Problems of Global Ocean Averages and Integrals

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Abstract

The ocean circulation exhibits a great regional variation in the temporal behavior of physical parameters such as temperature or salt or carbon. Some of the variety is very familiar, e.g. equatorial versus mid-latitude, but in addition much heterogeneity appears within those very large-scale geographical regions. Questions arise in computing and understanding the meaning of such quantities as the global average temperature or heat uptake and their corresponding uncertainties. Heat content through space and time in the abyssal ocean, below 2000m, as seen in a 26-year state estimate, is used as an example.

1 Introduction

For many purposes, both descriptive and analytic, a knowledge of characteristics of quantities applying globally to the Earth is useful. An example is the mean temperature of the global ocean, acting both as a description relative e.g., to geological-past-states, and as an element of energy components of the present climate system. This utility persists despite what is quite clearly vast regional differences that require description and analysis in many geographical variations. Similar statements can be made about descriptive elements of global health and general econometric variables, whose statistical problems overlap the oceanographic one.

The project called Estimating the Circulation and Climate of the Ocean (ECCO) has produced a 26 year-estimate (starting in 1992) of the global ocean circulation that combines a

numerical model of the circulation with most of the available global data sets. This particular product, labelled version4 release 4 (ECCOv4r4), was constructed by procedures discussed by Wunsch and Heimbach (2014), and Forget et al. (2015). The process can be summarized as one treating the inverse problem of estimating the ocean circulation from the numerical representation of the appropriate equations of motion in quantitative combination with observations, all subject to quantitative estimates of the errors inevitably present in both data and numerical models.

Extended descriptions of the results can be found in Fukumori et al. (2018), ECCO Consortium et al.. (2021), Wunsch, 2024, hereafter W24; Wunsch , 2025, hereafter W25) and will not be repeated here. For present purposes, the main properties are that the estimate, computed with a nominal 1 hour time-step, 50-vertical layers, and a horizontal resolution of nominal order 1 deg of latitude and longitude although varying, satisfies a free-running version of the MITgcm (Adcroft et al., 2004) and is generally consistent within error estimates of the observations as described in ECCO Consortium et al. (2021). Because the model contains "controls" (including initial and surface boundary conditions) which have been adjusted so that the consequent free-running model is generally consistent with the data, one is assured that the physical conservation laws (energy, mass, potential vorticity, etc.), vitally important to physical interpretation, are satisfied up to the numerical accuracy of the GCM.[1]

In W24,W25, the ECCOv4r4 estimate was time-averaged over the full 26-year period and some of the elements of the average were displayed and described. For present purposes, a few of the gross properties that emerged are of interest: (1) Over much of the ocean many of the physical parameters (e.g., temperature, velocity) exhibit long period changes precluding the inference that the time-mean represents a steady-state. (2) Along with that inference was the indication of underlying temporal variability inconsistent with any assumption of a *statistically* steady state. (3) Much regional structure in the physics of the time-mean is apparent. (4) Conventional default assumptions of near-Gaussian statistics of the temporal and spatial variability are often violated.

Taken together, these results suggest difficulties in the best computation of quantities such as the global oceanic mean temperature and its trends. The ECCOv4r4 result remains an estimate of the ocean circulation, and is not the circulation itself. The literature cited above discusses the extent to which elements have been observed and the degree to which those data are reproduced

[1]One might compare the ECCO estimate to the apparently similar meteorologically-originated "reanalyses." As discussed e.g., by Wunsch, Williamson, Heimbach (2023), those products are subject to a kind of uncertainty principle, in which the sequential injection of observations changes the estimated state—in the process violating the conservation laws.

within error estimates—themselves subject to uncertainties. For present purposes, it will only be assumed that the estimate reproduces much of the character of the real temporal fluctuations and their multi-decadal means, with the intention of exploring a few of the implications. Statisticians (e.g., Wilcox, 2010) emphasize that such problems do not have a "right" answer—merely ones that can be useful as descriptors or predictors or providers of insight.[2]

Description of the global ocean circulation and its variability is a challenging task. Here, intending only to be exemplary, the focus is put on the water column heat content below 2000m—a subject of intense interest in the wider climate science framework, and one that to some extent amplifies the problems. (See e.g., Hakuba et al., 2024, for a general review of the climate framework.)

Oceanic mean temperatures, and its divisions along with salinity defining oceanographic "water masses", have long been a central descriptor of oceanic structure (e.g. Worthington, 1981). The ECCOv4r4 state estimate employes almost all of the extant temperature measurements in the interval following 1992 including those from the traditional global ship observations and the Argo data from many thousands of free drifting floats. It remains true however, that the great bulk of those latter data pertain to the upper 1-2km of the ocean with the abyssal estimates relying heavily upon the model physics plus such other data as the altimetry and meteorology.[3]

W25 has a more extended summary of some of the statistical issues and as applied to sea level, the total water column heat content, and the abyssal sub-element. The purpose of the present note is to elaborate, and partially correct, some of that discussion as it applies to the abyss.

2 The Ocean Below 2000m. Heat Content

Let $\theta\,(\mathbf{r},z,t)$ be oceanic potential temperature, where $\mathbf{r}$ is the lateral position vector (latitude, longitude), z the vertical coordinate, and t is time. All variables are discrete. Heat content below 2000m is approximately,

$$Q_a\,(\mathbf{r}_i,t) = c_p\rho_0 \int_h^{-2000} \theta_a\,(\mathbf{r}_i,z,t)\,dz,\ \mathrm{J/m^2} \tag{1}$$ {Qa1}

c_p =3994J/kg/°C an approximate heat capacity, $\rho_0 = 1038\mathrm{kg/m^3}$ a reference density, and h is the local water depth, taken as negative. (The Q_a values computed here are based upon degrees

[2]The language used in much of the statistics literature, with its overlay of philosophy, is commonly very much stronger, sometimes startlingly so, than is commonly found in the wider scientific literature. See for example, Jaynes (2003).

[3]See Hennon et al. (2019) as one example of myriad data issues.

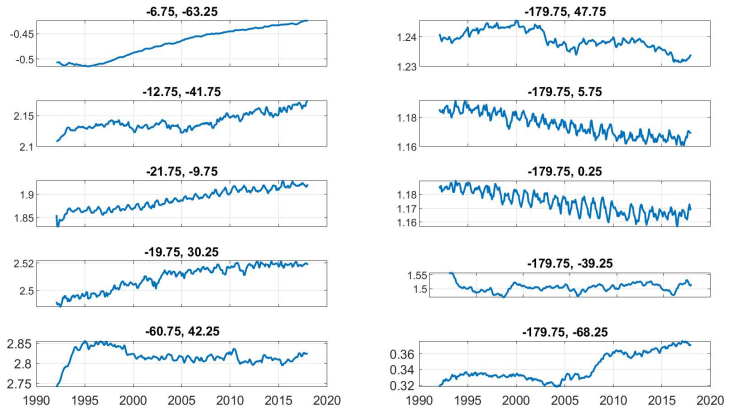

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Figure 1: (left panel) Example time series of the vertical abyssal average (2000m to the bottom) temperature $\bar{\theta}_a$, at 5 locations in the Atlantic Ocean. Units °C. South to north. Labels are longitude, latitude respectively. See Fig. 2 for positions. W25b displayed some of the total heat content time series.(Right panel) Same as left panel except for the Pacific Ocean along 180°W and showing some notable cooling. W25, Fig. 16, provides some comment on the appearance in some regions of an annual cycle in the abyss. {temper_timese

Celsius, which permits negative values. Absolute temperatures, degrees Kelvin, required for energy discussions, preclude the negative values displayed here in some areas. For a full discussion of the energetics of a water column see, for example, Vallis, 2017). A cogent argument can be made that the upper limit of the integral should be an isopycnal. That however, introduces another space-time variable that is not important for the present purposes Vertical average temperature is,

$$\bar{\theta}_a\left(\mathbf{r}_i,t\right) = \frac{Q_a\left(\mathbf{r}_i,t\right)}{c_p\rho_0}/(-2000-h). \tag{2}$$

These values were computed at grid points spaced at 1° of longitude and 1/2° of latitude. Some coincide with the model grid and some are interpolated values. Fig. 1 displays a few of the time series of vertically averaged potential temperatures at fixed positions in the Atlantic and Pacific Oceans. The visual melange, which is not exhaustive, is an indication of the numerous ways in which the temporal variability influences any time average. Positions are shown in Fig. 2. This spatial and temporal inhomogeneity raises questions of the physical interpretation of global averages in space and/or time.

An overbar will be used to denote vertical averages, and a bracket, $\langle . \rangle$, for temporal ones. In some cases, the operation will refer to a median rather than a mean. Thus $\langle Q_a\left(\mathbf{r},t\right)\rangle$ is a function of position, $\mathbf{r}$, alone. Note that areas corresponding to the grid points here are a strong function of latitude, vanishing at the north pole, and values of means/medians and trends on the grid must be distingushed from those that are appropriately area-weighted.

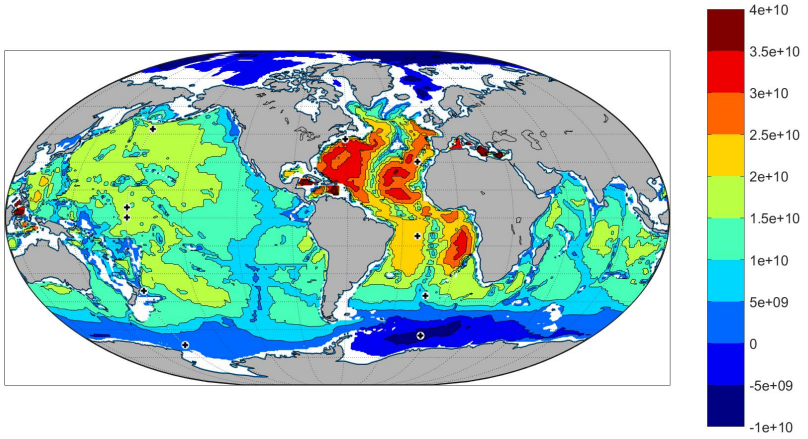

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Figure 2: Median in time of Q_a at each grid point (J/m^2). A map of the time average is visually indistinguishable from this one. + marks the positions of the time series in Fig. 1. Strong topographic control of heat content is apparent.

{qa_median_gri

Default discussions of stochastic processes in an oceanographic context, but also in the wider geophysical/climate literature, almost inevitably assume near-Gaussian, statistically stationary processes. The statistics literature (e.g., Efron and Tibshirani, 1993) forcefully emphasizes that such assumptions can lead to seriously incorrect estimates from samples, particularly of the accuracy of values, should deviations, even sometimes slight, occur.

2.1 A Statistical Framework

A major issue in capturing the flavor of the great variations in the behavior of time (or space) series, such as shown in Fig. 1, is the extent to which the processes can be regarded as statistically stationary. That character is commonly assumed by default e.g., in computing confidence limits on such quantities as time-means and spectral estimates. As emphasized e.g., by Beran (1994), deviations from stationarity[4] generally produce radically different statistical interpretations. To deal with the potential lack of stationarity, and the varying extent to which it can be violated, the statistical literature commonly relies upon a generalization of the mathematical machinery of autoregressive moving average (ARMA) representations used for fully stationary processes; see Chatfield (2004) or Box et al. (2008); or the summary in W25b). Mudelsee (2010) is a textbook treating some of the general peculiarities of climate space-time statistics.

The important characteristic of stationarity can fail e.g., if a linear trend is present. Defining

[4]We distinguish "stationary" here from its sometimes common use as implying steady flows.

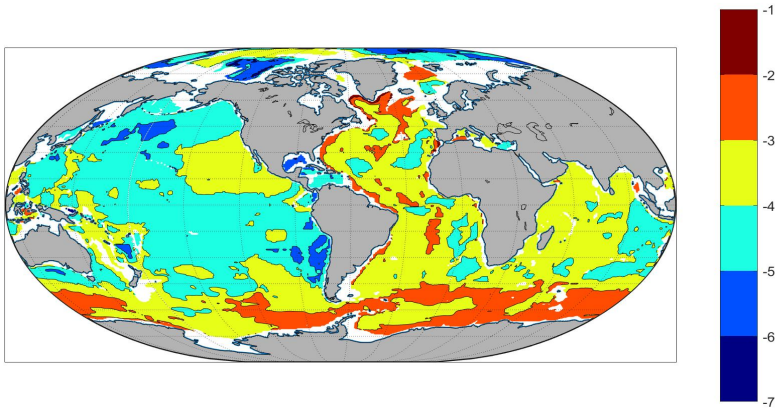

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Figure 3: $\log_{10}$ of the conventional variance of vertical average abyssal temperature (degrees C). High Southern and North Atlantic Ocean values appear, as does a comparatively quiet Western and South Pacific Ocean.

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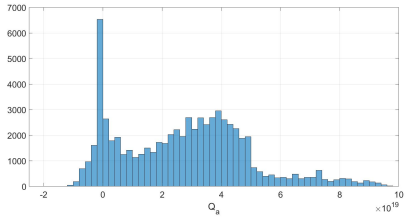

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Figure 4: Histogram of the time median values Q_a weighted by area at each grid point (Joules). The time-mean values are nearly identical. Non-normal (non-Gaussian) behavior appears. Here and elsewhere a very small number of outlying values are omitted.

{histograms_qa

the backwards difference operator $B\xi_t = \xi_{t-1}$, a stationary process in that situation can then be generated by taking the difference,

$$w_i = \xi_i - \xi_{i-1} = (1 - B)\,\xi_i$$

an ARMA(p, q) representation including ξ_t is then possible,

$$\sum_{i=1}^{p} \phi_i w_i = \sum_{i=1}^{q} \psi_i \varepsilon_i \tag{3}$$ {A1}

where ε_i is commonly a white noise process and the constants ϕ_i and ψ_i are the coefficients of the autoregressive and moving average representations respectively. The result is generally labelled an autoregressive integrated move average (ARIMA$(p, 1, q)$). The desirable properties of the polynomials generated from ϕ_i, ψ_i are discussed in the references, as are the best choices choices of p, q from various statistical tests. Solving Eq. (3) for w_i,

$$w_i = \left(\sum_{i=1}^{p} \phi_i B^i \right)^{-1} \sum_{i=1}^{q} \psi_i \varepsilon_i, \tag{4}$$

existence of the polynomial inverse being guaranteed by the requirements on its determination.

A simple backwards difference is too restrictive for present purposes. The system is generalized by defining,

$$\xi_i = (1 - B)^d\, w_i, \tag{5}$$ {A2}

where d need not be an integer and,

$$(1 - B)^d = \sum_{j=0}^{\infty} (-1)^j \frac{\Gamma(d + 1)\, B}{\Gamma(j + 1)\,\Gamma(d - j + 1)} \tag{6}$$ {one_minus_B}

where Γ is the gamma function (Beran et al., 2013, P.47) . The original, non-stationary, w_i, is,

$$w_i = (1 - B)^{-d} \left(\sum_{i=1}^{p} \phi_i B^i \right)^{-1} \sum_{i=1}^{q} \psi_i \varepsilon_i. \tag{7}$$

Behavior of the series varies greatly with d, ranging from stationary ($d \leq 0$), to explosively growing for large values, $d > 1$. The general formulation with arbitrary d in Eq. (3) is labelled an ARIMA(p, d, q). Here d will be used as a descriptor of oceanic variability.

Note that in some contexts, the Hurst parameter, originating in hydrology, $H = d + 1/2$, is more familiar. Table 1, from W25 and Brockwell and Davis (1991), provides a listing of the behavior of time series as a function of the values of d, H. As in W25, a distinction is made between "long autocovariance memory" and "long physical memory," where in the terminology

d	$H = d + 1/2$	**Character**
$0 < d \leq 1/2$	$1/2 < H \leq 1$	Stationary, long autocovariance memory
0	$1/2$	Stationary white noise
$-1/2 < d \leq 0$	$0 < H \leq 1/2$	Stationary, short autocovariance memory
$d \leq -1/2$	$H < 0$	Stationary
$d > 1/2$	$H > 1$	Non-stationary, long physical memory

Table 1: General behavior of a time series for a given value of d. H is the corresponding Hurst parameter used particularly in the hydrological literature. From W25. Stationarity implies a local statistical equilibrium not dependent upon earlier initial conditions, while non- stationarity likely means that dependence is still present in the ocean. Discussion can be found in Brockwell and Davis (1991), Beran (1994), Chatfield (2004), Box et al. (2008) and elsewhere. Long autocovariance memory refers to processes whose autocovariances decay much more slowly than those for conventional processes and are distinguished here from long physical memory processes.

of Beran (1994), long autocovariance memory refers to the slow decay of the autocovariance relative to that expected for a stationary normal process, and is distinguished here from long physical memory involving oceanic processes having time-scales longer than the duration of observations—extending to thousands of years.

Among numerous other properties, as described in W25, d determines the estimated variance of e.g., an estimated sample mean. Thus for $d = 1/2$, (a random walk) the uncertainty of a sample mean does *not* diminish with increasing numbers of samples, M. For larger values of d, variance *increases* with M. $d = 0$ produces the familiar conventional behavior of the variance of a Gaussian process sample mean decreasing with M.

Determination of d can be made by different methods. The one with the most immediate physical interpretation derives it from the estimated power spectrum of the process,

$$\Phi(s) \approx s^{-2d}, \; s \to 0, \tag{8}$$

where s is the frequency or equivalent. Values of d become descriptors of the familiar spectral behavior of "blue noise" ($d < 0$), "white noise" ($d = 0$), "red noise" ($d > 0$), "non-stationary rednoise" ($d > 1/2$) etc. (see Table 1). The general subject of spectral behavior of climate variables, not discussed here, has been the focus of considerable effort. von Storch et al. (2001) describe many of the general properties.

Here, estimated values, $\tilde{d}$, are determined from a least-squares fit to the 10 lowest frequencies in the periodograms of monthly-varying temperature (periods longer than the annual). Hurvich and Ray (1995) discuss the sometimes complicated biases in this calculation, called by them the

GPH test, and which can be a strong function of d. If the periodograms have been computed using a bell taper, they suggest omiting the periodogram estimate at the lowest non-zero frequency. In the present case, no taper has been used, and no bias correction has been attempted. As with many results recorded here, estimates of d should be regarded as tentative, with a primarily qualitative connotation.

2.2 Spatial Means/Medians

By integrating over the entire global ocean, one has the complete population, and not a sample, and which renders moot much of the conventional statistics discussion (see for example, Rice, 2007, Ch. 7) applicable to finite limited samples. In the abstract, one might regard the existing oceanic state as simply one member taken from an ensemble of hypothetical Earths each subject to statistically varying climate systems. That assumption, for example, can justify the use, as in W25, of a bootstrap for discussing statistics. In the present case, that possibilty is set aside as it ultimately demands a statement of the statistics governing the ensemble. The approach assumed that all values come from an independent identically distributed (iid) collection (Efron and Tibshirani, 1993), and which in practice cannot be justified for subsamples of $\langle Q_A(\mathbf{r}_i)\rangle$.

(1) *The Mean and Median*

Let ΔA_i be the area represented by the grid point at $\mathbf{r}_i$. Total area is $A = \sum_1^N \Delta A_i$ Then the mean value is simply,

$$m_1 = \frac{1}{A}\sum_{i=1}^{N} \langle Q_A(\mathbf{r}_i)\rangle \Delta A_i \tag{9}$$

producing, $m_1 = 1.26 \times 10^{10}\mathrm{J/m^2}$, the 26-year average value.

A median value (median $(Q_A(\mathbf{r}_i,t)\Delta A_i/\text{median}(\Delta A_i))$ is nearly indistinguishable at 1.16×10^{10} J/m^2 and see Fig. 2. Estimating the accuracy of such values will depend upon any systematic and stochastic errors both in the underlying model representations and in the multiple data sets used there, and is not attempted. (A covariance of an estimated uncertainty is a square matrix of dimension of the state vector, which in this case exceeds 10^7 values at any one time.) Whether the small difference between the mean and median here is of any importance can only be decided in the context of their use.

Fig. 5 displays the logarithm of the area weighted temporal variance in $\langle Q_a(\mathbf{r}_i,t)\Delta A_i\rangle$. The very small values in the Arctic Sea are prominent, arising primarily from the very small grid spacing near the pole, but also from the small temporal range of estimated temperatures there. An inference that variance-weighted forms of these values should dominate the global average

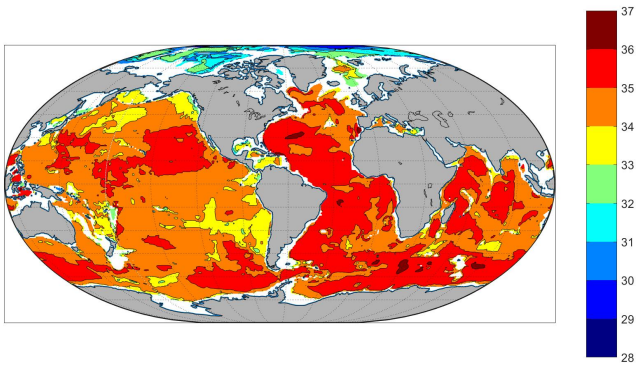

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Figure 5: $\log_{10}$ of the area weighted variance of the temporal variability in Q_a (joules). The very low values in the high Arctic Sea are notable. These would give a high weight to $\langle Q_a \rangle$ in a variance weighted spatial mean—not discussed here.

{rqa_map.jpg}

(it is negative with a very small uncertainty) is rejected on physical grounds. Furthermore, any assumption of spatially uncorrelated fields cannot be defended.

(2) *Means/medians stratified by* d

An estimate of d was made in W25 from the low-frequency least-squares fit slopes of the logarithms of the power spectra/periodograms at each grid point and a spatially smoothed estimate is shown here in Fig. 6. Because $\tilde{d}$ is itself an estimate, it has a corresponding uncertainty measured by the residual in the periodgram fits. Fig. 7 is the unsmoothed result except that regions where $\tilde{d}$ is indistinguishable from 0—ordinary stationarity—are shown as blank areas. Much of the visual complexity, as with the ordinary temporal variances, is undoubtedly due to the complexity of the model bottom-topography, subject to the caveat that it remains greatly under-resolved.

"Stratified" means and medians (e.g.,see Kendall and Stuart, 1973 or Rice, 2007) are computed for the different ranges of $\tilde{d}$, noting that these regions are generally not geographically contiguous and the samples are complete—that is they are not a random sample. Using ordinary sampling statistics in what is a homogeneous environment—at least as defined by $\tilde{d}$—the result is shown in Table 2. Despite the full population, twice the ordinary standard deviations of the means are listed for each region as a measure of noisiness. If interpreted as a random sample, neither of the largest areas produces a mean or median differing statistically from zero. The corresponding areas show that the bulk of the abyssal variability has long autocovariance memory or more general non-stationarity. Time averages in such a situation are not easy to interpret. In particular, the system must be regarded, despite the multi-decadal averaging, as

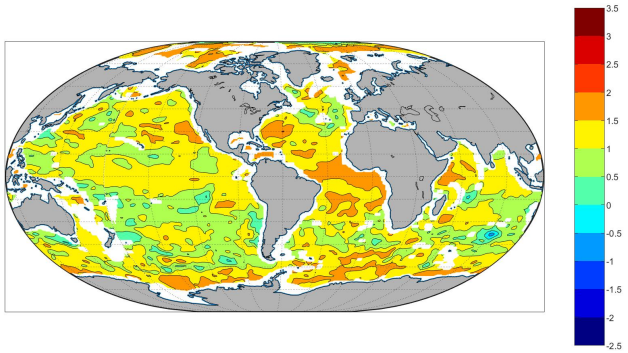

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Figure 6: $\tilde{d}$ for the ARIMA(p, d, q) determined by least-squares from the periodograms at each point and smoothed over 4 degrees of longitude and 2 degrees of latititude. (Unsmoothed version is in W25, Fig. 14.)

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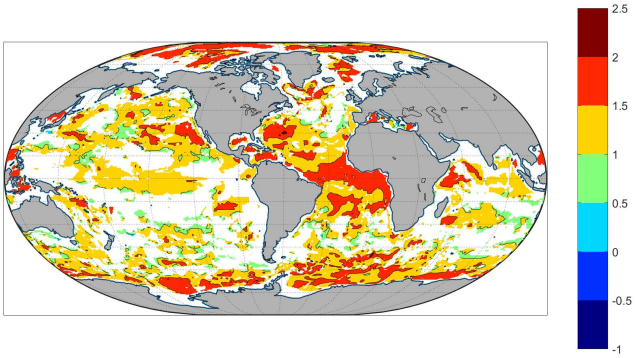

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Figure 7: Unsmoothed version of Fig. 6 except regions not differing at two standard deviations from $\tilde{d} = 0$, an ordinary stationary time series are left as blank and that value assigned everywhere there. Values of $\tilde{d} < 0$ are nearly non-existent; see Table 2. (Bottom topography remains a simplified version of the fully resolved field.)

{dqasig_ls.jpg

$\tilde{d}$	mean/10^{10}J	median/10^{10}J	No. Vals.	Area/10^{13}m^2
$d \leq -1/2$	1.00±0.40	1.09	27	0.0065
$-1/2 \leq \tilde{d} < 0$	0.97±0.88	0.98	324	0.078
$0 \leq \tilde{d} < 1/2$	1.11±1.66	1.01	3689	0.765
$1/2 \leq \tilde{d} < 1$	1.19±1.32	1.13	18619	4.439
$1 \leq \tilde{d} < 2$	1.30±2.38	1.20	45547	9.655
$2 \leq d$	1.43±4.09	0.27	88	0.0142

Table 2: Values of the mean and median of Qa in the different ranges of d. Mean and median differ significantly only for the small population d¿=2. Uncertainty values are 2 standard deviations of the means. Uncertainty of the median can be computed from a bootstrap, but is omitted here. Modified from W25.

being far from anything resembling a steady-state.

Taken altogether, and coupled with the charts of values, the simplest inference here is that the heat content of the abyss, *measured in degrees Celsius*, is indistinguishable from zero. Further stratification by combined d and basin geography is obviously possible. Because of the variable area weighting and number of points, the global mean, $\tilde{m}_1$, is not simply related to the means of Table 2.

3 Global Trends

Of more immediate interest in the climate change context is the extent to which the abyssal ocean has been a sink of heat from a warming atmosphere. Consider first Fig. 8. Medians of Q_a ($\mathbf{r}$,t) on each grid point for year 26 and for year 1 were subtracted and divided by 26 years. A generally large-scale pattern in the rates is evident, including the major regions of cooling. A value of 10^7J/m^{2}y corresponds to 0.32W/m^2. The figure demonstrates a fundamental difficulty—that whatever global mean or median is found, it will be a small residual of large numbers. That large regions in the Southern and Atlantic Oceans have been undergoing strong warming is evident and uncontroversial. Cooling in the abyssal Pacific Ocean has been discussed e.g., by Bindoff and Mcdougall (1994), Meza and Gebbie (2025), and the Southern Ocean warming by Johnson and Purkey (2024) and others.

One criticism of the simple difference approach is that determining a trend by taking the difference of two averaged years ignores the information contained in the 312 monthly values occuring between the start and finish of the state estimate. To address that shortcoming, resort is now had to the so-called Theil-Sen (ThS) median, following the discussion in Wilcox (2010).

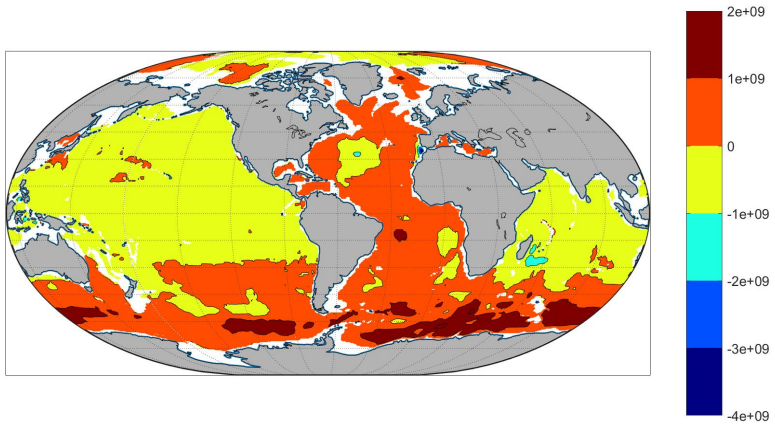

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Figure 8: Rate of change of Q_a J/m^{2}y from the difference of the time median values over the last and first years. Large regions of both positive and negative values appear. 10^7J/m^{2}y=0.32W/m^2 {qalastyr_minu

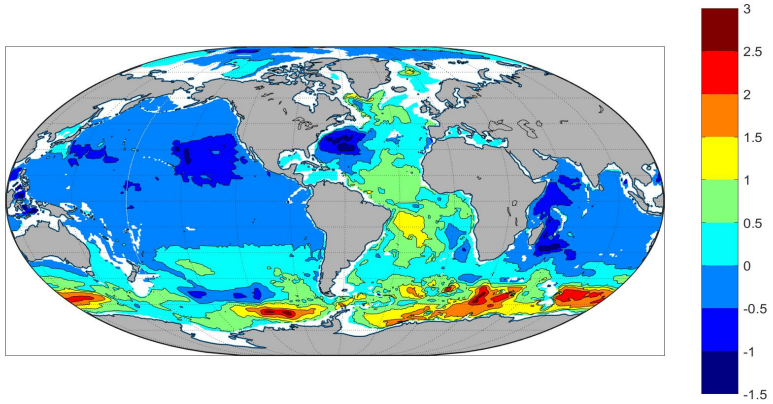

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Figure 9: Grid point values, converted to W/m^{2}of the Theil-Sen median trend in Q_a. Gross patterns are similar to those of Fig. 8 but much difference in detail appears. {qatimetrend_t

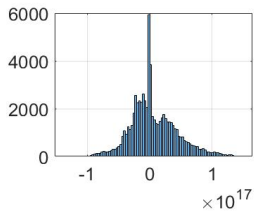

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Figure 10: Area weighted Theil-Sen median values of the time trend, Joules/y, at each grid point. Non-normality is again apparent.

{histogram_qat

The ordinary median estimator, itself more robust than least-squares is, for a sequence ξ_j, the middle value of successive pairs, $\xi_j - \xi_{j-1}$. In contrast, the Theil-Sen estimator computes the median of *all* pairs $\left(\xi_j - \xi_{j'}\right)/(j - j')$, $j \neq j'$, which in a sequence of M values produces $M(M-1)/2$ pairs for estimation in each sequence.

Applying the Theil-Sen algorithm as encoded in Matlab (Tilgencamp, 2025) to each grid point results in the trend/year estimates shown in Fig. 9, but converted to W/m^2. Although there exists a strong resemblance to Fig. 8, many detailed pattern differences appear The ordinary spatial average of the area weighted estimate of $\Delta Q_{ai}\Delta A_i/\Delta t = 2.79 \times 10^6 \mathrm{J/m^2/y}$ If an ordinary least-squares fit is used to determine the temporal trend instead of the ThS estimate, the values are indistinguishable within the uncertainty estimate of Gaussian least-squares. In a spatial map of the difference of the mean and median values of $\Delta Q_{ai}\Delta A_i/\Delta t$, the region of largest difference is in the western North Atlantic Ocean just east of the Gulf Stream.

Stratified values

Values when stratified by $\tilde{d}$ as above, produce the results shown in Table 3. As with the time means, the values apply primarily to non-statistically stationary regions, or those with long autocovariance memory. In a regional discussion, not offered here, this different behavior would have to be accommodated.

4 Discussion

As already noted above, the statistics literature reminds one that definitive answers to statistical problems rarely exist—procedures must be analyzed and interpreted within the physical context of any particular problem. Determining the global ocean average of any quantity, whether it be via a state estimate, or a particular instrument set, faces the special problems of a continuum with highly heterogeneous statistical properties. Not discussed here is the issue of how much

$\tilde{d}$	mean ThS/10^6J/m^{2}y	median ThS/10^6J/m^{2}y	No. Vals.	Area/10^{13}m^2
$d \leq -1/2$	1.16±0.0019	1.36	27	0.0065
$-1/2 \leq \tilde{d} < 0$	0.36±4.29	-0.015	324	0.078
$0 \leq \tilde{d} < 1/2$	-0.18±12.7	-0.016	3689	0.765
$1/2 \leq \tilde{d} < 1$	0.36±2.16	-1.14	18619	4.439
$1 \leq \tilde{d} < 2$	4.16±39.59	0.50	45547	9.655
$2 \leq d$	6.7±7.29	2.39	88	0.0142

Table 3: Values of the mean and median of the Theil-Sen (ThS) time trend in the different ranges of d. Except for the minute area of d¡=-1/2, none of the median or mean values is significant at 2 standard deviations. $10^6 J/my = 0.03 W/m$

of these changing fields should be regarded as deterministic e.g., by a warming atmosphere, and how much stochastic. Calculation of global mean or median values is straightforward, but determining their accuracy and the signficance of any apparent change with time or space is subject to a long list of assumptions—combining the purely physical with the purely statistical. As the result of a state estimate of a state vector of very large dimension, present values cannot be readily evaluated in absolute accuracy terms. These problems are generic.

In a fundamental way, description of the ocean circulation appears to require methods irretrievably regional in nature. The great heterogeneity in the underlying variability precludes regarding even a near-three-decade averaging interval as representing a system close to a steady-state overall. But the physics of regional values and changes nonetheless do depend upon the global values and changes.

The state estimate has much less noise than does any particular near-global observational data set—lacking explicit internal waves, mesoscale (balanced), and sub-mesoscale eddies, as well as a simplified set of boundary conditions. As such, much of the expected error is systematic—lying in the various parameterizations of internal waves, and of interior and boundary turbulence. Systematic errors should have a greater influence on e.g., the time means/medians than on temporal differences—whether explicit year differences, or through the determination of trends by successive differencing of the (here) monthly values. Spatial differences entail many different potential systematic problems. Full understanding of systematic errors involves evaluation of the impacts of the various parameterizations, the limited resolution of the model representation, and calibration problems of the underlying observational data.

Whether a mean or median is a more meaningful measure of the physical parameter is often not very clear when the values differ significantly. The common use of time- or space-averaged equations of motion tends to dictate use of the mean. But given situations in which

the distributions of properties are qualitatively non-normal, use of median value equations ought at least to be considered. In the present state estimate, one not producing extreme numerous outliers, means and medians are nearly interchangeable. Underlying the results is the measure used within the state estimate of least-squares fits to time dependent data sets—data sets which themselves may have highly non-normal distributions with outliers. That likely dictates ultimate use of more robust measures of misfit.

With its primary focus on the spatial heterogeneity, this paper refrains from examining existing published estimates of mean ocean abyssal heat content and which evidently demands a region-by-region approach. That strong warming has taken place over much of the Southern and Atlantic Oceans is not in dispute, but the meaning of the global averages remains unclear. Such diverse fields as human health care and econometrics share many of the issues raised here, but no sufficiently general statistical tools appear to exist accommodating all of the complexity found.

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References

Adcroft, A., J. M. Campin, C. Hill and J. Marshall (2004). Implementation of an atmosphere-ocean general circulation model on the expanded spherical cube.Monthly Weather Review 132(12): 2845-2863.

Anderson, D. L. T., K. Bryan, A. E. Gill and R. C. Pacanowski (1979). Transient-response of the North Atlantic-some model studies.Journal of Geophysical Research-Oceans and Atmospheres 84(NC8): 4795-4815.

Beran, J. (1994). Statistics for Long Memory Processes. Chapman and Hall, New York.

Bindoff, N. L. and T. J. Mcdougall (1994). Diagnosing climate change and ocean ventilation using hydrographic data.J. Phys. Oc. 24(6): 1137-1152.

Box, G. E. P., G. M. Jenkins and G. C. Reinsel (2008). Time Series Analysis: Forecasting and Control. Hoboken, N.J., John Wiley.

Brockwell, P. J. and R. A. Davis (1991). Time Series: Theory and Methods 2nd Ed. New York, Springer-Verlag.

Chatfield, C. (2004). The Analysis of Time Series. An Introduction, Sixth Ed. Boca Raton, Chapman and Hall/CRC.

ECCOConsortium, I. F., O. Wang, I. Fenty1, G. Forget, P. Heimbach, R. M. Ponte (2021) Synopsis of the ECCO Central Production Global Ocean and Sea-Ice State Estimate, Version 4 Release 4. https://zenodo.org/records/4533349

Efron, B. and R. Tibshirani (1993). An Introduction to the Bootstrap. New York, Chapman & Hall.

Forget, G., J.-M. Campin, P. Heimbach, C. Hill, R. Ponte and C. Wunsch (2015). ECCO version 4: an integrated framework for non-linear inverse modeling and global ocean state estimation. Geosci. Model Dev. 8: 3071-3104.

Fukumori, I., P. Heimbach, R. M. Ponte and C. Wunsch (2018). A dynamically-consistent ocean climatology and its temporal variations. Bull. Am. Met. Soc., Oct.: 2107-2127.

Hakuba, M. Z., Fourest, S., Boyer, T. et al. (2024). Trends and variability in Earth's energy imbalance and ocean heat uptake since 2005. Surv. Geophys. 45: 1721-1756.

Hennon, T. D., M. H. Alford and Z. X. Zhao (2019). Global assessment of semidiurnal internal tide aliasing in Argo profiles. J. Phys. Oc. 49(10): 2523-2533.

Hurvich, C. M. and B. K. Ray (1995). Estimation of the memory parameter for nonstationary or noninvertible fractionally integrated processes. J. Time Series Anal. 16(1): 17-41.

Johnson, G. C. and S. G. Purkey (2024). Refined estimates of global ocean deep and abyssal decadal warming trends. Geophys. Res. Lett. 51, c2024GL.111229: https://doi.org.10.1029/2024GL.111229.

Kendall, M. G. and A. Stuart (1973). The Advanced Theory of Statistics, 3rd ed, Griffin, London.

Mudelsee, M. (2014). Climate Time Series Analysis: Classical Statistical and Bootstrap Methods, Springer.

Rice, J. A. (2007). Mathematical Statistics and Data Analysis, 3rd Ed. Belmont, CA, Thomson.

Vallis, G. K. (2017). Atmospheric and Oceanic Fluid Dynamics: Fundamentals and Large-Scale Circulation. Cambridge UK, Cambridge Un. Press.

Wilcox, R. R. (2010). Fundamentals of Modern Statistical Methods. Substantially Improving Power and Accuracy, 2nd Ed. New York, Springer.

Worthington, L. V. (1981). The water masses of the world ocean: some results of a fine-scale census. in, Evolution of Physical Oceanography. Scientific Surveys in Honor of Henry Stommel. B. A. Warren and C. Wunsch, The MIT Press, Cambridge (http://ocw.mit.edu/ans7870/textbooks/Wunsch/wunscht 42-69.

Wunsch, C. (2024). A time-average ocean: Thermal wind and flow spirals. Prog. oceanog. 221.

Wunsch, C. (2025). Statistical features of an ECCO time-averaged state estimate: Sea level and abyssal heat content. J. Phys. Oc. 55: 1859-1879.

Wunsch, C. and P. Heimbach (2014). Bidecadal thermal changes in the abyssal ocean and the observational challenge.J. Phys. Oc. 44: 2013-2030.

Wunsch, C., S. Williamson and P. Heimbach (2023). Potential artifacts in conservation laws and invariants inferred from sequential state estimation. Ocean Sci. 19: 1253-1275.