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A Unified Treatment of SKS Splitting and Surface-Wave Anisotropy for Media with Arbitrary Elastic Symmetry

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SUMMARY

SKS splitting and azimuthal anisotropy of surface-wave phase speed are two of the primary observational constraints on crustal and upper-mantle anisotropy, yet anisotropy models derived from surface waves commonly underpredict the magnitude of observed SKS splitting. We present a unified treatment of the two observables for weakly anisotropic media with arbitrary elastic symmetry. We derive a new first-order asymptotic theory relating anisotropy to SKS splitting intensity, validate it against numerical simulations, and compare it directly with quasi-degenerate surface-wave theory that includes Rayleigh–Love coupling. Including Rayleigh–Love coupling in the surface-wave theory used to interpret azimuthal anisotropy changes the inferred elastic tensor in ways that can strongly affect predicted SKS splitting. We demonstrate these effects using surface-wave and SKS observations and anisotropy models from Alaska. First, accounting for Rayleigh–Love coupling produces substantially larger $G_{c,s}$, increasing splitting intensity even for vertically incident SKS waves. Second, at the small non-zero incidence angles characteristic of SKS propagation, the parameters $K_{c,s}$, $M_{c,s}$, and $D_{c,s}$, which are important for Rayleigh–Love coupling in surface-wave theory, also contribute directly to SKS splitting, with $M_{c,s}$ particularly important. We show that at five example points in Alaska, a surface-wave anisotropy model derived using theory that includes Rayleigh–Love coupling reproduces

the observed SKS splitting intensities. A reference model constructed without accounting for coupling systematically underpredicts them. Both models reproduce the fast-axis orientations reasonably well. We conclude that neglecting Rayleigh–Love coupling in surface-wave inversions can lead to substantial underestimation of the anisotropy relevant to SKS splitting, particularly where Rayleigh–Love coupling is strong, which may help explain the apparent need for anisotropy below the depth range sampled by surface waves.

Key words: Seismic anisotropy; Surface waves and free oscillations; Wave propagation; Body waves; Crust and mantle structure.

1 INTRODUCTION

Observations of SKS splitting and azimuthal anisotropy of surface-wave phase speed are two of the principal constraints on anisotropy in the crust and upper mantle. Fast-axis orientations inferred from SKS splitting and surface-wave anisotropy are often similar (e.g., Feng et al., 2020; Zhu et al., 2020), but despite decades of application, estimates of the strength and depth distribution of anisotropy can differ substantially. In particular, anisotropy models derived from surface-wave observations commonly underpredict the magnitude of observed SKS splitting, a discrepancy that has often been interpreted as evidence for significant anisotropy below the depth range sampled by surface waves. Recent quasi-degenerate surface-wave theory has shown that explicitly modeling Rayleigh–Love coupling can substantially change the elastic tensor inferred for the crust and uppermost mantle from observations of surface-wave phase-speed azimuthal anisotropy (X. Liu & Ritzwoller, 2025). Observations of Rayleigh- and Love-wave azimuthal anisotropy across Alaska provide evidence that such coupling is strong (X. Liu et al., 2025). This raises the central question addressed here: how does accounting for Rayleigh–Love coupling in the surface-wave inversion affect the SKS splitting intensity predicted from the resulting elastic-tensor model, and what are the implications for inferences about the strength and depth distribution of anisotropy?

To address this question, we require a formulation of SKS splitting that is sufficiently general to describe the anisotropic structure represented by the elastic tensor inferred from surface-wave observations. We therefore derive a first-order asymptotic theory for SKS splitting intensity in weakly anisotropic media with arbitrary elastic symmetry, applicable to vertical incidence and to the small non-vertical incidence angles characteristic of SKS propagation. This formulation permits a direct comparison with the quasi-degenerate surface-wave theory that includes Rayleigh–Love coupling (X.

Liu & Ritzwoller, 2025). As we show below, the two theories are closely linked through their dependence on overlapping components of the elastic tensor, providing a unified framework for relating surface-wave anisotropy to SKS splitting.

Surface-wave inversions for anisotropy have evolved in two related but distinct ways. First, the elastic parameterizations used in the inversions have become progressively more general. Most studies have assumed transverse isotropy with either a vertical (VTI) or horizontal (HTI) symmetry axis, although some have allowed the symmetry axis to tilt (TTI) (e.g., Montagner & Jobert, 1988; Xie et al., 2015, 2017; C. Liu & Ritzwoller, 2024). To our knowledge, published surface-wave inversions have not considered symmetry lower than TTI, although such models are now being developed (e.g., X. Liu & Ritzwoller, 2026b). Second, the forward theory used in the inversion must be sufficiently general to predict the effects of low-symmetry anisotropic structure on the surface-wave observations. Rayleigh–Love coupling can occur in transversely isotropic media with non-vertical symmetry axes and in lower-symmetry media, but its effects are neglected if the forward theory assumes that Rayleigh and Love waves propagate independently and are not coupled, as is common practice. We have argued elsewhere (X. Liu & Ritzwoller, 2025) that fully exploiting such general elastic parameterizations requires a surface-wave theory that explicitly accounts for Rayleigh–Love coupling.

Historically, observational studies of surface-wave azimuthal anisotropy included both Rayleigh and Love waves. Tanimoto & Anderson (1985) and Montagner & Tanimoto (1990), for example, presented global maps of azimuthal anisotropy from both Rayleigh and Love waves, including a Love-wave 2ψ component. Subsequent studies increasingly emphasized Rayleigh-wave 2ψ anisotropy, which became the most commonly used surface-wave constraint on mantle azimuthal anisotropy. More recently, observations of both Rayleigh- and Love-wave 2ψ and 4ψ anisotropy (X. Liu et al., 2025) have been interpreted using a quasi-degenerate theory that incorporates Rayleigh–Love coupling (X. Liu & Ritzwoller, 2025), which affects the azimuthal distributions of both Rayleigh- and Love-wave phase speeds. These studies demonstrate that neglecting Rayleigh–Love coupling in the forward theory can significantly bias the amplitude and depth distribution of inferred anisotropy. In particular, X. Liu & Ritzwoller (2025) showed that strong Love-wave 2ψ anisotropy indicates significant Rayleigh–Love coupling, and accounting for that coupling can yield substantially larger inferred anisotropy amplitudes. The present study builds on these results by determining how the elastic tensor inferred using this more complete surface-wave theory affects predicted SKS splitting and its interpretation.

The relatively large anisotropy amplitudes inferred when Rayleigh–Love coupling is taken into account are consistent with growing evidence that seismic anisotropy in some parts of the mantle may be substantially stronger than has commonly been assumed. Li et al. (2018) inferred shear-wave anisotropy of about 25% in intermediate-depth portions of subducting slabs from the non-double-

couple components of earthquake moment tensors. More recently, Appini et al. (2025a, 2025b) showed through full-wave modeling that approximately 30% intra-slab shear-wave anisotropy can reproduce observed shear-wave splitting patterns in both the Ryukyu and Alaska subduction zones. These studies provide complementary evidence for the existence of substantially stronger mantle anisotropy than is commonly represented in seismic models.

SKS splitting provides an important complementary constraint on mantle anisotropy. Shear-wave splitting has been used to investigate seismic anisotropy dating back several decades (e.g., Crampin, 1978; Ando et al., 1980; Fukao, 1984), with teleseismic shear phases becoming a particularly important probe of mantle anisotropy (e.g., Vinnik et al., 1984; Silver & Chan, 1991; Long & Silver, 2009). SKS splitting is commonly characterized either by estimating a fast polarization direction and splitting delay time, often through transverse-component minimization (e.g., Silver & Chan, 1991), or through the splitting-intensity method of Chevrot (2000). We focus on splitting intensity because it is well suited to the first-order theoretical treatment developed below and permits predictions from depth-dependent elastic-tensor models to be compared directly with SKS observations.

Montagner et al. (2000) demonstrated that, under the assumptions of HTI symmetry and vertical propagation adopted in their analysis, SKS splitting and Rayleigh-wave azimuthal anisotropy are principally sensitive to the same anisotropy parameters, $G_{c,s}$ (Appendix B). This implies that SKS splitting can be predicted from anisotropy models derived from surface-wave observations alone (e.g., Marone & Romanowicz, 2007; Yuan & Romanowicz, 2010). In practice, comparisons at both global and regional scales commonly find better agreement in fast-axis orientation than in splitting amplitude, with observed SKS splitting often substantially stronger than predicted from surface-wave anisotropy models (Wüstefeld et al., 2009; Kosarian et al., 2011; Becker et al., 2012; Wagner & Long, 2013; Feng et al., 2020; Zhu et al., 2020; Feng & Díaz, 2022).

This amplitude discrepancy has often been interpreted as evidence for additional anisotropy at depths greater than those well constrained by the surface waves used in the comparison. Li et al. (2003), for example, inferred that much of the observed SKS splitting beneath eastern North America originates below about 200 km; Gaherty (2004) favored layered anisotropy in which a deeper layer produces much of the splitting; and other studies have invoked deeper upper-mantle or transition-zone anisotropy to improve agreement with SKS observations (Yuan & Romanowicz, 2010; Lin et al., 2011; Yuan & Beghein, 2013). At the same time, Wagner & Long (2013) found reasonably similar anisotropy geometry from SKS and Rayleigh waves beneath the High Lava Plains but substantially larger amplitudes from SKS, and argued that the strong small-scale spatial variability of the SKS observations is difficult to explain primarily by deeper mantle structure. Their result points instead toward limitations in the shallower anisotropy inferred from surface waves.

The results presented here suggest that inferences about the depth distribution of anisotropy drawn from comparisons of surface-wave and SKS splitting observations may be biased if Rayleigh–Love coupling is neglected. Rayleigh–Love coupling affects the comparison between the two observables in two distinct ways. For vertical incidence, only the anisotropy parameter pair $G_{c,s}$ contributes to splitting intensity in the first-order SKS theory developed here. Nevertheless, accounting for Rayleigh–Love coupling in the surface-wave inversion can change the $G_{c,s}$ inferred from the observations and therefore change the predicted SKS splitting even at vertical incidence. For the small but non-zero incidence angles typical of SKS observations, the connection is more general: splitting intensity also depends on the $K_{c,s}$, $M_{c,s}$, and $D_{c,s}$ anisotropy parameters, which also play important roles in Rayleigh–Love coupling in surface-wave theory. Thus, a consistent comparison of the two observables requires both a surface-wave inversion that accounts for Rayleigh–Love coupling and an SKS theory that describes the effects of an elastic tensor with general symmetry, including the anisotropy parameters that also affect Rayleigh–Love coupling.

In this paper, we first derive a first-order asymptotic expression for SKS splitting intensity in weakly anisotropic media with arbitrary elastic symmetry and validate it against numerical simulations for both single- and multi-layer models. We then compare the SKS formulation directly with the quasi-degenerate surface-wave theory of X. Liu & Ritzwoller (2025) to establish the theoretical connection between SKS splitting intensity and surface-wave phase-speed anisotropy. Finally, we use observations of Rayleigh- and Love-wave 2ψ and 4ψ phase-speed anisotropy in Alaska (X. Liu et al., 2025) and an elastic-tensor anisotropy model derived from those observations using the surface-wave theory of X. Liu & Ritzwoller (2025), which accounts for Rayleigh–Love coupling, to predict SKS splitting. We compare these predictions with those from the reference anisotropy model of C. Liu & Ritzwoller (2024), which was derived without accounting for Rayleigh–Love coupling. We compare the predictions from both models with observed SKS splitting and examine the implications for estimates of the strength and depth distribution of mantle anisotropy. All observational applications presented here are in Alaska.

2 THEORY OF SKS SPLITTING-INTENSITY IN A GENERAL ANISOTROPIC MEDIUM

2.1 Vertically homogeneous anisotropic medium

We begin by considering an SKS wave incident on and then modified by propagation through a single homogeneous anisotropic medium – a single layer. The discussion that follows is subsequently generalized to a vertically heterogeneous anisotropic medium in Section 2.2.

Consider an SKS wave propagating upward in an isotropic medium in direction $\vec{n} = (\sin I \cos \psi, \sin I \sin \psi, \cos I)$

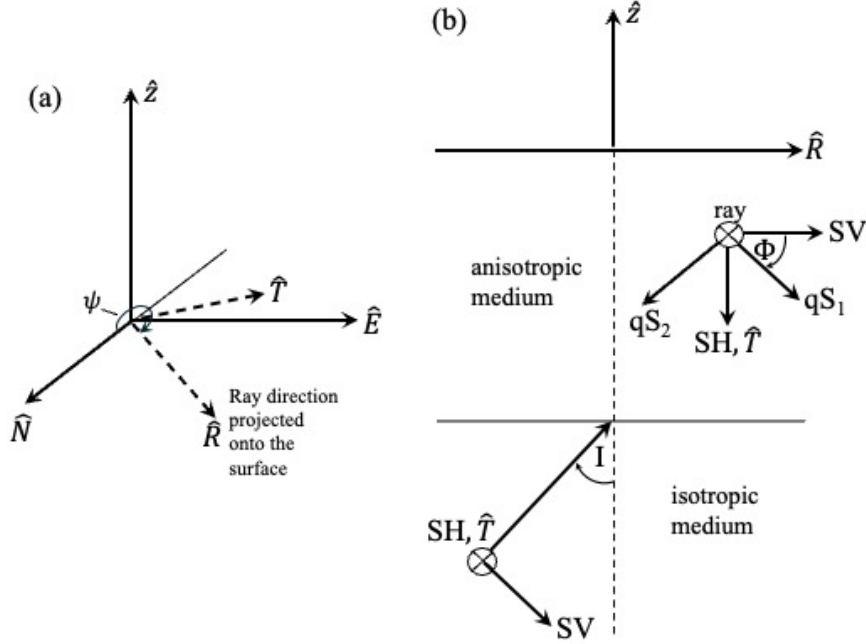


Figure 1. (a) Coordinate axes (north $\hat{N}$, east $\hat{E}$, vertical $\hat{Z}$), radial ($\hat{R}$) and transverse ($\hat{T}$) axes, and azimuthal angle (ψ), measured positive clockwise from North. (b) An SKS wave propagating in an isotropic medium is polarized in the SV direction with an incidence angle I . The SV and SH ($\hat{T}$) directions are both perpendicular to the ray. In the anisotropic medium, the ray's polarization rotates by an angle Φ and there are both quasi-S₁ and quasi-S₂ components.

(Fig. 1), where I is the incidence angle and ψ is the azimuth. SKS incidence angles in the uppermost mantle are typically around 10° (e.g., Davis 2003; Rumpker & Silver 2000), motivating the near-vertical approximation developed below. We therefore use $I = 10^\circ$ as a representative non-vertical incidence angle in the numerical examples. Because the initial polarization of the SKS wave lies in the SV wave direction, the vector displacement is in the plane perpendicular to $\vec{n}$:

$$\mathbf{u}_{VH}^{iso} = \begin{pmatrix} u_{SV}^{iso} \\ u_{SH}^{iso} \end{pmatrix} = \begin{pmatrix} u_{SV}^{iso} \\ 0 \end{pmatrix} \quad (1)$$

where u_{SV}^{iso} is the initial SKS scalar displacement in the isotropic medium. We refer to these as the $V-H$ coordinates, which lie in the transverse plane in ray-centered coordinates. The H -axis coincides with the SH direction and the V -axis with the SV direction. Following Montagner et al. (2000), the displacement vector of the SKS wave that propagates in the isotropic medium in the $V-H$ coordinates is given by

$$u_{SV}^{iso} = a_0 e^{i\omega(t - \frac{z-z_0}{V_{iso}})} \quad (2)$$

where ω is angular frequency, z is the vertical coordinate measured positive upward from the surface

(and therefore negative below the surface), and z_0 is the vertical coordinate at the base of the isotropic medium. Assuming that z lies above z_0 , $z - z_0$ is positive and is the propagation distance in that isotropic medium.

Either by degenerate perturbation theory (e.g. Jech & Psencik 1989; Chen & Tromp 2007) or a variational method (Liu & Ritzwoller, 2025), when the SKS wave enters a general anisotropic medium it is split into two quasi-shear waves with polarizations rotated by angle Φ , as shown in Figure 1b, such that $V_1 = V_{iso} + \delta c_1$ and $V_2 = V_{iso} + \delta c_2$. V_1 and V_2 depend on the incidence angle I , the azimuth ψ , and the elastic tensor, which may have arbitrary elastic symmetry. The perturbations to the two quasi-shear wave speeds and the polarization angle Φ are determined by

$$\delta c_{1,2} = \frac{B_{11} + B_{22} \pm B}{4V_{iso}} \quad (3)$$

$$\tan 2\Phi = \frac{2B_{12}}{B_{11} - B_{22}} \quad (4)$$

where $B = \sqrt{(B_{11} - B_{22})^2 + 4B_{12}^2}$ and B_{11} , B_{22} , and B_{12} are given by equations (36)-(38) of Chen & Tromp (2007), and are functions of incidence angle I and azimuth ψ . δc_1 is the faster shear wave computed with the plus sign in front of B in equation (3) and δc_2 is the slower shear wave computed with the negative sign. There is a $\frac{\pi}{2}$ ambiguity of Φ in equation (4) because $\tan 2\Phi = \tan 2(\Phi + \frac{\pi}{2})$, so we define $2\Phi \equiv \arg[(B_{11} - B_{22}) + i2B_{12}] = \text{atan2}(2B_{12}, B_{11} - B_{22})$.

To simplify the calculation, we define a new reference shear wave speed, V_0 , as

$$V_0(\psi, I) \equiv V_{iso} + \frac{\delta c_1 + \delta c_2}{2} = V_{iso} + \frac{B_{11} + B_{22}}{4V_{iso}} \quad (5)$$

which is a function of both azimuth and incidence angle. In general, $V_0 \neq V_{iso}$. This choice removes the mean anisotropic velocity perturbation and isolates the splitting associated with the velocity difference between the quasi-shear modes. Thus, the speeds of the quasi- S_1 and quasi- S_2 waves are $V_1 = V_0 + \frac{1}{2}\delta V$ and $V_2 = V_0 - \frac{1}{2}\delta V$, respectively, where δV is the speed difference between the two quasi-shear waves

$$\delta V = \delta c_1 - \delta c_2 = \frac{B}{2V_{iso}}. \quad (6)$$

We also define a reference displacement field around which we perturb, which is polarized in the SV-direction and propagates with average phase speed V_0 in the anisotropic medium:

$$u_{SV}^0 \equiv a_0 e^{i\omega(t - \frac{z - z_0}{V_0})} \quad (7)$$

Employing these definitions, the phase part of quasi-shear waves in the anisotropic medium can be approximated as

$$e^{i\pm} = e^{i\omega(t - \frac{z - z_0}{V_0 \pm \frac{1}{2}\delta V})} \approx e^{i\omega(t - \frac{z - z_0}{V_0})} e^{\pm \frac{1}{2}i\omega\delta t} \quad (8)$$

where δt is given by

$$\delta t \equiv \frac{(z - z_0)\delta V}{V_0^2}. \quad (9)$$

In equation (8), we have applied the weak anisotropy approximation, which is accurate to first-order in $\delta V/V_0$. The expansion of phase speed assumes $\delta V/V_0 \ll 1$, while the subsequent splitting intensity approximation assumes $\omega\delta t/2 \ll 1$. The condition $\omega\delta t/2 \ll 1$ is generally more restrictive than $\delta V/V_0 \ll 1$, because splitting accumulates with propagation distance.

Based on equations (6) and (9), we could derive a fully general expression for SKS splitting time in a general single-layer anisotropic medium given the definitions of B_{11} , B_{22} , and B_{12} by Chen & Tromp (2007). However, the complexity of the result would obscure both the direct theoretical comparison between SKS and surface-waves and its physical meaning. For clarity, we instead continue to apply the weak-anisotropy approximation in deriving a simpler expression for splitting intensity.

Because the two quasi-shear waves have different speeds in a general anisotropic medium, the displacement vector in $V - H$ coordinates is

$$\mathbf{u}_{VH} = R(\Phi) \begin{pmatrix} e^{i+} & 0 \\ 0 & e^{i-} \end{pmatrix} R(-\Phi) \mathbf{a}_0 = \begin{pmatrix} [\cos(\frac{1}{2}\omega\delta t) + i \sin(\frac{1}{2}\omega\delta t) \cos 2\Phi] u_{SV}^0 \\ [i \sin(\frac{1}{2}\omega\delta t) \sin 2\Phi] u_{SV}^0 \end{pmatrix} \quad (10)$$

where $\mathbf{a}_0 = \begin{pmatrix} a_0 \\ 0 \end{pmatrix}$ is the initial polarization of the wave as it emerges from the isotropic medium

and $R(\Phi) = \begin{pmatrix} \cos \Phi & -\sin \Phi \\ \sin \Phi & \cos \Phi \end{pmatrix}$ is the 2-D rotation that defines the polarization in the anisotropic medium.

Imposing the second approximation, which is first-order in $\omega\delta t/2$, equation (10) becomes

$$\mathbf{u}_{VH} \approx \begin{pmatrix} u_{SV}^0 + \frac{1}{2}\delta t \cos 2\Phi \dot{u}_{SV}^0 \\ \frac{1}{2}\delta t \sin 2\Phi \dot{u}_{SV}^0 \end{pmatrix} \quad (11)$$

where $\dot{u}_{SV}^0$ is the time derivative of u_{SV}^0 . Using Φ from equation (4), δV from equation (6), and δt from equation (9), gives

$$\delta t \cos 2\Phi = \frac{(z - z_0)}{V_0^2} \cdot \frac{B}{2V_{iso}} \cdot \frac{B_{11} - B_{22}}{B} = \frac{(z - z_0)(B_{11} - B_{22})}{2V_0^2 V_{iso}} \quad (12)$$

$$\delta t \sin 2\Phi = \frac{(z - z_0)}{V_0^2} \cdot \frac{B}{2V_{iso}} \cdot \frac{2B_{12}}{B} = \frac{(z - z_0)B_{12}}{V_0^2 V_{iso}} \quad (13)$$

Therefore, equation (11) can be rewritten as

$$\mathbf{u}_{VH} \approx \begin{pmatrix} u_{SV}^0 + \frac{(z-z_0)(B_{11}-B_{22})}{4V_0^2 V_{iso}} \dot{u}_{SV}^0 \\ \frac{(z-z_0)B_{12}}{2V_0^2 V_{iso}} \dot{u}_{SV}^0 \end{pmatrix} \approx \begin{pmatrix} u_{SV}^0 + \frac{(z-z_0)(B_{11}-B_{22})}{4V_{iso}^3} \dot{u}_{SV}^0 \\ \frac{(z-z_0)B_{12}}{2V_{iso}^3} \dot{u}_{SV}^0 \end{pmatrix} \quad (14)$$

where we have invoked the weak anisotropy approximation in setting $V_{iso}^3 \approx V_0^2 V_{iso}$. Finally, we apply

the third approximation. Because the perturbation to the SV component enters as a correction to an existing $O(1)$ displacement, whereas the SH component is itself $O(\epsilon)$ where ϵ is the characteristic fractional anisotropy, we retain only the leading-order contribution to u_{SV} :

$$\mathbf{u}_{VH} \approx \begin{pmatrix} u_{SV}^0 \\ \frac{(z-z_0)B_{12}}{2V_{iso}^3} \dot{u}_{SV}^0 \end{pmatrix} \quad (15)$$

Rotating the displacement vector $\mathbf{u}_{VH}$ from the $V-H$ back to the $R-T-Z$ (radial-transverse-vertical) coordinates (Fig. 1a), we obtain

$$\mathbf{u}_{RTZ} = \begin{pmatrix} u_R \\ u_T \\ u_Z \end{pmatrix} = \begin{pmatrix} \cos I & 0 & \sin I \\ 0 & 1 & 0 \\ -\sin I & 0 & \cos I \end{pmatrix} \begin{pmatrix} u_{SV}^0 \\ \frac{(z-z_0)B_{12}}{2V_{iso}^3} \dot{u}_{SV}^0 \\ 0 \end{pmatrix} = \begin{pmatrix} u_{SV}^0 \cos I \\ \frac{(z-z_0)B_{12}}{2V_{iso}^3} \dot{u}_{SV}^0 \\ -u_{SV}^0 \sin I \end{pmatrix} \quad (16)$$

For a general anisotropic medium with an arbitrary incidence angle I and azimuth ψ , the expression for B_{12} (Chen & Tromp, 2007) is:

$$\begin{aligned} \rho B_{12}(\psi, I) = & \frac{1}{2}(\sin I + \sin 3I)(K_c \sin \psi - K_s \cos \psi) - \sin I(M_c \sin \psi + M_s \cos \psi) \\ & + \frac{1}{2}(\cos I + \cos 3I)(G_c \sin 2\psi + G_s \cos 2\psi) + \frac{1}{8}(\cos I - \cos 3I)(B_c \sin 2\psi + B_s \cos 2\psi) \\ & - \frac{1}{4}(\cos I - \cos 3I)(H_c \sin 2\psi + H_s \cos 2\psi) - \frac{1}{4}(\sin I - 3 \sin 3I)(D_c \sin 3\psi - D_s \cos 3\psi) \\ & + \frac{1}{4}(\cos I - \cos 3I)(E_c \sin 4\psi + E_s \cos 4\psi) \end{aligned} \quad (17)$$

where expressions for the anisotropy parameters ($K_{c,s}$, $M_{c,s}$, etc) are presented in Appendix B here.

For a nearly vertically propagating SKS wave ($I \ll 1$ rad), we obtain the following approximation to equation (17):

$$\begin{aligned} \rho B_{12}(\psi, I) \approx & 2I(K_c \sin \psi - K_s \cos \psi) - I(M_c \sin \psi + M_s \cos \psi) \\ & + (G_c \sin 2\psi + G_s \cos 2\psi) + 2I(D_c \sin 3\psi - D_s \cos 3\psi) \end{aligned} \quad (18)$$

Equation (18) is the near-vertical expression used in the applications presented later in this paper. In the limit as $I \rightarrow 0$, only the $G_{c,s}$ contribution remains in equation (18). The $G_{c,s}$ term varies with azimuth as 2ψ . The remaining sensitivity to anisotropy is through the $K_{c,s}$, $M_{c,s}$, and $D_{c,s}$ parameters. These parameters are particularly important because they also play a central role in the description of Rayleigh–Love coupling in surface-wave theory, as discussed below. These terms are all proportional to the incidence angle I , which is small for SKS waves. Thus, their effect on B_{12} is expected to be smaller than the $G_{c,s}$ term, but may be significant, depending on the structure of the elastic tensor. Each of these terms has an odd-order dependence on ψ . Consequently, for non-zero incidence the 1ψ and 3ψ terms break the 180° periodicity of the vertical-incidence splitting-intensity pattern, so that

$SI(\psi + \pi)$ need not equal $SI(\psi)$. This effect of non-vertical incidence was demonstrated numerically by Chevrot et al. (2004). The sensitivity of SKS splitting intensity to the four anisotropy-parameter pairs $K_{c,s}$, $M_{c,s}$, $G_{c,s}$, and $D_{c,s}$ at small incidence angles was demonstrated numerically by Sieminski et al. (2008) using finite-frequency sensitivity kernels. Their analysis did not provide a closed-form expression showing the explicit dependence of splitting intensity on these parameters, incidence angle, and azimuth. Equation (18) provides such a first-order asymptotic expression and makes explicit the overlap between the parameters controlling SKS splitting and those important for Rayleigh–Love coupling in surface-wave theory.

Following Chevrot (2000), we define *splitting intensity* as $SI \equiv -2u_T/\dot{u}_R$ where u_T and u_R are the transverse and radial components of displacement. Therefore, using equation (16) splitting intensity is

$$SI(\psi) = -\frac{(z - z_0)B_{12}}{V_{iso}^3 \cos I} \approx -\delta t \sin 2\Phi \quad (19)$$

where again we have invoked the weak anisotropy approximation when setting $V_{iso}^3 \approx V_0^2 V_{iso}$ in equation (13) and near vertical incidence in setting $\cos I \approx 1$.

Equations (18) and (19) describe the behavior of splitting intensity in a homogeneous anisotropic layer with arbitrary elastic symmetry for a nearly vertically propagating SKS wave and provide the foundation for the multilayer generalization developed below. They show that, at small but finite incidence angles, splitting intensity depends not only on the classical $G_{c,s}$ parameters but also on $K_{c,s}$, $M_{c,s}$, and $D_{c,s}$. In earlier studies, such as Montagner et al. (2000), expressions presented for δt and Φ depended only on $G_{c,s}$ parameters and not the others that appear in equation (18). Equations (18) and (19) show that the same anisotropy parameters that influence Rayleigh–Love coupling in surface-wave theory also contribute to SKS splitting. This common dependence provides the foundation for the unified treatment of SKS splitting and surface-wave anisotropy developed in the remainder of the paper.

2.2 Vertically heterogeneous anisotropic medium

To generalize the single-layer results to a vertically heterogeneous anisotropic medium, we divide the medium into n homogeneous anisotropic layers and then take the continuum limit. We follow the approach of Montagner et al. (2000), from which we obtain

$$\mathbf{u}_{VH} = R_n H_n R_n^{-1} \cdots R_2 H_2 R_2^{-1} R_1 H_1 R_1^{-1} \mathbf{a}_0 \quad (20)$$

where $R_n = R(\Phi_n)$, $R_n^{-1} = R(-\Phi_n)$, and $H_n = \begin{pmatrix} e_n^{i+} & 0 \\ 0 & e_n^{i-} \end{pmatrix}$.

Expanding the matrix product in equation (20) to second order in $\omega\delta t$ and collecting terms, we obtain the following:

$$\mathbf{u}_{VH} \approx \begin{pmatrix} u_{SV}^0 + \sum_{i=1}^{i=n} \frac{i\omega\delta t_i}{2} \cos 2\Phi_i u_{SV}^0 - \omega^2 \sum_{i=1}^{i=n} \frac{\delta t_i^2}{8} u_{SV}^0 - \omega^2 \sum_{i=1}^{i=n} \sum_{j=i+1}^{j=n} \frac{\delta t_i}{2} \frac{\delta t_j}{2} \cos 2(\Phi_j - \Phi_i) u_{SV}^0 \\ \sum_{i=1}^{i=n} \frac{i\omega\delta t_i}{2} \sin 2\Phi_i u_{SV}^0 - \omega^2 \sum_{i=1}^{i=n} \sum_{j=i+1}^{j=n} \frac{\delta t_i}{2} \frac{\delta t_j}{2} \sin 2(\Phi_j - \Phi_i) u_{SV}^0 \end{pmatrix} \quad (21)$$

Retaining terms only to first order in $\omega\delta t$ and using $i\omega u_{SV}^0 = \dot{u}_{SV}^0$, equation (21) reduces to

$$\mathbf{u}_{VH} \approx \begin{pmatrix} u_{SV}^0 + \sum_{i=1}^{i=n} \frac{1}{2} \delta t_i \cos 2\Phi_i \dot{u}_{SV}^0 \\ \sum_{i=1}^{i=n} \frac{1}{2} \delta t_i \sin 2\Phi_i \dot{u}_{SV}^0 \end{pmatrix} \quad (22)$$

where $\delta t_i \cos 2\Phi_i$ and $\delta t_i \sin 2\Phi_i$ are defined by equations (12) and (13), respectively.

Using the same definition of splitting intensity introduced in the single-layer case (equation (18) combined with (19)), equation (22) gives, for a discrete stack of layers, $SI \approx -\sum_i \delta t_i \sin 2\Phi_i$. In the continuum limit, we define the differential splitting-delay increment

$$d\tau(z) \equiv \frac{\delta V(z)}{V_0^2(z)} dz,$$

which is the infinitesimal counterpart of the finite-layer splitting time δt_i . Replacing the sum over layers by an integral then gives the splitting intensity for a multilayer model:

$$\begin{aligned} SI &\approx - \int \sin 2\Phi(z) d\tau(z) \\ &= - \int \frac{dz}{\rho V_{iso}^3} [2I(K_c \sin \psi - K_s \cos \psi) - I(M_c \sin \psi + M_s \cos \psi) + \\ &\quad + (G_c \sin 2\psi + G_s \cos 2\psi) + 2I(D_c \sin 3\psi - D_s \cos 3\psi)] . \end{aligned} \quad (23)$$

Here, I is generally a function of depth, $I(z)$. To apply equation (23) most accurately, bending of the ray should be tracked at each interface. Equation (23) shows that, to first order in the amplitude of anisotropy and accumulated splitting, the contributions of individual layers to splitting intensity add linearly along the ray path. This linear superposition is the basis for the comparison between multilayer SKS splitting and surface-wave anisotropy models presented below.

The linear superposition of layer contributions in equation (23) is valid only because the second-order terms in equation (21) have been discarded. Thus, equation (23) is not only first-order in anisotropy, but also first-order in accumulated splitting. This linear superposition provides the framework for predicting SKS splitting intensity directly from depth-dependent anisotropy models derived from surface-wave observations. Romanowicz & Yuan (2012) showed explicitly that layer-order dependence enters through higher-order cross terms, whereas the first-order contribution to splitting intensity is a linear sum of the individual layer contributions irrespective of their order in the summation.

The theory developed above relies on five principal assumptions. (1) Anisotropy is weak, such that

$\delta V/V_0 \ll 1$. This is an assumption of the asymptotic theory; its accuracy for models relevant to the present study is evaluated with the numerical benchmarks in Section 2.3. (2) Accumulated splitting is sufficiently small that $\omega\delta t/2 \ll 1$. For a representative splitting time $\delta t \approx 1$ s and period of 10 s, $\omega\delta t/2 \approx 0.3$. Although not negligibly small, this value is sufficiently below unity for the first-order approximation to capture the leading dependence of splitting intensity on anisotropy, as demonstrated by the numerical benchmarks below. (3) The analysis is based on ray theory and therefore neglects finite-frequency effects and scattering caused by structural heterogeneity. Numerical experiments by Chevrot et al. (2004) demonstrate the importance of finite-frequency effects when anisotropic structure varies on scales comparable to or smaller than the SKS Fresnel zone. (4) Interfaces between layers are horizontal. (5) Coupling between S and P waves and P–SV conversion across interfaces are neglected. Finite-frequency effects, scattering, and coupling to P waves can be incorporated using first-Born approaches (e.g., Chevrot 2006; Sieminski et al., 2008).

Montagner et al. (2000) and Romanowicz & Yuan (2012) also presented results for SKS splitting intensity in anisotropic media, but under more restrictive assumptions than we consider here. In Appendix D, we show that their results can be derived from ours when we apply their assumptions.

2.3 Numerical benchmarks

To assess the accuracy of the asymptotic theory, we compare predictions from equations (18) and (19), and from equation (23), with numerical simulations for both single-layer and multilayer anisotropic models. The single-layer benchmark isolates the contributions of the individual anisotropy parameters, whereas the multilayer benchmark evaluates the theory in a model more representative of anisotropy structures inferred from surface-wave observations.

To verify the dependence of equations (18) and (19) on the individual anisotropy parameters, we consider four homogeneous single-layer anisotropy models in which only one of the parameters K_c , M_c , G_c , or D_c is non-zero. These four models isolate the anisotropy parameters that play central roles in the SKS and surface-wave formalisms. Numerical splitting intensities are measured on waveforms computed using the SEM-FK hybrid method of Tong et al. (2014) and are then compared with predictions from equations (18) and (19).

The isotropic reference model consists of a single 200 km thick layer with density $\rho = 2720$ kg m⁻³, $V_p = 5.80$ km/s, and $V_s = 3.46$ km/s. Four anisotropic perturbations to this reference model are considered, each independent of depth: Model 1, $K_c/A = 3\%$; Model 2, $M_c/A = 3\%$; Model 3, $G_c/L = 3\%$; and Model 4, $D_c/A = 3\%$, with all remaining anisotropy parameters set to zero in each case. Here, $A = \rho V_p^2$ and $L = \rho V_s^2$. For all simulations the incidence angle is fixed at $I = 10^\circ$. This deliberately idealized configuration, including the 200-km homogeneous layer extending to the

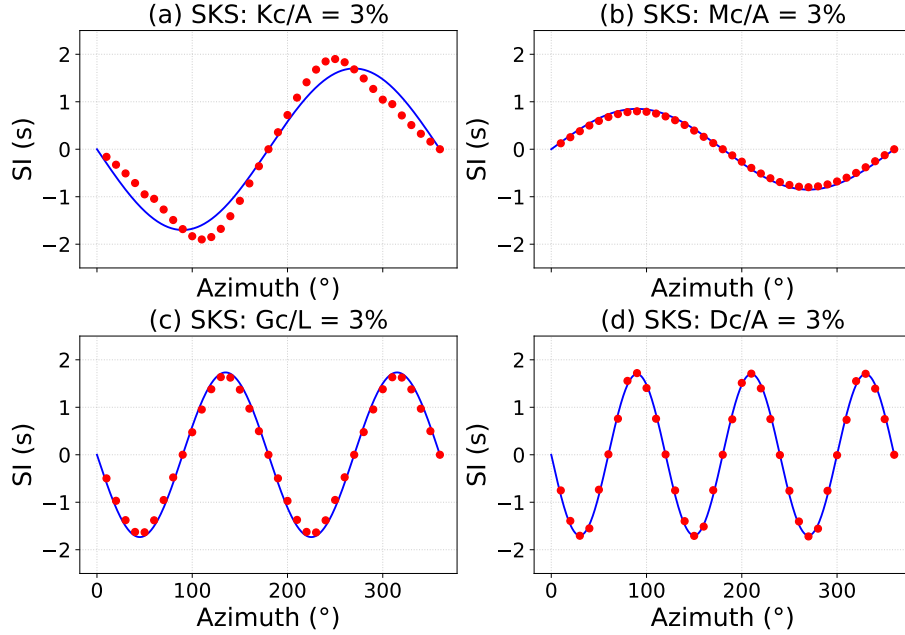


Figure 2. Single-layer numerical benchmark. The blue solid line is the theoretical prediction of splitting intensity using equations (18) and (19), and the red dots are numerical results from the SEM-FK hybrid method (Tong et al., 2014). (a) Model 1, $K_c/A = 3\%$; (b) Model 2, $M_c/A = 3\%$; (c) Model 3, $G_c/L = 3\%$; and (d) Model 4, $D_c/A = 3\%$. The incidence angle is $I = 10^\circ$.

surface, is not intended to represent realistic crust and mantle structure; its purpose is to isolate the dependence of splitting intensity on the individual K_c , M_c , G_c , and D_c parameters.

Comparisons between numerical and theoretical splitting intensities are shown in Figure 2, which demonstrates that equations (18) and (19) reproduce the numerical splitting intensities with high accuracy for all four single-layer benchmarks. Agreement is particularly good for the M_c , G_c , and D_c models. A small discrepancy is observed only for Model 1, the K_c simulation. This small discrepancy may result from S–P coupling in the numerical simulations, which is neglected in the asymptotic theory presented here. Although each perturbation is specified as 3% of its reference modulus, G_c has the smallest absolute magnitude because it is normalized by $L = \rho V_s^2$, whereas K_c , M_c , and D_c are normalized by $A = \rho V_p^2 > L$. For the reference model considered here, A is approximately 2.8 times larger than L .

For the multilayer benchmark, we test the asymptotic theory using a simplified but representative model of anisotropy based on models inferred from Rayleigh- and Love-wave azimuthal anisotropy across western Alaska (X. Liu & Ritzwoller, 2025, their Fig. 13; X. Liu & Ritzwoller, 2026b). Those models incorporate observations of both Rayleigh- and Love-wave 2ψ and 4ψ azimuthal anisotropy and explicitly account for Rayleigh–Love coupling. The benchmark therefore provides a test of the theory closer to actual applications than the single-layer examples.

The multilayer benchmark model is TTI, with properties summarized in Figure 3. S-wave anisotropy, γ , is set to 6% in the crust and 12% in the mantle, while the dip angle is 30° in the crust and 40° in the mantle. We further assume elliptical anisotropy ($\eta_X = 4L/(A + C - 2F) = 1$, or equivalently $E = A + C - 2F - 4L = 0$), which implies $K_{c,s} = D_{c,s} = 0$ (Appendix C). Because P-wave anisotropy is not independently constrained, we assume $(A - C)/(2C) = (N - L)/(2L)$ and set the strike angle to 90° . The elliptical assumption is introduced only to simplify the numerical benchmark; it is not imposed on the anisotropy models inferred from the Alaska surface-wave observations discussed in Section 4. The full 21-component elastic tensor is then constructed from the TTI parameters using the coordinate transformation described in Appendix C and supplied to SPECFEM3D. The computational domain extends from -4120 to 4120 km in both horizontal directions and from the surface to 200 km depth, with approximate grid spacing $20 \times 20 \times 5$ km. From the simulated wavefields, we measure the phase speeds of the quasi-Rayleigh and quasi-Love waves and the SKS splitting intensity for comparison with the theoretical predictions. Consequently, among the anisotropy-parameter pairs retained in equation (18), only $G_{c,s}$ and $M_{c,s}$ are non-zero, and their depth dependence is shown in Figure 3d. The model extends to 200 km depth and is represented by 100 layers that are thinner in the crust than in the mantle; Figure 3 displays only the upper 100 km.

Equation (23) reproduces the principal azimuthal dependence and amplitude of splitting intensity in the multilayer model (Figure 4a,b). Agreement between the theoretical and numerical results is good over most azimuths, although the first-order theory underpredicts the magnitude near the extrema. This underprediction is consistent with the neglect of higher-order terms in both anisotropy and accumulated splitting.

For this benchmark model, vertical incidence produces a purely 2ψ variation arising from $G_{c,s}$. At $I = 10^\circ$, the $M_{c,s}$ contribution introduces an additional 1ψ variation. The benchmark therefore demonstrates that the asymptotic theory captures the principal amplitude and azimuthal dependence of SKS splitting while also illustrating a finite-incidence contribution that vanishes for vertical propagation.

Panels (c) and (d) of Figure 4 provide a corresponding numerical benchmark for the quasi-degenerate surface-wave theory at a period of 50 s. The theoretical predictions reproduce the principal azimuthal variations of both quasi-Rayleigh and quasi-Love phase speeds obtained with SPECFEM. Thus, Figure 4 provides numerical benchmarks for both the SKS and surface-wave theories in the same anisotropic model.

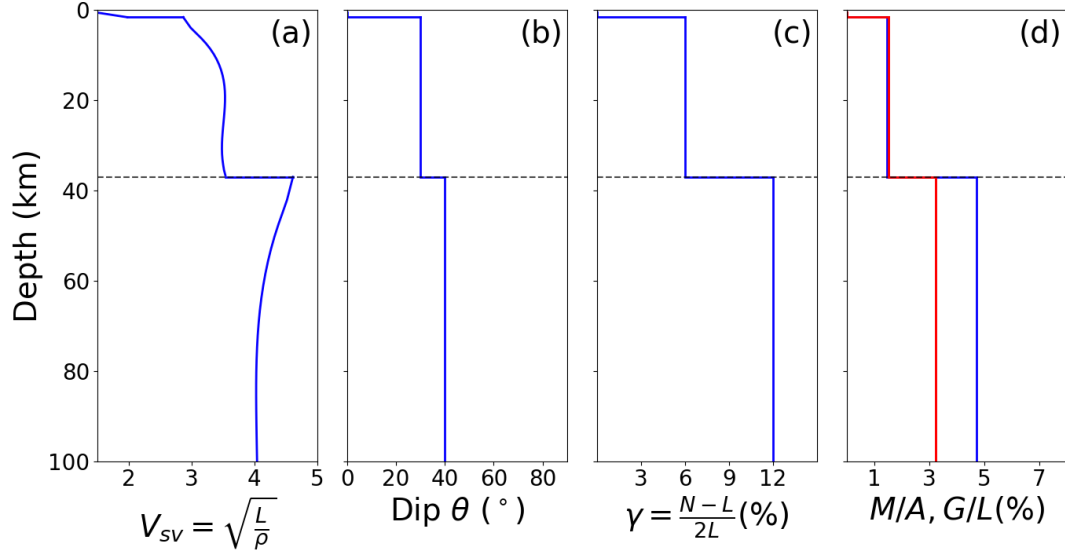


Figure 3. Aspects of the multilayer TTI model for the numerical benchmark. Shown are (a) $\sqrt{L/\rho}$, (b) dip angle θ , (c) S-wave anisotropy $\gamma = (N - L)/(2L)$, and (d) $G = \sqrt{G_c^2 + G_s^2}$ (blue) and $M = \sqrt{M_c^2 + M_s^2}$ (red), normalized by the Love moduli L and A , respectively. The model is similar to models presented by X. Liu & Ritzwoller (2026b) for western Alaska, but is simplified; in particular, elliptical anisotropy is assumed, implying $K_{c,s} = D_{c,s} = 0$. The full benchmark model extends to 200 km depth; only the upper 100 km are shown.

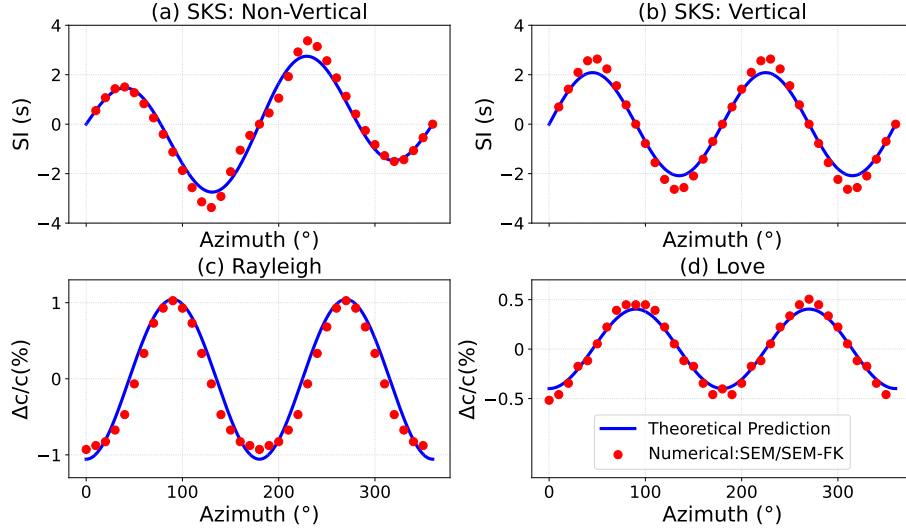


Figure 4. Multilayer numerical benchmark for the model described in the text and shown in Figure 3. (a)–(b) SKS splitting intensity. The blue solid line is the theoretical prediction from equation (23), and the red dots are numerical results from the SEM-FK hybrid method (Tong et al., 2014). Panel (a) is for non-vertical incidence, $I = 10^\circ$, and panel (b) is for vertical incidence. (c)–(d) Theoretical and numerical azimuthal variations of quasi-Rayleigh and quasi-Love phase speeds at a period of 50 s. The blue line is the theoretical prediction from X. Liu & Ritzwoller (2025; Appendix A), and the red dots are SPECFEM results (Komatitsch & Tromp, 1999).

3 DISCUSSION OF SURFACE-WAVE THEORY AND THE SKS FORMALISM

The connection between Rayleigh–Love coupling in surface-wave theory and SKS splitting at the small non-zero incidence angles characteristic of SKS propagation has two components. First, accounting for Rayleigh–Love coupling in the surface-wave inversion changes the inferred elastic tensor, including $G_{c,s}$, and therefore changes the predicted SKS splitting. Second, the parameters $K_{c,s}$, $M_{c,s}$, and $D_{c,s}$, which are important for Rayleigh–Love coupling, also contribute directly to SKS splitting at finite incidence. The vertical-incidence limit provides a useful diagnostic because the latter contributions vanish, isolating the effect of changes in the inferred $G_{c,s}$. This connection provides the basis for the unified treatment of SKS splitting and surface-wave anisotropy presented here and implies that the two observables should be interpreted jointly when constraining the elastic tensor. To clarify this connection, we briefly review the quasi-degenerate surface-wave theory of X. Liu & Ritzwoller (2025) and compare it with the SKS formalism developed here.

A quasi-degenerate theory for the effect of anisotropy with arbitrary symmetry on surface-wave phase speeds and polarizations was presented by X. Liu & Ritzwoller (2025) and is summarized briefly in Appendix A. In a laterally homogeneous isotropic reference model, Rayleigh and Love waves propagate independently and are polarized in the vertical plane and transverse directions, respectively. Introducing anisotropy with sufficient strength and low symmetry couples Rayleigh and Love waves, producing quasi-Rayleigh and quasi-Love waves whose polarizations are rotated relative to those in the reference medium. Transversely isotropic media with either a vertical (VTI) or horizontal (HTI) symmetry axis do not couple Rayleigh and Love waves or alter their polarizations strongly. HTI media do produce azimuthal anisotropy, however, with Rayleigh-wave phase speeds displaying predominantly a 2ψ pattern and Love waves a 4ψ pattern (Montagner and Nataf, 1986). For tilted transverse isotropy (TTI) or lower-symmetry anisotropy, Rayleigh–Love coupling modifies these patterns, introducing a 2ψ component in Love waves and a 4ψ component in Rayleigh waves.

The observational history of Love-wave 2ψ anisotropy extends back several decades. Tanimoto & Anderson (1985) presented global maps of Rayleigh- and Love-wave azimuthal anisotropy at periods of 100–250 s, including Love-wave 2ψ . Montagner & Anderson (1989) subsequently argued, based on the mineralogical models available at the time, that a substantial Love-wave 2ψ component was not generally expected. Using a substantially larger surface-wave data set, Montagner & Tanimoto (1990) again found evidence for Love-wave 2ψ anisotropy, particularly at periods longer than about 100 s. Subsequent analyses using still larger data sets did not find Love-wave 2ψ to be unequivocally resolved at global scales (Trampert & Woodhouse, 2003; Ekström, 2011). Trampert & Woodhouse (2003) noted, however, that if a significant Love-wave 2ψ signal were present, strong Rayleigh–Love coupling would provide a natural explanation. At much shorter periods, Russell et al. (2019) pro-

vided clear regional evidence for Love-wave 2ψ anisotropy below about 10 s. The strong and spatially coherent Love-wave 2ψ observed across Alaska (X. Liu et al., 2025) at mantle-sensitive periods therefore provides particularly direct observational evidence for significant Rayleigh–Love coupling. Although the Rayleigh-wave 4ψ component observed across Alaska is relatively weak, Rayleigh-wave 4ψ anisotropy has been reported in global studies dating back to Montagner & Tanimoto (1990), and subsequently by Trampert & Woodhouse (2003) and Beucier & Montagner (2006).

We use the same multilayer anisotropic model as in the numerical benchmark, aspects of which are shown in Figure 3, to compute the azimuthal variation of quasi-Rayleigh and quasi-Love waves, shown in Figure 4c,d. Predictions from the theory of X. Liu & Ritzwoller (2025) are compared with numerical calculations from SPECFEM3D at a period of 50 s. As expected, the quasi-Rayleigh wave exhibits a dominant 2ψ azimuthal pattern. The quasi-Love wave, however, also displays a strong 2ψ variation, which is not expected in the absence of significant Rayleigh–Love coupling. Observations of a strong 2ψ component in Love-wave azimuthal anisotropy across Alaska therefore provide evidence for strong Rayleigh–Love coupling (X. Liu et al., 2025).

The connection between surface-wave anisotropy and SKS splitting arises in part because the Rayleigh–Love coupling integral X depends primarily on the anisotropy parameters $D_{c,s}$, $M_{c,s}$, and $K_{c,s}$, three of the four parameter pairs that contribute to splitting intensity for SKS waves at small non-zero incidence (eq. (18)); the fourth is $G_{c,s}$. The coupling integral X also depends on $J_{c,s}$, which does not appear in the SKS formalism because coupling between S and P waves is neglected. The parameters $D_{c,s}$ and $M_{c,s}$ are particularly important for Rayleigh–Love coupling, whereas $K_{c,s}$ is more significant for coupling to higher modes. The parameter pair $G_{c,s}$ does not enter the X -integral, but plays an important role in surface-wave anisotropy through the A -, B -, and E -integrals, particularly through the B -integral defined in Appendix A. In the absence of Rayleigh–Love coupling (integrals $E = X = 0$), the B -integral governs the azimuthal dependence of Rayleigh-wave phase speeds.

Although the same anisotropy parameters enter both formalisms, their effect on the azimuthal variation of the two observables differs. At small non-zero incidence, some parameters generate 1ψ and 3ψ variations in SKS splitting intensity while contributing primarily to 2ψ and 4ψ azimuthal variations in surface-wave phase speeds (X. Liu & Ritzwoller, 2025). The same parameters can also generate 1ψ and 3ψ azimuthal variations in surface-wave polarization (X. Liu & Ritzwoller, 2026a), although polarization is not considered here. In media with strong Rayleigh–Love coupling, anisotropy parameters beyond $G_{c,s}$ may therefore be important for SKS splitting, particularly in the presence of azimuthal variations in SKS splitting beyond the familiar 2ψ component.

The fast directions inferred from SKS splitting and quasi-Rayleigh waves are expected to align closely when Rayleigh–Love coupling is weak; that is, when the $D_{c,s}$ and $M_{c,s}$ parameters are small.

In this limit, both observables are controlled primarily by the $G_{c,s}$ parameters. Equation (18) shows that the SKS fast direction depends principally on $G_{c,s}$, while equation (A.3) indicates that the B -integral, which governs Rayleigh-wave anisotropy in the absence of coupling, is also controlled by $G_{c,s}$. As the magnitudes of $D_{c,s}$ and $M_{c,s}$ increase, however, SKS splitting develops significant odd-harmonic (1ψ , 3ψ) contributions and quasi-Rayleigh-wave anisotropy may rotate away from the direction determined solely by G_c and G_s .

At the small non-zero incidence angles characteristic of observed SKS waves, splitting acquires additional sensitivity to the $D_{c,s}$, $M_{c,s}$, and $K_{c,s}$ parameters that also play important roles in Rayleigh–Love coupling. SKS splitting and surface-wave observations therefore provide complementary constraints on overlapping components of the elastic tensor. The two observables are consequently strongly linked and, when interpreted jointly, provide stronger constraints on the elastic tensor than either data set alone.

Another consequence of this dependence on multiple anisotropy parameters is that SKS splitting and surface-wave azimuthal anisotropy do not necessarily need to coexist. For example, when $G_{c,s} = 0$ but $M_{c,s} \neq 0$, strong surface-wave azimuthal anisotropy may be present even when SKS splitting is weak. Conversely, when both $G_{c,s}$ and $M_{c,s}$ are nonzero, Rayleigh-wave 2ψ azimuthal anisotropy may be effectively canceled by Rayleigh–Love coupling. In this case, SKS splitting may be present even when Rayleigh-wave azimuthal anisotropy is weak or absent. Both scenarios extend previous theoretical analyses in which SKS splitting and surface-wave azimuthal anisotropy were expected to coexist because both were controlled by the $G_{c,s}$ parameters, and therefore provide a possible explanation for their non-coexistence in real seismic observations (e.g. Zhu et al. 2020; Tao and Li 2025). Neglecting Rayleigh–Love coupling in the surface-wave inversion usually reduces the predicted SKS splitting intensity, but in some cases it can instead increase it. Such behavior, which may also be accompanied by changes in the predicted fast-axis orientation, is illustrated by the example in Figure 8 of X. Liu & Ritzwoller (2025).

In summary, Rayleigh–Love coupling affects the interpretation of SKS splitting in two related ways. Accounting for Rayleigh–Love coupling in the surface-wave inversion changes the inferred elastic tensor, including $G_{c,s}$, while at the small non-zero incidence angles characteristic of observed SKS waves, the parameters $D_{c,s}$, $M_{c,s}$, and $K_{c,s}$ also contribute directly to splitting intensity. The vertical-incidence limit provides a useful diagnostic that isolates the first of these effects. The widespread occurrence of strong 2ψ Love-wave anisotropy across Alaska, characteristic of Rayleigh–Love coupling, indicates that the parameters $D_{c,s}$, $M_{c,s}$, and to a lesser extent $K_{c,s}$ should not be neglected when interpreting SKS observations. These parameters, together with $G_{c,s}$, should therefore be in-

cluded in calculations of SKS splitting intensity whenever evidence for significant Rayleigh–Love coupling exists. The theory presented here provides the framework for such joint interpretation.

4 OBSERVATIONAL EXAMPLES

Here we examine how including Rayleigh–Love coupling in surface-wave inversions alters the inferred crustal and mantle anisotropy and the resulting predictions for SKS splitting. We begin with Point A identified in Figure 8a as a representative example and then extend the analysis to five locations distributed across Alaska.

X. Liu et al. (2025) present observations of the azimuthal anisotropy of Rayleigh and Love waves across Alaska, including Rayleigh- and Love-wave 2ψ and 4ψ observations. A strong Love-wave 2ψ component to anisotropy indicates significant Rayleigh–Love coupling. An example of observations of surface-wave azimuthal anisotropy is shown in Figure 5 for point A, which is identified in Figure 8a. Based on these data, X. Liu and Ritzwoller (2026b) present models of anisotropy in the crust and uppermost mantle to a depth of 200 km across Alaska. The models are TTI in the crust, and both TTI and tilted orthorhombic (TO) inversion results are reported in the mantle. Both inversions include Rayleigh–Love coupling. An orthorhombic mantle is considered because it fits the data better, particularly Rayleigh-wave 4ψ , and requires a smaller amplitude of anisotropy than a TTI model. In these inversions the ellipticity parameter is a free variable, so the inferred models are not constrained to elliptical anisotropy and parameters such as $K_{c,s}$ and $D_{c,s}$ may be non-zero. For the comparisons presented below in Figures 6 and 7 and Table 2, we use the TTI mantle inversion of X. Liu & Ritzwoller (2026b), not the tilted-orthorhombic model. The details and comparisons between the TTI and TO models are discussed in that paper.

The role of dip in this comparison is subtle. Equations (C.7) and (C.8) show that, because the ellipticity parameter E is typically small in earth materials, $G_{c,s}$ for a TTI medium is approximately proportional to the product $(L - N) \sin^2 \theta$, where $L - N$ measures the strength of S-wave anisotropy and θ is the dip of the symmetry axis. Thus, when Rayleigh–Love coupling is neglected, an error in the assumed dip can be compensated by a corresponding change in inferred anisotropy strength while leaving $G_{c,s}$, and hence the predicted vertical-incidence SKS splitting intensity, approximately unchanged. The principal importance of dip here is instead that it generates Rayleigh–Love coupling. Once this coupling is included in the surface-wave forward theory, the observations map into a different elastic tensor, including larger inferred $G_{c,s}$ and substantial coupling-related parameters such as $M_{c,s}$.

Examples of four of the anisotropy parameters from the model of X. Liu and Ritzwoller (2026b) at point A are shown in Figure 6 with blue lines. At this point, the strike angle ϕ of anisotropy is

Table 1. Amplitude of S-wave anisotropy γ at 100 km depth at selected points.

Point	Coordinates (X, Y)	X. Liu & Ritzwoller (2026b) (%)	C. Liu & Ritzwoller (2024) (%)
A	(147°, 64.0°)	8.8	5.3
B	(155°, 62.0°)	8.6	3.8
C	(142°, 65.5°)	11.5	3.2
D	(141°, 62.0°)	11.7	2.6
E	(155°, 66.5°)	10.5	4.6
Average	–	10.2	3.9

non-zero, so both parameters of each pair may be non-zero; e.g., both G_c and G_s . Figure 6 presents the square root of the sum of the squares of each pair; e.g., $G = \sqrt{G_c^2 + G_s^2}$. The largest parameters in this model in the mantle are G and M . For TTI media, Appendix C shows that K and D diminish as the ellipticity parameter E approaches zero, whereas M/L and G/L remain proportional to the strength of S-wave anisotropy, γ , for a given tilt. Formally, we define γ as

$$\gamma = \frac{\max C_{ij} - \min C_{ij}}{\max C_{ij} + \min C_{ij}} \quad (24)$$

where $\max C_{ij}$ and $\min C_{ij}$ denote the maximum and minimum, respectively, of the shear moduli C_{44} , C_{55} , and C_{66} of the unrotated elastic tensor. For a TTI medium, these quantities are evaluated in the underlying VTI model prior to tilting; the maximum and minimum typically are N and L , respectively. Thus, $\gamma = (N - L)/(N + L) \approx (N - L)/2L$, the usual Thomsen parameter γ in the weak-anisotropy limit (Thomsen, 1986).

For comparison, Figure 6 also presents the four anisotropy parameters at point A for the reference anisotropy model of C. Liu and Ritzwoller (2024). This model is based on observations of Rayleigh-wave 2ψ anisotropy and isotropic Love-wave phase speed, with both the crust and mantle modeled as TTI. Observations of Love-wave 2ψ and 4ψ anisotropy and Rayleigh-wave 4ψ anisotropy were not included in their inversion, and Rayleigh–Love coupling was not modeled. The amplitudes of all four anisotropy parameters are smaller in the mantle than those in the model of X. Liu and Ritzwoller (2026b), particularly G and M . As seen in Figure 5, the Rayleigh-wave 2ψ observations can be fit by the model of C. Liu and Ritzwoller (2024), but the remaining observations of anisotropy cannot. Those observations are indicative of significant Rayleigh–Love coupling. In contrast, both the Rayleigh-wave and Love-wave 2ψ and 4ψ components of azimuthal anisotropy are fit by the model of X. Liu and Ritzwoller (2026b), which includes Rayleigh–Love coupling. Table 1 illustrates that including Rayleigh–Love coupling increases inferred mantle anisotropy substantially, from an average of 3.9% to 10.2% at the five locations considered.

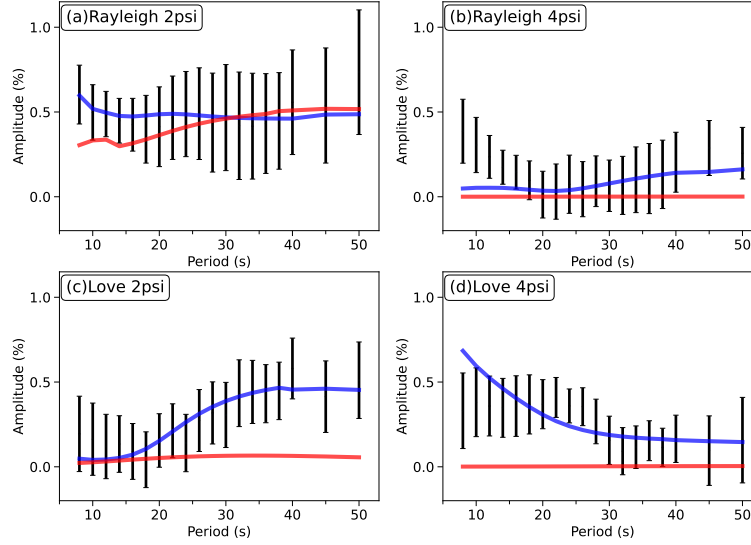


Figure 5. Surface-wave observations (error bars) of Rayleigh- and Love-wave 2ψ and 4ψ azimuthal anisotropy at point A identified in Figure 8a. Blue line: prediction from the anisotropy model of X. Liu and Ritzwoller (2026b), in which Rayleigh–Love coupling is considered in its construction. Red line: prediction from the anisotropy model of C. Liu and Ritzwoller (2024), which is constructed without considering Rayleigh–Love coupling.

Including Rayleigh–Love coupling, therefore, allows the azimuthal variations of both Rayleigh and Love waves to be interpreted accurately. We now examine how the resulting differences in inferred anisotropy affect predictions of SKS splitting intensity. The azimuthal variation of SKS splitting intensity computed at point A using the model of X. Liu and Ritzwoller (2026b) is shown by the blue curves in Figure 7, while predictions from the reference model of C. Liu and Ritzwoller (2024) are shown by the red curves. The contributions of the individual anisotropy-parameter pairs are summarized in Table 2. For vertical incidence ($I = 0^\circ$), only $G_{c,s}$ contributes to splitting intensity, and the maximum predicted SI increases from 0.62 s in the reference model to 1.10 s in the model inferred using the Rayleigh–Love-coupled surface-wave theory. Thus, even in the vertical-incidence limit, where $K_{c,s}$, $M_{c,s}$, and $D_{c,s}$ do not contribute directly, accounting for Rayleigh–Love coupling in the surface-wave interpretation leads to substantially larger predicted SKS splitting because the inferred $G_{c,s}$ is larger. An inversion based only on the observed Rayleigh-wave 2ψ anisotropy and an uncoupled theory therefore underestimates $G_{c,s}$ at point A, as illustrated in Figure 6.

At the representative incidence angle $I = 10^\circ$, the difference between the two models is larger because $K_{c,s}$, $M_{c,s}$, and $D_{c,s}$ contribute in addition to $G_{c,s}$. The maximum total splitting intensity

increases from 0.91 s for the reference model to 1.79 s for the model inferred using the Rayleigh–Love-coupled theory (Table 2). For the latter model, $G_{c,s}$ and $M_{c,s}$ provide the largest individual contributions, with $M_{c,s}$ providing a particularly large additional contribution at $I = 10^\circ$. Thus, for observed SKS waves, accounting for Rayleigh–Love coupling in the surface-wave inversion affects the predicted splitting in two ways: it changes the elastic tensor inferred from the surface-wave observations, including $G_{c,s}$, and the additional $K_{c,s}$, $M_{c,s}$, and $D_{c,s}$ parameters also contribute directly to splitting intensity. The general expression linking SKS splitting intensity to the elastic tensor derived in this paper is therefore required for a consistent comparison of SKS splitting with surface-wave anisotropy. The individual G , K , M , and D entries in Table 2 are the maximum amplitudes of their isolated azimuthal contributions, whereas SI (Total) is the maximum of the summed splitting-intensity curve. Because these harmonics generally reach their maxima at different azimuths, the individual contributions do not sum to SI (Total).

To assess whether the conclusions drawn at point A are representative, we examine four additional locations distributed across Alaska where both SKS observations and strong Love-wave 2ψ anisotropy are observed. Strong Love-wave 2ψ anisotropy occurs widely across Alaska at mantle-sensitive periods, motivating this broader comparison. The surface-wave phase-speed maps and their azimuthal variations are constructed using eikonal tomography (Lin et al., 2009), rather than by solving a damped or regularized tomographic inverse problem of the type commonly used in traditional surface-wave tomography (e.g., Barmin et al., 2001). At each period, travel-time surfaces are obtained by spline interpolation of the measurements, and local phase speeds are determined from their spatial gradients. Measurements near each location are then averaged as a function of azimuth to estimate the isotropic and azimuthally varying phase speeds. The effective spatial resolution therefore principally reflects interpolation and spatial averaging. Because SKS and surface-wave observations have intrinsically different spatial sensitivities, we compare predicted SKS splitting from the surface-wave-derived models with spatial averages of the SKS measurements to give the two data types a comparable spatial resolution.

In particular, we use the SKS observations from Venereau et al. (2019) and form an average within a 200 km radius of each target location. We first select all SKS measurements within this radius. Because their spatial distribution is uneven, we divide the surrounding region into six equal azimuthal sectors, each spanning 60° . Within each occupied sector, we calculate the arithmetic mean of the splitting delay times, δt , and the axial circular mean of the fast-axis orientations, ϕ , accounting for the 180° symmetry of the fast axis. The occupied sectors are then given equal weight in forming the regional mean, thereby reducing the influence of sectors containing a disproportionately large number of measurements. The reported standard deviations are calculated from the individual SKS

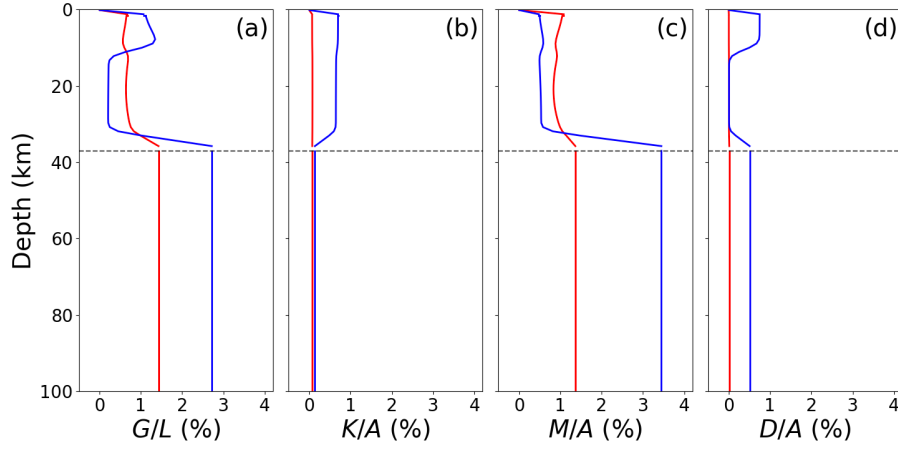


Figure 6. Examples of anisotropy parameters computed from the models of (blue lines) X. Liu and Ritzwoller (2026b) and (red lines) C. Liu and Ritzwoller (2024). Each parameter-pair ($G_{c,s}$, $K_{c,s}$, $M_{c,s}$, $D_{c,s}$) is combined and normalized by a Love parameter (L or A); e.g., $G = \sqrt{G_c^2 + G_s^2}$.

measurements within the 200 km radius: the ordinary standard deviation for δt and the axial circular standard deviation for ϕ .

As shown in Figure 8b, the models of C. Liu and Ritzwoller (2024) and X. Liu and Ritzwoller (2026b) both fairly well reproduce the observed SKS fast-axis orientations. Rayleigh–Love coupling often does not need to be included for surface-wave models to predict SKS fast-axis orientations accurately. However, Rayleigh–Love coupling substantially affects the inferred strength of mantle anisotropy and, consequently, the predicted splitting intensity, as shown in Figure 8c. These results are summarized in Table 3. The model of C. Liu and Ritzwoller (2024), which neglects Rayleigh–Love coupling, systematically underpredicts SKS splitting intensity, whereas the model of X. Liu and Ritzwoller (2026b), which includes Rayleigh–Love coupling, reproduces the observed splitting intensities. Both models possess anisotropy that extends to 200 km depth. The underprediction of splitting intensity by the model that neglects Rayleigh–Love coupling reflects its weaker inferred mantle anisotropy and not an incorrect fast-axis orientation.

These examples from Alaska demonstrate that incorporating Rayleigh–Love coupling into the interpretation of surface-wave anisotropy increases the inferred strength of upper-mantle anisotropy and yields SKS splitting predictions that agree with observations. They indicate that the apparent discrepancy between surface-wave and SKS estimates of anisotropy can be largely resolved without requiring anisotropy to extend substantially below 200 km depth, at least beneath Alaska. Other continents may be different, although numerous studies at both global and regional scales have reported SKS splitting amplitudes larger than those predicted from surface-wave anisotropy models (e.g., Wüstefeld et al., 2009; Kosarian et al., 2011; Becker et al., 2012; Feng et al., 2020; Zhu et al., 2020; Feng & Díaz, 2022).

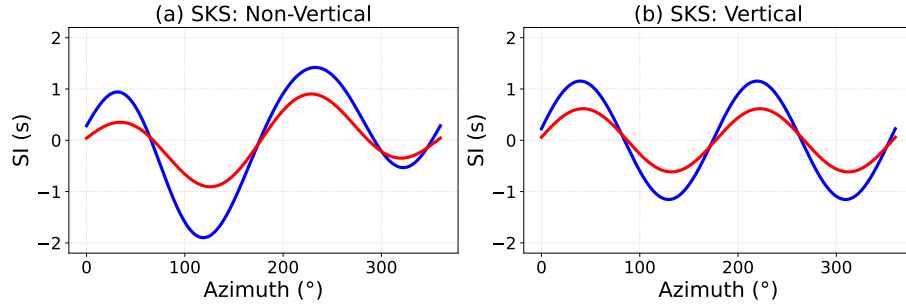


Figure 7. Examples of the azimuthal variation of the splitting intensity computed from the models of (blue lines) X. Liu and Ritzwoller (2026b) and (red lines) C. Liu and Ritzwoller (2024) using the theory presented here. (a) Incidence angle $I = 10^\circ$. (b) Incidence angle $I = 0^\circ$.

This discrepancy has often motivated interpretations involving anisotropy at greater depth (e.g., Li et al., 2003; Gaherty, 2004; Yuan & Romanowicz, 2010; Lin et al., 2011; Yuan & Beghein, 2013). The results presented here suggest that at least part of this discrepancy may instead arise from neglecting Rayleigh–Love coupling in surface-wave inversions. More generally, wherever a strong Love-wave 2ψ component of azimuthal anisotropy indicates significant Rayleigh–Love coupling, SKS splitting and surface-wave anisotropy should be interpreted jointly rather than used separately to constrain mantle structure. The results presented here provide a framework for such a unified interpretation.

5 CONCLUSION

SKS splitting and surface-wave anisotropy are two of the principal observational constraints on seismic anisotropy in the crust and upper mantle. For decades, however, comparisons between these observables have revealed an apparent inconsistency. Although fast-axis orientations inferred from SKS splitting and surface waves are often observed to be similar, anisotropy models derived from surface-

Table 2. Contributions to splitting intensity (SI) at point A (Figure 7) for Model A (C. Liu & Ritzwoller, 2024) and Model B (X. Liu & Ritzwoller, 2026b) for incidence angles $I = 0^\circ$ and $I = 10^\circ$.

	$I = 0^\circ$		$I = 10^\circ$	
	Model A	Model B	Model A	Model B
SI (G)	0.62	1.10	0.62	1.10
SI (K)	0	0	0.04	0.05
SI (M)	0	0	0.36	0.75
SI (D)	0	0	0.01	0.20
SI (Total)	0.62	1.10	0.91	1.79

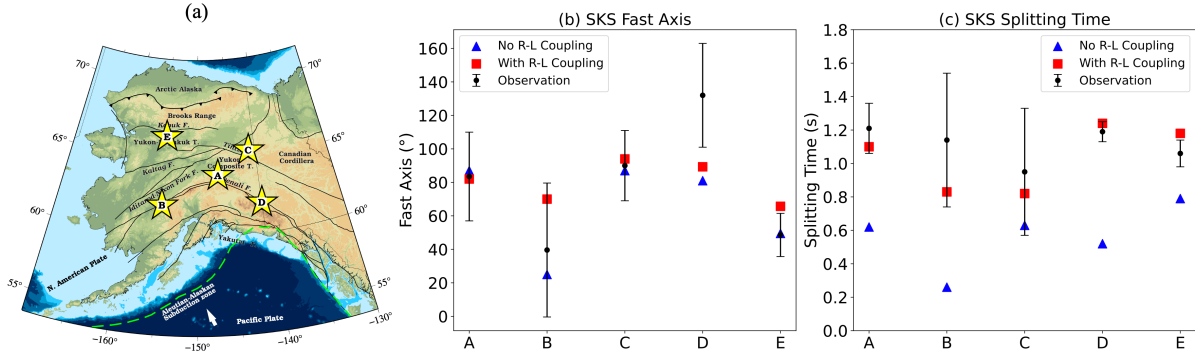


Figure 8. (a) The location of the five example locations A-E. (b) Comparison of observed SKS fast-axis orientation and (c) splitting time at the five locations with predictions from the crustal and uppermost mantle anisotropy models of X. Liu & Ritzwoller (2026b) and C. Liu & Ritzwoller (2024), constructed with and without considering Rayleigh–Love coupling, respectively. Both models fit SKS fast-axis orientation. Including Rayleigh–Love coupling in the inversion allows the SKS splitting time to be fit. (Data are summarized in Table 3.)

wave observations have typically predicted SKS splitting intensities that are substantially smaller than those observed. This discrepancy has commonly been interpreted as evidence that a significant fraction of SKS splitting originates from anisotropy at depths greater than those sampled by surface waves.

In this study, we present a unified treatment of SKS splitting and surface-wave anisotropy for media with arbitrary elastic symmetry. We derive a new asymptotic theory for SKS splitting in weakly anisotropic media and compare it directly with the quasi-degenerate theory of surface-wave anisotropy that includes Rayleigh–Love coupling. This comparison reveals that the two observables are linked through a common set of anisotropy parameters. Our results lead to two principal conclusions.

First, SKS splitting and surface-wave anisotropy depend on a common set of anisotropy parameters. In particular, the parameters $M_{c,s}$ and $D_{c,s}$, which play important roles in Rayleigh–Love cou-

Table 3. SKS splitting observations and predictions from models derived from surface-wave observations.

	Pt A (9)	Pt B (10)	Pt C (11)	Pt D (21)	Pt E (8)
δt_{obs} (s)	1.21 ± 0.15	1.14 ± 0.40	0.95 ± 0.38	1.19 ± 0.06	1.06 ± 0.08
$\delta t^{\dagger}$ (s)	0.62	0.26	0.63	0.52	0.79
$\delta t^{\dagger\dagger}$ (s)	1.10	0.83	0.82	1.24	1.18
ψ_{obs} (°)	83.5 ± 26.5	39.6 ± 40	90 ± 21	132 ± 31	48.6 ± 12.9
$\psi^{\dagger}$ (°)	87	30	87	81	49.5
$\psi^{\dagger\dagger}$ (°)	82	70	94	89.3	65.7

$\dagger$ – SI and fast-azimuths from model of C. Liu and Ritzwoller (2024), without Rayleigh–Love coupling.

$\dagger\dagger$ – SI and fast-azimuths from model of X. Liu and Ritzwoller (2026b), with Rayleigh–Love coupling.

pling in surface-wave theory, also contribute significantly to SKS splitting intensity for non-vertical incidence. Because these parameters, particularly $M_{c,s}$, contribute to Rayleigh–Love coupling, observations of Love-wave 2ψ and Rayleigh-wave 4ψ azimuthal anisotropy provide evidence that they may also contribute significantly to SKS splitting. Because both data sets constrain overlapping but distinct combinations of anisotropy parameters, their joint interpretation provides stronger constraints on the strength, geometry, and depth distribution of anisotropy than either observable alone.

Second, accounting for Rayleigh–Love coupling in the interpretation of surface-wave anisotropy plays a central role in the comparison between SKS splitting and surface-wave anisotropy. In the Alaska examples presented here, the model inferred using surface-wave theory that includes Rayleigh–Love coupling contains substantially stronger anisotropy in the uppermost mantle than the reference model derived without accounting for coupling. The resulting increase in inferred anisotropy strength brings predicted SKS splitting intensities into agreement with observations without requiring anisotropy to extend below the uppermost mantle (200 km depth). These results indicate that neglecting Rayleigh–Love coupling in surface-wave inversions may contribute significantly to the long-standing discrepancy between SKS splitting and surface-wave anisotropy.

The observational results we present are all from Alaska, and it remains uncertain how well anisotropy in Alaska represents continental regions more generally. However, observations of Love-wave 2ψ or Rayleigh-wave 4ψ azimuthal anisotropy have been reported in other regions, both at global scales (e.g., Tanimoto & Anderson, 1985; Montagner & Tanimoto, 1990; Trampert & Woodhouse, 2003; Ekström, 2011) and regional scales (e.g., Weeraratne et al., 2007; Polat et al., 2012; Zigone et al., 2015; Russell et al., 2019). Thus, where strong Love-wave 2ψ anisotropy provides direct evidence for significant Rayleigh–Love coupling, previous inferences concerning the strength and depth extent of mantle anisotropy based on discrepancies between surface-wave and SKS observations may warrant reexamination. In addition to observations of Love-wave 2ψ anisotropy, the extent of Rayleigh–Love coupling can also be assessed from surface-wave polarization measurements (X. Liu & Ritzwoller, 2026a).

More broadly, the framework developed here provides a basis for the joint interpretation and eventual joint inversion of SKS splitting together with both Rayleigh- and Love-wave phase-speed azimuthal anisotropy (e.g., X. Liu et al., 2025) and surface-wave polarization measurements (X. Liu & Ritzwoller, 2026a). Because these observables depend on overlapping but distinct combinations of anisotropy parameters, their combined analysis offers the prospect of substantially improved constraints on the strength, geometry, and depth distribution of seismic anisotropy in the crust and upper mantle.

Several observational extensions of this work are important. First, the comparison between SKS

splitting predicted from surface-wave-derived anisotropy models and observed SKS splitting should be extended systematically across Alaska, beyond the five representative locations considered here. Second, Rayleigh- and Love-wave 2ψ and 4ψ azimuthal anisotropy should be measured in other regions, with particular attention to whether a significant Love-wave 2ψ component is present. Where such observations indicate strong Rayleigh–Love coupling, anisotropy models constructed using the coupled surface-wave theory can be used to predict SKS splitting for direct comparison with observations. Third, our results motivate explicit searches for a 1ψ component in observed SKS splitting intensity, particularly for non-vertical incidence and in regions where independent surface-wave observations, such as strong Love-wave 2ψ anisotropy, indicate substantial Rayleigh–Love coupling. Finally, P-wave observations could be incorporated jointly with surface-wave anisotropy and SKS splitting because they can provide complementary constraints on the dip of the anisotropic symmetry axis (e.g., Beller & Chevrot, 2020).

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Data Availability Statement

Original seismic waveform data were obtained from the Data Management Center of IRIS (www.iris.edu). The complete surface-wave dataset used to generate the seismic models is publicly available on Zenodo (X. Liu, 2025, <https://zenodo.org/records/17118270>). Seismic models based primarily on Rayleigh-wave anisotropy were taken from C. Liu and Ritzwoller (2024) and are available through the

EarthScope Earth Model Collaboration repository (<https://ds.iris.edu/ds/products/emc-earthmodels/>).

ObsPy (Beyreuther *et al.* 2010) is used in data processing.

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APPENDIX A: SURFACE-WAVE THEORY IN A GENERAL ANISOTROPIC MEDIUM

Finding the phase speeds and polarizations of coupled quasi-Rayleigh and quasi-Love waves in a general anisotropic medium can be approximated as an eigenvalue problem (X. Liu & Ritzwoller, 2025):

$$\begin{pmatrix} A & E + iX \\ E - iX & B \end{pmatrix} \begin{pmatrix} a_L^* \\ a_R^* \end{pmatrix} = \omega^2 \begin{pmatrix} a_L^* \\ a_R^* \end{pmatrix} \quad (\text{A.1})$$

where A, B, E , and X are

$$A(\psi) = k^2 \int_0^\infty dz [(\mathcal{N} - E_c \cos 4\psi + E_s \sin 4\psi)W^2 + (\mathcal{L} - G_c \cos 2\psi + G_s \sin 2\psi)W'^2/k^2] \quad (\text{A.2})$$

$$\begin{aligned} B(\psi) = & k^2 \int_0^\infty dz [(\mathcal{A} + B_c \cos 2\psi - B_s \sin 2\psi + E_c \cos 4\psi - E_s \sin 4\psi)V^2 \\ & + (\mathcal{L} + G_c \cos 2\psi - G_s \sin 2\psi)(U - \frac{V'}{k})^2 \\ & + 2(\mathcal{F} + H_c \cos 2\psi - H_s \sin 2\psi)VU'/k + \mathcal{C}U'^2/k^2] \end{aligned} \quad (\text{A.3})$$

$$\begin{aligned} E(\psi) = & k^2 \int_0^\infty dz [(-\frac{1}{2}B_c \sin 2\psi - \frac{1}{2}B_s \cos 2\psi - E_c \sin 4\psi - E_s \cos 4\psi)WV \\ & + (G_c \sin 2\psi + G_s \cos 2\psi)(U - \frac{V'}{k})W'/k + (-H_c \sin 2\psi - H_s \cos 2\psi)WU'/k] \end{aligned} \quad (\text{A.4})$$

$$\begin{aligned} X(\psi) = & k^2 \int_0^\infty dz \{ [2(J_c - M_c) \sin \psi - 2(J_s + M_s) \cos \psi + D_c \sin 3\psi - D_s \cos 3\psi]VW'/k \\ & + (M_c \sin \psi + M_s \cos \psi + D_c \sin 3\psi - D_s \cos 3\psi)W(U - \frac{V'}{k}) \\ & + 2[(J_c - K_c) \sin \psi - (J_s - K_s) \cos \psi]W'U'/k^2 \} \end{aligned} \quad (\text{A.5})$$

and where ψ is the wave propagation direction, k is the wavenumber, W, U and V are the surface-wave eigenfunctions, and $\mathcal{A}, \mathcal{C}, \mathcal{N}, \mathcal{L}, \mathcal{F}$, and so forth are anisotropy parameters defined in Appendix B. The solution to this eigenvalue problem gives the coupled quasi-Love and quasi-Rayleigh wave eigenfrequencies

$$\omega^2 = \frac{A + B \pm \sqrt{(A - B)^2 + 4(E^2 + X^2)}}{2} \equiv \frac{1}{2} [A + B \pm D] \quad (\text{A.6})$$

or phase speeds given by

$$V_{qL}^2 = \frac{1}{2k^2} [A + B + D] \quad (\text{A.7})$$

$$V_{qR}^2 = \frac{1}{2k^2} [A + B - D] \quad (\text{A.8})$$

where

$$D \equiv ((A - B)^2 + 4(E^2 + X^2))^{1/2}. \quad (\text{A.9})$$

In addition to ignoring finite frequency effects, eq.(A.1) only considers the coupling between one Rayleigh wave mode and one Love wave mode from a reference medium, and assumes that the surface-wave mode in the anisotropic medium can be expressed as a superposition of these two modes, ignoring the coupling to all other modes. Based on these assumptions, equation (A.1) is a first-order approximation of the exact solution, which is appropriate for many applications.

APPENDIX B: THE 21 ANISOTROPY PARAMETERS

Montagner & Nataf (1986) introduced linear recombinations of the elastic tensor components to define certain anisotropic parameters needed to analyze surface-wave azimuthal anisotropy. Chen & Tromp (2007) introduced others that also are needed for body waves. We follow Chen & Tromp (2007) by

including the negative sign in the definition of G_s , B_s , H_s and E_s as follows:

$$\mathcal{A} = \frac{1}{8}(3C_{11} + 3C_{22} + 2C_{12} + 4C_{66}) \quad (\text{B.1})$$

$$\mathcal{C} = C_{33} \quad (\text{B.2})$$

$$\mathcal{N} = \frac{1}{8}(C_{11} + C_{22} - 2C_{12} + 4C_{66}) \quad (\text{B.3})$$

$$\mathcal{L} = \frac{1}{2}(C_{44} + C_{55}) \quad (\text{B.4})$$

$$\mathcal{F} = \frac{1}{2}(C_{13} + C_{23}) \quad (\text{B.5})$$

$$J_c = \frac{1}{8}(3C_{15} + C_{25} + 2C_{46}) \quad (\text{B.6})$$

$$J_s = \frac{1}{8}(C_{14} + 3C_{24} + 2C_{56}) \quad (\text{B.7})$$

$$K_c = \frac{1}{8}(3C_{15} + C_{25} + 2C_{46} - 4C_{35}) \quad (\text{B.8})$$

$$K_s = \frac{1}{8}(C_{14} + 3C_{24} + 2C_{56} - 4C_{34}) \quad (\text{B.9})$$

$$M_c = \frac{1}{4}(C_{15} - C_{25} + 2C_{46}) \quad (\text{B.10})$$

$$M_s = \frac{1}{4}(C_{14} - C_{24} - 2C_{56}) \quad (\text{B.11})$$

$$G_c = \frac{1}{2}(C_{55} - C_{44}) \quad (\text{B.12})$$

$$G_s = -C_{45} \quad (\text{B.13})$$

$$B_c = \frac{1}{2}(C_{11} - C_{22}) \quad (\text{B.14})$$

$$B_s = -(C_{16} + C_{26}) \quad (\text{B.15})$$

$$H_c = \frac{1}{2}(C_{13} - C_{23}) \quad (\text{B.16})$$

$$H_s = -C_{36} \quad (\text{B.17})$$

$$D_c = \frac{1}{4}(C_{15} - C_{25} - 2C_{46}) \quad (\text{B.18})$$

$$D_s = \frac{1}{4}(C_{14} - C_{24} + 2C_{56}) \quad (\text{B.19})$$

$$E_c = \frac{1}{8}(C_{11} + C_{22} - 2C_{12} - 4C_{66}) \quad (\text{B.20})$$

$$E_s = -\frac{1}{2}(C_{16} - C_{26}) \quad (\text{B.21})$$

We use the script notation for $\mathcal{A}, \mathcal{C}, \mathcal{N}, \mathcal{L}$ and $\mathcal{F}$ (using these five parameters to construct an effective VTI or transversely isotropic medium) to distinguish them from the Love moduli A, C, N, L and F that define the reference VTI medium that is the basis for the elastic tensor for a TTI medium in Appendix A.

J_c (J_s), K_c (K_s) and M_c (M_s) are body wave 1ψ azimuthal anisotropy parameters and D_c (D_s) is the body wave 3ψ azimuthal anisotropy parameter, which were not included by Montagner &

Nataf (1986). G_c (G_s), B_c (B_s) and H_c (H_s) are 2ψ azimuthal anisotropic parameters for both body waves and surface-waves. E_c (E_s) is the 4ψ azimuthal anisotropic parameter for both body waves and surface-waves.

For a TTI medium with strike along the x_1 -axis ($\phi = 0$ in the convention adopted in Appendix C), the parameters J_c , K_c , M_c , G_s , B_s , H_s , D_c , and E_s vanish. The remaining 13 anisotropy parameters may be non-zero, forming a medium with effective monoclinic symmetry.

APPENDIX C: THE ANISOTROPY PARAMETERS IN A TTI MEDIUM

The elastic tensor for a TTI medium obtained by a rotation about the y -axis, with no subsequent rotation about the vertical axis, is presented in Appendix A of X. Liu & Ritzwoller (2025). Here we adopt instead a strike–dip convention for a general TTI medium, consistent with Xie et al. (2015): the reference VTI tensor is first rotated by the dip angle θ about the x -axis and then by the strike angle ϕ about the vertical z -axis. Thus, ϕ denotes the strike, which is perpendicular to the azimuth of the

horizontal projection of the symmetry axis. With this convention, the anisotropy parameters are:

$$J_c = \left[\frac{1}{8} E \sin \theta \cos \theta (3 \sin^2 \theta - 2) + \frac{1}{4} (C - A) \sin \theta \cos \theta \right] \sin \phi \quad (\text{C.1})$$

$$J_s = -\left[\frac{1}{8} E \sin \theta \cos \theta (3 \sin^2 \theta - 2) + \frac{1}{4} (C - A) \sin \theta \cos \theta \right] \cos \phi \quad (\text{C.2})$$

$$K_c = \left[\frac{1}{8} E \sin \theta \cos \theta (7 \sin^2 \theta - 4) \right] \sin \phi \quad (\text{C.3})$$

$$K_s = -\left[\frac{1}{8} E \sin \theta \cos \theta (7 \sin^2 \theta - 4) \right] \cos \phi \quad (\text{C.4})$$

$$M_c = \left[\frac{1}{4} E \sin^3 \theta \cos \theta + (L - N) \sin \theta \cos \theta \right] \sin \phi \quad (\text{C.5})$$

$$M_s = \left[\frac{1}{4} E \sin^3 \theta \cos \theta + (L - N) \sin \theta \cos \theta \right] \cos \phi \quad (\text{C.6})$$

$$G_c = -\left[\frac{1}{2} E \sin^2 \theta \cos^2 \theta + \frac{1}{2} (L - N) \sin^2 \theta \right] \cos 2\phi \quad (\text{C.7})$$

$$G_s = \left[\frac{1}{2} E \sin^2 \theta \cos^2 \theta + \frac{1}{2} (L - N) \sin^2 \theta \right] \sin 2\phi \quad (\text{C.8})$$

$$B_c = -\left[-\frac{1}{2} E \sin^2 \theta \cos^2 \theta + \frac{1}{2} (C - A) \sin^2 \theta \right] \cos 2\phi \quad (\text{C.9})$$

$$B_s = \left[-\frac{1}{2} E \sin^2 \theta \cos^2 \theta + \frac{1}{2} (C - A) \sin^2 \theta \right] \sin 2\phi \quad (\text{C.10})$$

$$H_c = -\left[\frac{1}{2} E \sin^2 \theta \cos^2 \theta - \frac{1}{2} (A - F - 2N) \sin^2 \theta \right] \cos 2\phi \quad (\text{C.11})$$

$$H_s = \left[\frac{1}{2} E \sin^2 \theta \cos^2 \theta - \frac{1}{2} (A - F - 2N) \sin^2 \theta \right] \sin 2\phi \quad (\text{C.12})$$

$$D_c = -\left[\frac{1}{4} E \sin^3 \theta \cos \theta \right] \sin 3\phi \quad (\text{C.13})$$

$$D_s = \left[\frac{1}{4} E \sin^3 \theta \cos \theta \right] \cos 3\phi \quad (\text{C.14})$$

$$E_c = \left[\frac{1}{8} E \sin^4 \theta \right] \cos 4\phi \quad (\text{C.15})$$

$$E_s = -\left[\frac{1}{8} E \sin^4 \theta \right] \sin 4\phi \quad (\text{C.16})$$

where A, C, N, L , and F are the Love moduli for the reference VTI medium that is tilted by angle θ and rotated by strike angle ϕ to produce the TTI medium, and $E = A + C - 2F - 4L$.

APPENDIX D: COMPARISON WITH PREVIOUS THEORETICAL STUDIES

The theory of Montagner et al. (2000) is for an HTI medium ($\theta = 90^\circ$) with vertical propagation ($I = 0^\circ$). In this case, let $\rho V_{iso}^2 \approx L$, $\psi = \Psi$, $G = \sqrt{G_c^2 + G_s^2}$, $G_c = G \cos 2\Psi_G$, and $G_s = -G \sin 2\Psi_G$, as the G_s in Chen & Tromp (2007) which we follow is the opposite of the G_s in Montagner et al.

(2000), from equation (23) we obtain

$$SI \approx -\sin 2\Psi \int \sqrt{\frac{\rho}{L}} \frac{G_c}{L} dz - \cos 2\Psi \int \sqrt{\frac{\rho}{L}} \frac{G_s}{L} dz = \delta\bar{t} \sin 2(\bar{\Psi} - \Psi) \quad (\text{D.1})$$

$$\delta\bar{t} = \sqrt{\left(\int \sqrt{\frac{\rho}{L}} \frac{G_c}{L} dz\right)^2 + \left(\int \sqrt{\frac{\rho}{L}} \frac{G_s}{L} dz\right)^2} \quad (\text{D.2})$$

$$\tan 2\bar{\Psi} = -\frac{\int \sqrt{\frac{\rho}{L}} \frac{G_s}{L} dz}{\int \sqrt{\frac{\rho}{L}} \frac{G_c}{L} dz} \quad (\text{D.3})$$

Equations (D.1 - D.3) are the equations (20) - (22) in Montagner et al. (2000) with a sign difference in the definition of $\bar{\Psi}$ because the G_s in Chen & Tromp (2007) is the opposite of the G_s in Montagner et al. (2000).

The theory of Romanowicz & Yuan (2012) is for a TTI medium with elliptical anisotropy. We present in Appendix C the anisotropy parameters for a TTI medium, which together with the assumption of elliptical anisotropy ($E = A + C - 2F - 4L = 0$) greatly simplify the anisotropy parameters. In this case, $K_c = K_s = D_c = D_s = E_c = E_s = 0$, and equation (23) becomes

$$SI = -\int \frac{dz}{\rho V_{iso}^3} [-I(M_c \sin \psi + M_s \cos \psi) + (G_c \sin 2\psi + G_s \cos 2\psi)]. \quad (\text{D.4})$$

Applying equations (C.5)–(C.8) to an elliptical TTI medium with dip angle θ and strike angle ϕ , as defined in Appendix C, gives

$$SI = \int \frac{(L - N)dz}{2\rho V_{iso}^3} [\sin^2 \theta \sin 2(\psi - \phi) + I \sin 2\theta \cos(\psi - \phi)]. \quad (\text{D.5})$$

Romanowicz & Yuan (2012) define Θ as the tilt of the symmetry axis from the horizontal and Φ as the azimuth of its horizontal projection. With the strike–dip convention adopted in Appendix C, the corresponding oriented angles are $\theta = \pi/2 + \Theta$ and $\phi = \Phi + \pi/2$. The second relation follows because ϕ is the strike, perpendicular to the horizontal projection of the symmetry axis. Thus $\sin^2 \theta = \cos^2 \Theta$ and $\sin 2\theta = -\sin 2\Theta$.

Using $\rho V_{iso}^2 \approx L$, we define the positive fast-minus-slow splitting delay in the convention of Romanowicz & Yuan for layer j as

$$\delta t_j^{\text{RY}} \approx \frac{\Delta z_j}{V_{iso}} \frac{N - L}{2L},$$

so that $(L - N)\Delta z_j/(2\rho V_{iso}^3) \approx -\delta t_j^{\text{RY}}$. Because $I = \tan I + O(I^3)$, equation (D.5) can therefore be written

$$\begin{aligned} SI &= \int \frac{(L - N)dz}{2\rho V_{iso}^3} [\cos^2 \Theta \sin 2(\Phi - \psi) + I \sin 2\Theta \sin(\Phi - \psi)] \\ &\approx -\sum_{j=0}^n \delta t_j^{\text{RY}} [\cos^2 \Theta_j \sin 2(\Phi_j - \psi) + \sin 2\Theta_j \sin(\Phi_j - \psi) \tan I]. \end{aligned} \quad (\text{D.6})$$

The bracketed geometric dependence in the second line of equation (D.6) is the same as in equation (31) of Romanowicz & Yuan (2012). Their first-order transverse waveform relation contains one-half of this coefficient, whereas our splitting intensity is defined as $SI = -2u_T/\dot{u}_R$. The two expressions are therefore equivalent after accounting for the different definitions.

We do not understand one aspect of the results obtained by Romanowicz & Yuan (2012). For vertical incidence ($I = 0^\circ$) in a TTI medium, our theory (eq. (16)) has no vertical component of displacement. Romanowicz & Yuan (2012) do obtain a vertical component even for vertical incidence as shown in their equation (24). A vertical component can only be generated if there is a P-SV conversion across interfaces or P and S waves are coupled, which are not included in their theory or ours.

Chevrot & van der Hilst (2003) present expressions for splitting time for a general TTI medium. Because their results are not for splitting intensity we do not derive their results from ours.