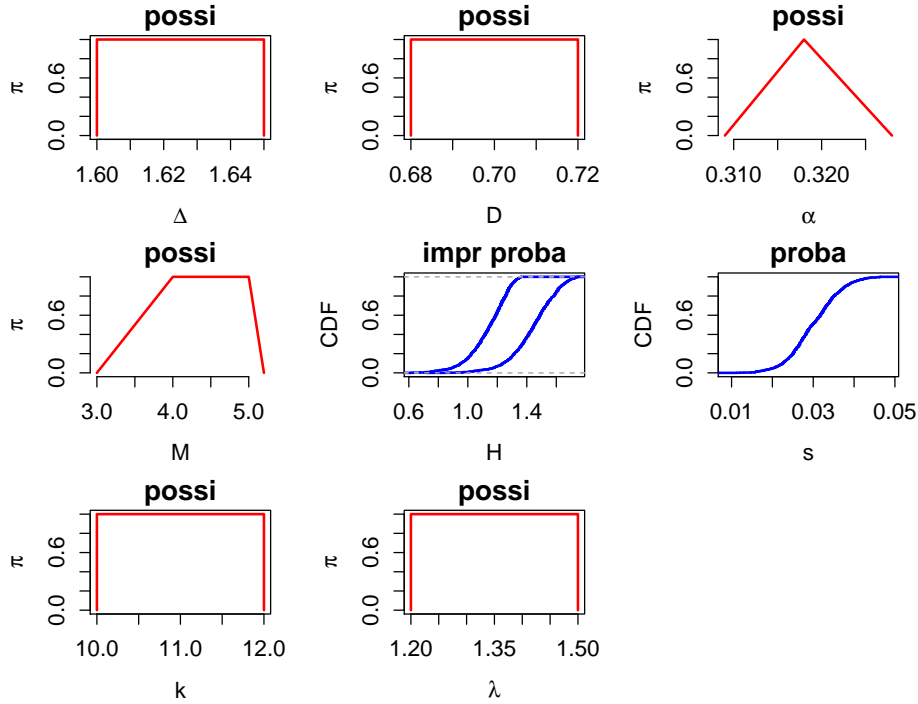
Figure 2:   Uncertainty representation of the input parameters of the dike stability case. The terms *proba*, *possi* and *impr proba* respectively designate probability distribution, possibility distribution, and imprecise probability (i.e. p-box).

```
                type="proba",
                distr="normal",
                param=c(0.03,0.006)
                )
R> input[[7]]=CREATE_INPUT(
                name="shape",
                type="possi",
                distr="interval",
                param=c(10,12)
                )
R> input[[8]]=CREATE_INPUT(
                name="scale",
                type="possi",
                distr="interval",
                param=c(1.2,1.5)
                )
```

Similarly to the situation described for representing imprecision, the user may also provide a possibility distribution to represent the uncertain knowledge on the parameters of the considered imprecise probability distribution following the approach of (Baudrit *et al.* 2008). Table 3 summarizes the assumptions on the uncertainty representation (represented in Fig. 2) for the dike example.

Parameter	Symbol	Uncertainty type	Representation
Significant wave height	H	Randomness	p-box of type Weibull with imprecise shape and scale
Weibull scale	λ	Imprecision	Interval $[1.2, 1.5]$
Weibull shape	k	Imprecision	Interval $[10, 12]$
Peak wave steepness	s	Randomness	Normal (Gaussian) probability distribution with mean=0.040 and standard deviation=0.0055
Revetment density	Δ	Imprecision	Interval $[1.60, 1.65]$
Revetment thickness	H	Imprecision	Interval $[0.68, 0.72]$
Slope angle	λ	Imprecision	Triangular possibility distribution of support $[0.309, 0.328]$ and core 0.318
Expert-defined factor	M	Imprecision	Trapezoidal possibility distribution of support $[3, 5.2]$ and core $[4, 5]$.

Table 3: Assumptions for uncertainty representation in the dike case.

5. Step 2: uncertainty propagation

5.1. Procedure

This step aims at evaluating the uncertainty on the model output given the uncertainty in the model input parameters. Different options are available:

- When using only probabilities, a classical approach can rely on Monte-Carlo random sampling.

- When probability and possibility are used, HYRISK implements the hybrid Monte-Carlo scheme developed by Baudrit, Couso, and Dubois (2007), which combines Monte-Carlo random sampling of probability distributions and interval analysis for possibility distributions. Details of the algorithm are provided in Appendix A. This can be used following two assumptions for the fuzzy interval analysis: (1) the sampling of the possibility distributions is done randomly and independently for each imprecise parameter (algorithm named IRS Independent Random Sampling method, see (Baudrit *et al.* 2007) and further described in Appendix A) or (2) the sampling is deterministic and the fuzzy interval analysis is done at the same level for all imprecision parameters (algorithm termed *HYBRID* described by Guyonnet, Bourgine, Dubois, Fargier, Come, and Chilès (2003) and further developed by Baudrit (2005)).

- When p-boxes are also present (and especially when possibility distributions are used to represent the uncertain knowledge on the parameters of the considered imprecise probability distribution), the sampling strategy developed by Baudrit, Dubois, and Perrot (2008) and further exploited by Pedroni, Zio, Ferrario, Pasanisi, and Couplet (2013) is used (see Appendix B);

- When imprecision on the parameters of the probability distributions are also represented by probability distributions (even though it would mix randomness with imprecision, see discussion by Baudrit, Dubois, and Perrot (2008), the propagation is conducted within a 2D Monte-Carlo scheme (e.g., (Kentel and Aral 2005)).

Since the hybrid algorithms ((Baudrit *et al.* 2007, 2008) are based on interval analysis, several algorithm options for solving the constrained optimization problem are available, namely:

- When the monotonicity of the input-output relationship is known for each variable, others being set at fixed values, (either increasing or decreasing), this information can be passed during the definition of the input (using the argument *monoton* of the function `CREATE_INPUT`). The optimisation problem is then solved by a combinational approach using only the values at the interval endpoints.

- When the monotony of the input-output relationship is unknown, the quasi-Newton method Limited-memory BFGS (e.g., Nocedal (1980)) is used with the **optimr** R package. Due to the sensitivity of the quasi-Newton algorithm to the initial best guest solution (vector of initial values for the parameters for which optimal values are to be found), a version with randomly generated multiple initial starting solutions is also provided.

- A more generic algorithm is also provided, which relies on the genetic algorithm as implemented in the **rgenoud** R package, see Mebane11.

For instance, the propagation in the dike stability case is done using the IRS algorithm with $N = 1,000$ Monte-Carlo random samples and the risk assessment function (denoted here `FUN` and defined in Sect. 3) as follows:

```
R> choice_opt="L-BFGS-B_MULTI" # quasiNewton with random multistart
R> param_opt=10 # 10 multistarts
R> Z0_IRS<-PROPAG(
                N=1000,
                input,
                FUN,
                choice_opt,
                param_opt,
                mode="IRS"
                )
```

5.2. Post-processing of the results for decision-making

The output of the propagation procedure then takes the form of N random intervals of the form $[\underline{Z}_k, \overline{Z}_k]$, with $k = 1, ..., N$. This information can be summarized in the form of a pair of upper and lower cumulative probability distributions (CDFs), in the form of a p-box which is closely related to upper and lower probabilities of Dempster (Dempster 1967), and belief functions of Shafer (Shafer *et al.* 1976) as proposed by Baudrit, Couso, and Dubois (2007). Figure 3 provides the output of the propagation phase using the `PLOT_CDF` function.
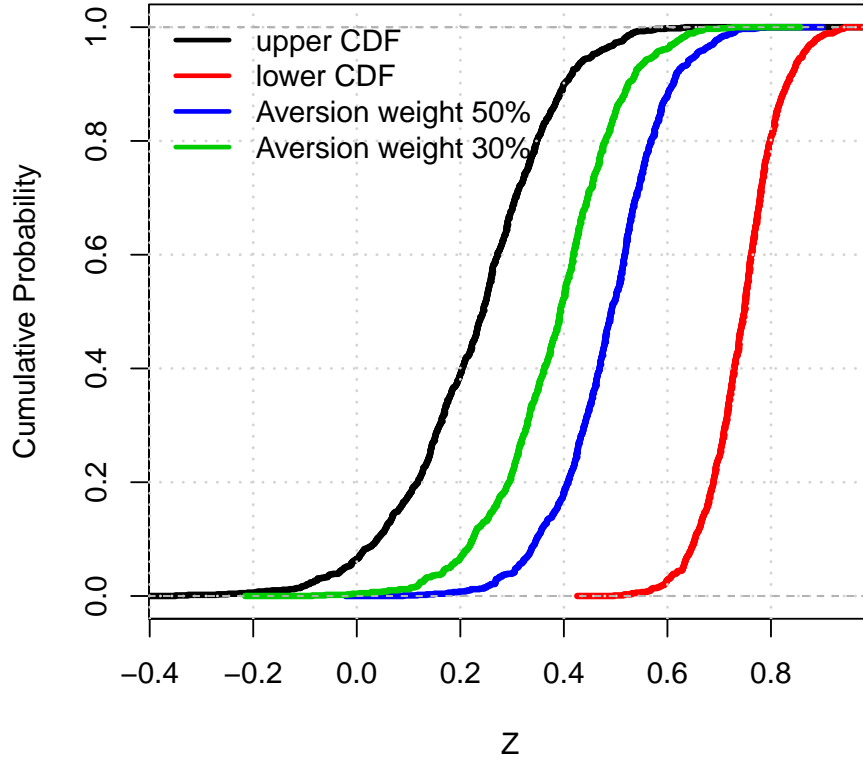
Figure 3: Results of the hybrid uncertainty propagation step for the dike stability case; the black and red lines respectively represent the lower and upper probability distribution, which defines the p-box related to the uncertainty on Z. The green and blue lines respectively represent the probability distribution determined by a weighing averaging of both bounds, here using a weight of 30 and 50% (following the approach of (Dubois and Guyonnet 2011)).

The functionalities of **HYRISK** enable to summarise the p-box of Figure 4 depending on the statistical quantity of interest supporting the decision making process.

- If the interest is the probability of Z being below the decision threshold at zero, the `PROBA_INTERVAL` function provides the corresponding upper and lower bound on this probability. In the dike case, this is comprised between $[0, 7]\%$.

- If the interest is a quantile at a given level (say 75%), the `QUAN_INTERVAL` function provides the corresponding interval, namely $[0.32, 0.79]$ in the dike case.

- If the interest is a global measure of epistemic uncertainty affecting the whole probability distribution of Z, the `UNCERTAINTY` function computes the area within the lower and upper probability distribution, namely of 51 in the dike case.

Finally in some situations, the analysts may be more comfortable in deriving a unique probability distribution. In order to support decision-making with a less extreme indicator than using either probability bounds, Dubois and Guyonnet (2011) proposed to weight the bounds by an index w, which reflects the attitude of the decision-maker to risk (i.e. the degree of risk aversion) so that the resulting distribution F holds as follows:

$$F^{-1}(x) = w.\overline{F}^{-1}(x) + (1-w).\underline{F}^{-1}(x) \tag{2}$$

where x is the quantile level ranging between 0 and 1. This can be done using the `SUMMARY_1CDF` function as exemplified in Fig. 3 using two aversion weight values of respectively 30 and 50%.

The afore-described summary functions can be implemented as follows:

```
# interval of quantiles
R> level=0.75##quantile level
R> quant<-QUAN_INTERVAL(Z0_IRS,level)
R> print(paste("Quantile inf: ",round(quant$Qlow,2)))
R> print(paste("Quantile sup: ",round(quant$Qupp,2)))

"Quantile inf:  0.33"
"Quantile sup:  0.79"


# interval of probabilities
R> thres=0.## decision threshold
R> prob<-PROBA_INTERVAL(Z0_IRS,thres)
R> print(paste("Probability inf: ",round(prob$Plow,2)))
R> print(paste("Probability sup: ",round(prob$Pupp,2)))

"Probability inf:  0"
"Probability sup:  0.07"


# Global indicator of uncertainty
R> unc<-UNCERTAINTY(Z0_IRS)
R> print(paste("Epistemic uncertainty: ",round(unc,2),sep=""))

"Epistemic uncertainty : 50.9"


# Summary using one CDF
R> PLOT_CDF(Z0_IRS,xlab="Z",ylab="Cumulative Probability",main="",lwd=3.5)
R> Z50<-SUMMARY_1CDF(Z0_IRS,aversion=0.5) ##risk aversion of 0.50
R> lines(ecdf(Z50),col=3,lwd=3.5)
```

6. Step 3: sensitivity

When mixed representations of uncertainty are present, a practical method for sensitivity analysis is the pinching approach proposed by Ferson and Tucker (2006), which consists of
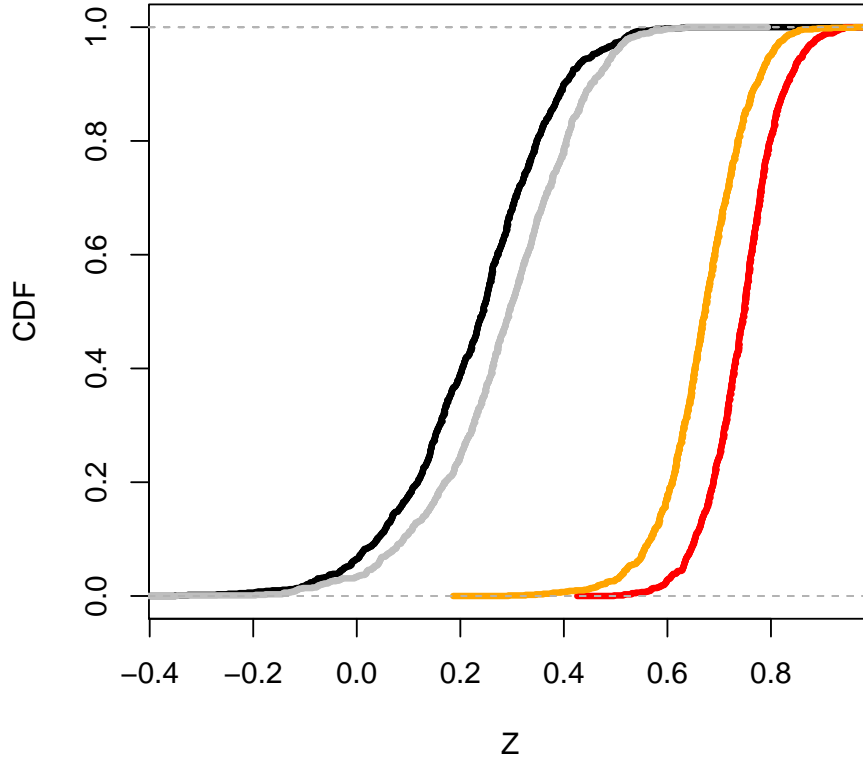
Figure 4: Lower and upper CDFs before (black and red lines) and after (grey and orange lines) pinching the scale parameter λ to the value of 1.4.

assessing how the imprecision on the output would reduce if additional information on the input parameters were available, i.e. if imprecision in the inputs were removed / reduced. For instance, this can be performed by transforming the interval for the scale parameter λ to a constant value (here of 1.4) or the triangular possibility distribution of D to a constant value, e.g. to the core 0.7. This can be performed using the `PINCHING_fun` function.

Figures 4 and 5 shows the evolution of the lower and upper CDF after respectively pinching λ and D.

After pinching λ and D, the impact of the epistemic reduction can be summarised using the `SENSI_PINCHING` function in terms of:

- Reduction of the area between both CDFs, which respectively yields $\approx 24\%$ and $\approx 13\%$.

- Reduction of the interval of probability of exceeding the zero threshold, which respectively yields 48% and 45%.

- Reduction of the interval of quantile at 75%, which respectively yields $\approx 26\%$ and $\approx 13\%$.
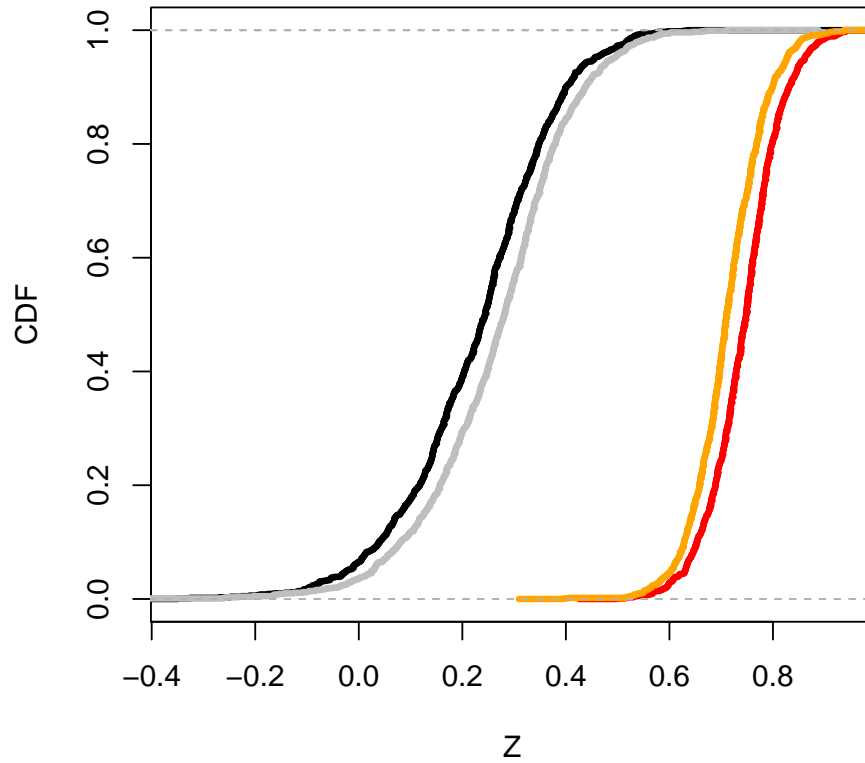
Figure 5: Lower and upper CDFs before (black and red lines) and after (grey and orange lines) pinching the block thickness D to the value of 0.7.

In this example, we can conclude that Z is more sensitive to the epistemic parameter λ than to D whatever the considered indicator (global, probability, quantile). The pinching procedure can be implemented as follows:

```
R> Z0p<-PINCHING_fun(
      which=8,##scale parameter = parameter 8
      value=1.4, ##pinched at the scalar value of 1.4
      N=1000,
      input,
      FUN,
      choice_opt,
      param_opt,
      mode="IRS"
      )
```

```
# VISU - PROPAGATION
R> PLOT_CDF(Z0_IRS,xlab="Z",ylab="CDF",main="",lwd=3.5)
R> PLOT_CDF(Z0p2,color1="gray75",color2="orange",new=FALSE,lwd=3.5)

## quantile mode
R> sensi.quan<-SENSI_PINCHING(Z0_IRS,Z0p,mode="quantile",level=0.75)
R> print(paste("Quantile-based sensitivity measure: ",round(sensi.quan,2),sep=""))

## proba mode
R> sensi.proba<-SENSI_PINCHING(Z0_IRS,Z0p,mode="proba",threshold=0)
R> print(paste("Probability-based sensitivity measure: ",round(sensi.proba,2),sep=""))

## global mode
R> sensi.global<-SENSI_PINCHING(Z0_IRS,Z0p,mode="global",disc=0.01)
R> print(paste("global sensitivity measure: ",round(sensi.global,2),sep=""))
```

```
"Quantile-based sensitivity measure: 26.05"
"Probability-based sensitivity measure: 48.76"
"global sensitivity measure: 24.93"
```

7. Summary and further developments

When dealing with incomplete, scarce or imprecise data, it is of primary importance to use flexible and appropriate mathematical representation tools for accounting for all data and pieces of information, but without introducing unwarranted assumptions Beer13. The **HYRISK** package, described in the present communication offers, within the open source statistical software R, the capability to handle a large spectrum of different tools, namely probabilities, possibility distributions and probabilities with imprecise parameters, for the different stages of uncertainty treatment in risk assessments (i.e. uncertainty representation; propagation, sensitivity analysis and decision-making).

These functionalities were illustrated using the case study of a dike stability analysis. To date, the package focuses on situations where the risk assessment model is costless to evaluate (i.e. low computation time cost). Yet a major additional barrier to the systematic use of these techniques is their practical implementation to numerical codes. Manipulating hybrid mathematical tools goes at the expense of additional computational efforts: in a broad sense, the hybrid uncertainty propagation imposes not only to randomly sample probability distributions, through Monte-Carlo-like techniques, but also to solve constrained optimization problems. The combination with meta-modelling techniques as proposed by Lockwood, Anitescu, and Mavriplis (2012) and Eldred, Swiler, and Tang (2011) is promising. Combining the HYRISK package with such techniques (for instance by taking advantages of the suite Dice of R packages based on kriging metamodeling, (Roustant, Ginsbourger, and Deville 2012) constitutes a line for future developments.

Acknowledgments

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A. IRS algorithm with possibilities and probabilities

Consider k random input variables X_i (i=1,...,k), each of them associated to a cumulative probability distribution F, and n-k imprecise input variables X_i (i=k+1,...,n), each of them associated to a possibility distribution π. In this situation, the IRS procedure proceeds as follows. First randomly generate from uniform probability distributions, m vectors of size n: α_i, i=1,...,n, such that $0 \leq \alpha_i \leq 1$. For each realisation:

1. Generate k values for the random input variables by using the inverse function of F_i: $x_i = F_i^{-1}(\alpha_i)$, i=1,... ,k

2. Sample n-k intervals I_i corresponding to the cuts of the possibility distributions with level of confidence 1-α_i, i=k+1,...,n

3. Evaluate the interval $[\underline{Z}, \overline{Z}]$ defined by the lower and upper bounds associated to the model output Z using the risk assessment model G using equation 3.

$$\underline{Z} = \inf_I(G(x_1, ..., x_k, I_{k+1}, ..., I_n)) \qquad \overline{Z} = \sup_I(G(x_1, ..., x_k, I_{k+1}, ..., I_n)) \qquad (3)$$

B. IRS algorithm with possibilities, probabilities, and with p-boxes

Consider k random input variables X_i (i=1,...,k), each of them associated to a cumulative probability distribution F, $n-k$ imprecise input variables X_i (i=k+1,...,n), each of them associated to a possibility distribution α and j-(n+k) random input variables tainted with imprecision X_i (i=n+1,...,j), each of them associated to a parametric probabilistic model with imprecise input variables. In this situation, the IRS procedure holds as follows. Randomly generate from uniform probability distributions, m vectors of size j: $alpha_i$, i=1,...,j, such that $0 \leq \alpha_i \leq 1$. For each realisation:

1. Generate k values for the random input variables by using the inverse function of F_i: $x_i = F_i^{-1}(alpha_i)$, i=1,...,k;

2. Sample n-k intervals I_i corresponding to the cuts of the possibility distributions with level of confidence: 1-α_i, i=k+1,...,n;

3. For each random input variables tainted with imprecision X_i (i=n+1,...,j), from the intervals $I_{parameters}$ sampled at the previous stage (2) for the imprecise parameters associated to this variable, evaluate the interval:

$$[inf_{I_{parametersofX_i}}(F_i^{-1}(\alpha_i)); sup_{I_{parametersofX_i}}(F_i^{-1}(\alpha_i))] \qquad (4)$$

4. Evaluate the interval $[\underline{Z}, \overline{Z}]$ defined by the lower and upper bounds associated to the model output Z using the risk assessment model G as follows:

$$\underline{Z} = \inf_I(G(x_1, ..., x_k, I_{k+1}, ..., I_n, I_{n+1}, ..., I_j)) \qquad \overline{Z} = \sup_I(G(x_1, ..., x_k, I_{k+1}, ..., I_n, I_{n+1}, ..., I_j))$$

$$(5)$$

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