Bootstrapped high quantile estimation — An experiment with scarce precipitation data*

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This paper details team SUTD's effort when participating in the "Prediction of extremal precipitation" challenge. We propose a framework that combines the generalized Pareto distribution, a bootstrap resampling scheme and inverse distance weights to capture spatial dependence. Our method reduces the quantile loss functions by 55.1% as compared to a naive benchmark, and shows improvement across all months and all stations. The method works well even for stations without training data. The framework is scalable and can be implemented easily by practising engineers.

Key words: Extreme precipitation; inverse distance weights; ungauged sites; generalized Pareto distribution

1 Introduction

Understanding the distribution of extreme precipitation is crucial for the design of hydraulic infrastructure, which, in turn, are critical for flood protection (Papalexiou and Koutsoyiannis, 2013). As societies grow more complex in spatial scales, it is no longer sufficient to study extreme precipitation one location at a time; we need to explore its spatial dependence. Such understanding helps mitigate concurrent floods at different locations in a region. It also has another major advantage: knowledge of stations with rich data can help us predict extremes at locations with limited data or at ungauged sites.

In this context, the challenge "Prediction of extremal precipitation" is timely. The challenge's task is to estimate monthly maximum of daily precipitation across 34 stations, a majority of which have little or no training data. The stations' coordinates are shifted such that relative distances are maintained, and no other information about them is available (e.g., we did not know that the stations are in the Netherlands). In addressing this challenge, we seek for a simple method that is scalable and can be implemented easily in practice. For the quantile estimates, we rely on the well-known generalized pareto distribution (GPD). For spatial dependence and estimation at sites with little data, we opt for the robust bootstrap resampling scheme combined with the simple yet useful inverse distance weighting. These techniques have been well tested in their own right, and here we conduct an experiment to combining them in a framework. We seek to see whether this simple framework works well so that it can be directly implemented by practising engineers.

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2 Methodology

We classified the stations into three distinct groups (Figure 1). Each of the 14 stations in the first group has less than 10% missing data: these are stations that have observations throughout the training period and into the test period (henceforth we refer to them as *old stations*). The second group (20 stations) started collecting data only either at the end of the training period (15 stations), or in the test period (5 stations); this group is called the *new stations*. The third group consists of six stations that terminated observations within the training period (*discontinued stations*); they are not in the test set and we are not interested in estimating their quantiles, but their data are still useful to learn abeout other stations. Station 32 is a special case: although it has observations throughout the training period, all observations before 1995 are zero and scattered with a lot of missing data in between. Only data in 1995 seem reliable. Thus, we treated it as a new station. Since there is a stark contrast in data availability between old and new stations, we estimated their quantiles separately.

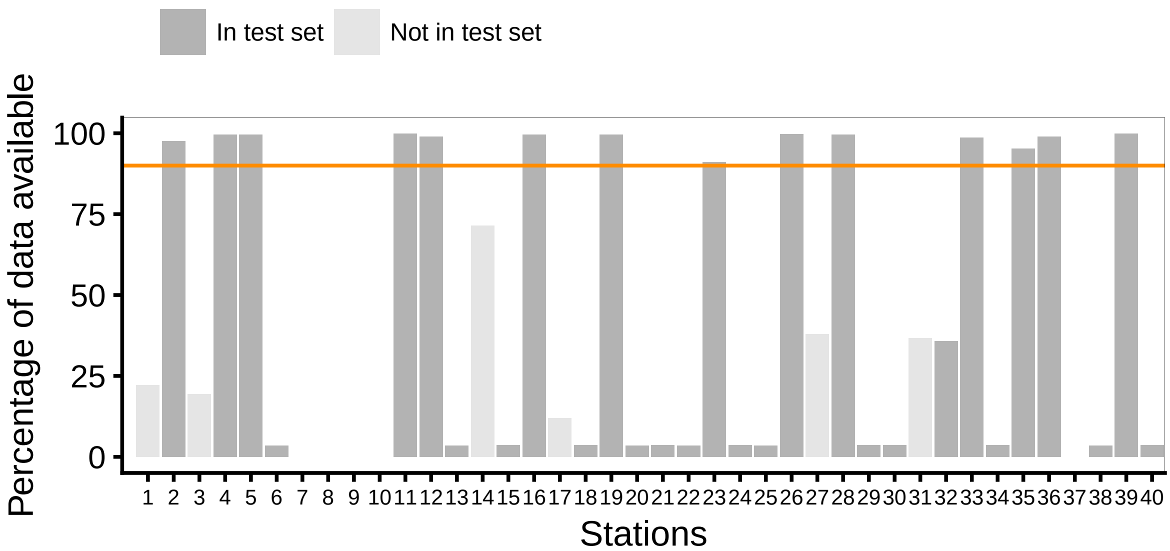


Figure 1. Data availability at different stations. The orange line indicates the 90% availability level. Percentage availability is the ratio between the number of days with recorded precipitation amount (including zeros) and the total number of days in the training period (8401 days from 31 December 1975 to 31 December 1995).

2.1 Old stations

Quantile estimation depends not only on the value of the data points, but also on the number of data points. Therefore, we attempted to fill in the missing data for the old stations using a correlation based linear regression. Let $\mathcal{O}$ be the set of old stations. For each missing data point on day t at station $j \in \mathcal{O}$, we selected another station $k \in \mathcal{O}$ that has the highest linear correlation with j (correlations are calculated using original training data). We then estimated the missing precipitation observation $p_{t,j}$ as

$$p_{t,j} = \beta_0 + \beta_1 p_{t,k}$$

where the regression parameters β_0 and β_1 were estimated using the precipitation time series at j and k using least square linear regression. These computations were carried out with the R package `hyfo` (Xu, 2017).

Once gap filling is completed, we can estimate the 0.998-quantile for each station-month pair by fitting a distribution into the data and derive a quantile from there. Due to the extreme nature of the quantile level, we decided to explore the possibility of working with a heavy-tailed distribution. Specifically, assuming the recorded precipitation levels for each station k and month m are iid realizations from the same distribution F_k^m, we examine whether this distribution can be approximated by a power law for sufficiently high observations. To be more precise: can we find $\alpha > 0, c \in \mathbb{R}, \bar{x} > 0$ such that $1 - F_k^m(x) \approx cx^{-\alpha},\ x > \bar{x}$.

Let $x_{(i)}^{m,k}, i = 1, ..., n_k^m$ denote the order statistics of the precipitation data for a given station-month pair (across all years), where n_k^m is the number of data points for each such pair (month m, station k). Let $\hat{F}$ denote the empirical distribution function defined as $\hat{F}_k^m(x_{(i)}^{m,k}) := \frac{i}{n_k^m+1}, i = 1, ..., n_k^m$. If the right hand tail of the distribution F_k^m can be approximated by a power law, then for large enough observations we expect the following to hold

$$1 - \hat{F}(x_{(i)}) \approx cx_{(i)}^{-\alpha}, \quad i \geq l,$$

or

$$-\log\left(1 - \frac{i}{n_k^m + 1}\right) \approx \alpha(\log x_{(i)} + \log c), \quad i \geq l, \tag{1}$$

where $\alpha > 0, c \in \mathbb{R}$ and $1 \leq l \leq n_k^m$ are suitably chosen constants.

To examine the validity of our assumption we plot the left hand side of equation (1) against the logarithms of the ordered log precipitation levels. If the highest values in the plot are all on a line with positive slope, our assumption cannot be rejected. We observed that this is not necessarily true for all station-month pairs. Figure 2a shows a station with a good fit for most months, whereas Figure 2b shows a station where a power does not seem appropriate, showcased by the “flat right tails”.

Based on this evidence we proceeded to fit a generalized Pareto distribution (GPD) (Embrechts et al, 2013, Chapter 3) to the data, which allows a fit to both heavy-tailed and light-tailed observations. GPD is commonly used with the “exceedance over threshold” approach, which assumes that extreme precipitations over a certain threshold follow a Poisson process (Davison et al, 2012). We decreed that the tail consists of the 25 largest observations for each station month pair (that is, we chose a tail fraction of roughly 3.5%). The choice is somewhat arbitrary and we found that it worked well. For a review of available choice methods, the reader is referred to Scarrott and MacDonald (2012). All computations involving fitting of the GPD and obtaining the quantile estimates were carried out using the R package `evir` (Pfaff et al, 2015).

2.2 New stations

With none or very little data, quantiles of the new stations must be derived from other stations. Poor training data also prohibits a parametric model. Hence, we opted for a nonparametric method, i.e., bootstrapping. This method estimates the sampling distribution of a random variable (in this case, the 0.998-quantile) using information of the observed data (Efron, 1979). It works under the assumption that precipitation at the new stations follow the same stochastic processes as the old ones.

The central idea of the bootstrapping algorithm is to construct for each new station an ensemble of sample daily precipitation time series, then estimating the 0.998-quantile for each sample using the GPD method (Section 2.1), and averaging these estimates over the ensemble to arrive at the final answer. The sampling pool $\mathcal{P}$ consists of all old stations and discontinued stations. In principle, sampling should be done using a weight function that reflects the similarities between stations, i.e., a station that has similar characteristics to the station of interests should be sampled with higher probability. Given that the sole source of information we have

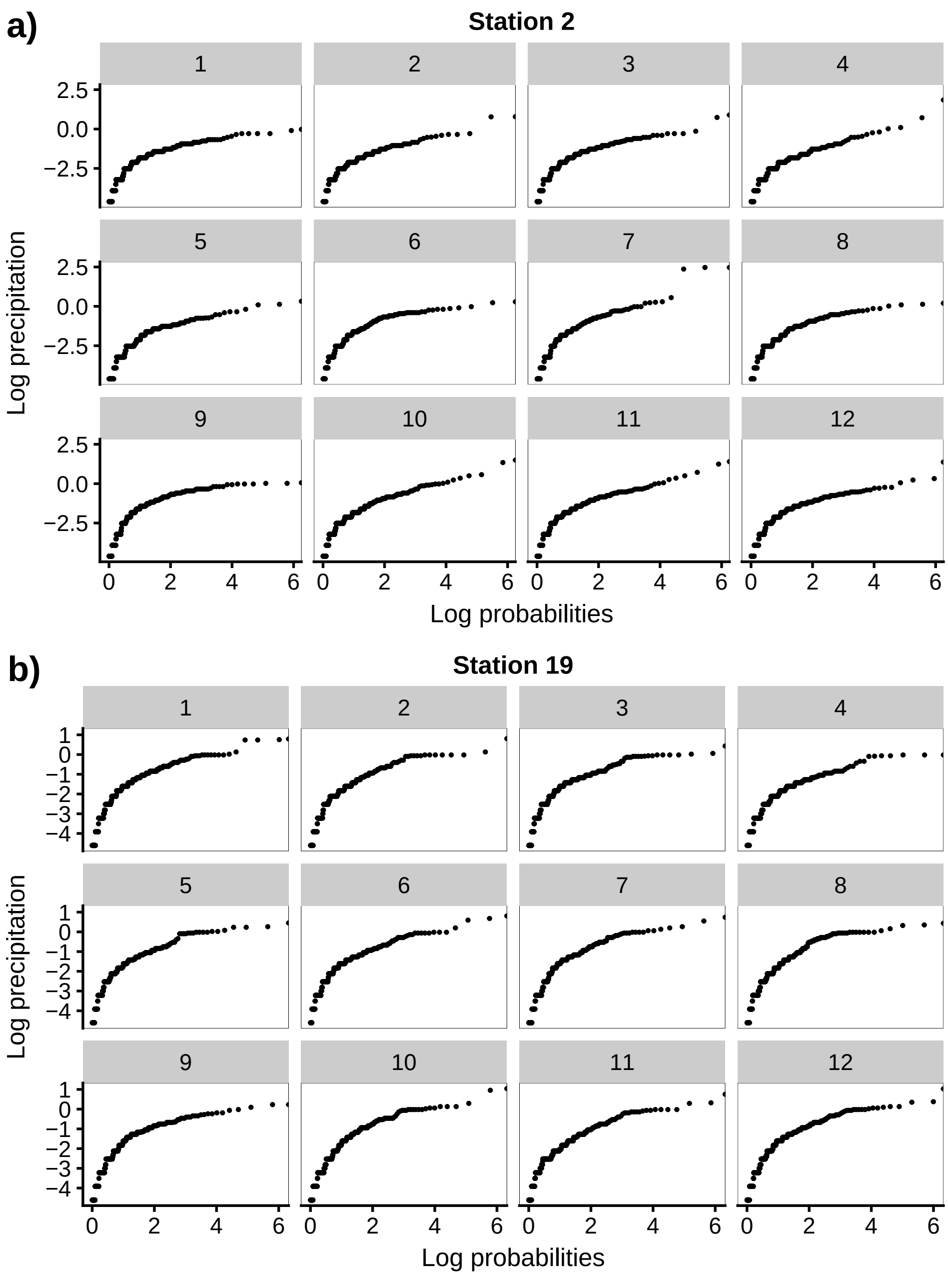


Figure 2. Log-log plot for examining whether GPD can be a good fit. a) A good example: for most months, the slopes show some linearity towards the right end. b) A less favourable example. Lots of months show flat parts with few outliers.

on the stations is the relative distances among them (calculated from their shifted coordinates), our sampling scheme uses the inverse-distance weights

$$w_{jk} = \frac{1/d_{jk}}{\sum\limits_{k \in \mathcal{N}} 1/d_{jk}} \tag{2}$$

where w_{jk} is the probability of station $k \in \mathcal{P}$ being selected when sampling for station $j \in \mathcal{N}$ and d_{jk} is the geodesic distance between stations j and k. Inverse distance weight is commonly used in hydrology for interpolation of spatial data (see e.g., Ahrens, 2006), but here we use it for sampling. A similar sampling scheme is the squared inverse distance function, where the terms d_{jk} in equation (2) are replaced by their squares. We discuss both schemes in Section 3.

As bootstrapping provides approximate frequency statements (Efron, 1979), it is an apt choice of methodology in our approach to the problem. Care must be taken so that the days with extreme precipitations are sampled at the right frequency. Initially, we experimented with monthly sampling, that is, once a station is selected, we resampled one entire month from that station, before moving on to the next month. This scheme resulted in extreme values being resampled too frequently because the number of months to be resampled is small. As a result, the quantile estimates appeared to be too high. We then modified the sampling scheme to allow for more variation within a month, such that the extreme days can be sampled less frequently, and the sampled time series resemble the original ones.

Let n_k^m be the number of days available in station k's data for month m. For example, if station k has 10 years of data, then $n_k^1 = 310$, $n_k^2 \approx 282$ depending on the number of leap years, and $n_k^4 = 300$. Then, each day in month m of station k's data is given a new weight

$$w_{jk}^m = \frac{w_{jk}}{n_{jk}^m} \tag{3}$$

Now, suppose we are resampling January. Instead of sampling 30 days from the same station all at once, we sample each day individually with the new weights according to equation (3). This sampling scheme allows each day to be sampled from a different station. The days are shuffled and the probability of an extreme day being sampled is lower as compared to the original scheme. Under this sampling scheme, the probability of an extreme day being sampled is approximately one over the total number of days, but adjusted for data availability. Thus, this scheme also favours stations with good data availability. The procedure is summarised in Algorithm 1.

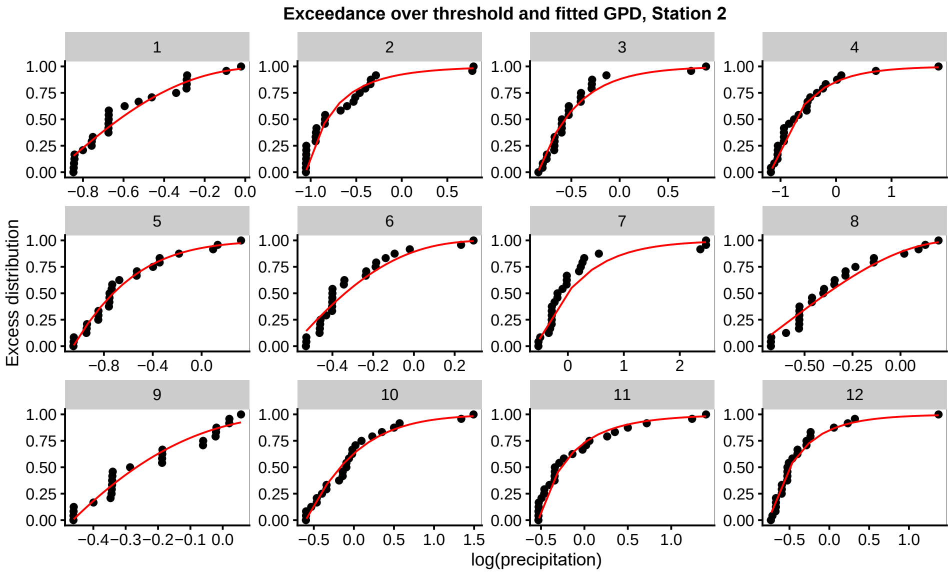
Algorithm 1 Bootstrapped quantile estimation

Build the sampling pool $\mathcal{P} = \{$old stations and discontinued stations$\}$
for each $j \in \mathcal{N}$ **do**
  for i = 1,...,1000 **do**
    for $y = 1976, ..., 1995$ **do**
      for $m = 1, ..., 12$ **do**
        Subset precipitation data of month m for all stations $k \in \mathcal{P}$
        Calculate w_{jk} for all $k \in \mathcal{P}$
        Calculate $w_{jk}^m = \dfrac{w_{jk}}{n_k^m}$
        Sample each day in month m with weights w_{jk}^m
      end for
    end for
    Calculate $\hat{q}_{i,j}$ using GPD
  end for
  $\hat{q}_j = \dfrac{1}{1000} \sum_{i=1}^{1000} \hat{q}_{i,j}$
end for

3 Results and discussion

We found that the GPD is a good fit for all old stations. To illustrate, we plot the 25 largest observations for each month of stations 2 and 19 together with the fitted distribution in Figure 3. As anticipated, the GPD fitted well for Station 2 (Figure 3a), but even for Station 19, the GPD fitted relatively well too (Figure 3b). Goodness-of-fit was also seen via the QQ-plots (not shown for brevity) where the relationship between empirical quantiles of precipitation and theoretical quantiles of GPD is close to a straight line (Das and Resnick, 2008). That GPD fitted well to precipitation extreme is crucial as it is our underlying distribution for all quantile estimations.

a)



b)

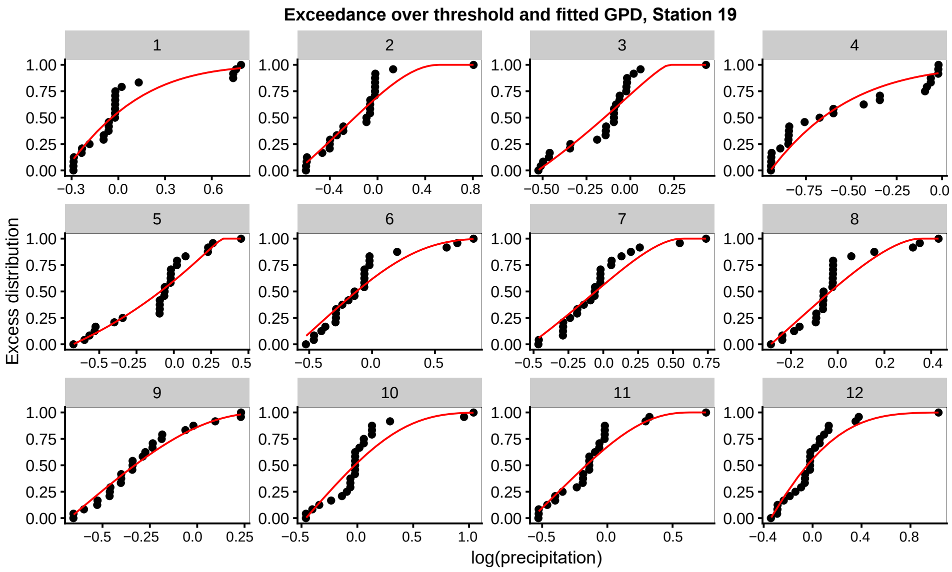


Figure 3. Plot of the 25 largest observations (black) by month for stations 2 (top) and 19 (bottom). The red line signifies the fitted GPD.

Overall, the SUTD method reduced the total quantile loss (summed over all months and stations) by 55.1% compared to the benchmark. Our method worked better than the benchmark in 77.1% of all station-month pairs (Figure 4a). More importantly, all the increased losses were small (the worse is 2.85 points), but many of the reduced losses were large (the best is -52.51 points). The difference in prediction skill became larger when summing quantile losses over all stations (Figure 4b) or over all months (Figure 4c): our prediction was always better than the benchmark. Furthermore, the SUTD method performed quite consistently, while the benchmark suffered from huge losses in several stations and month. Let us now examine these cases.

Station 32 is where quantile loss was highest for the benchmark (Figure 4b). This is a station

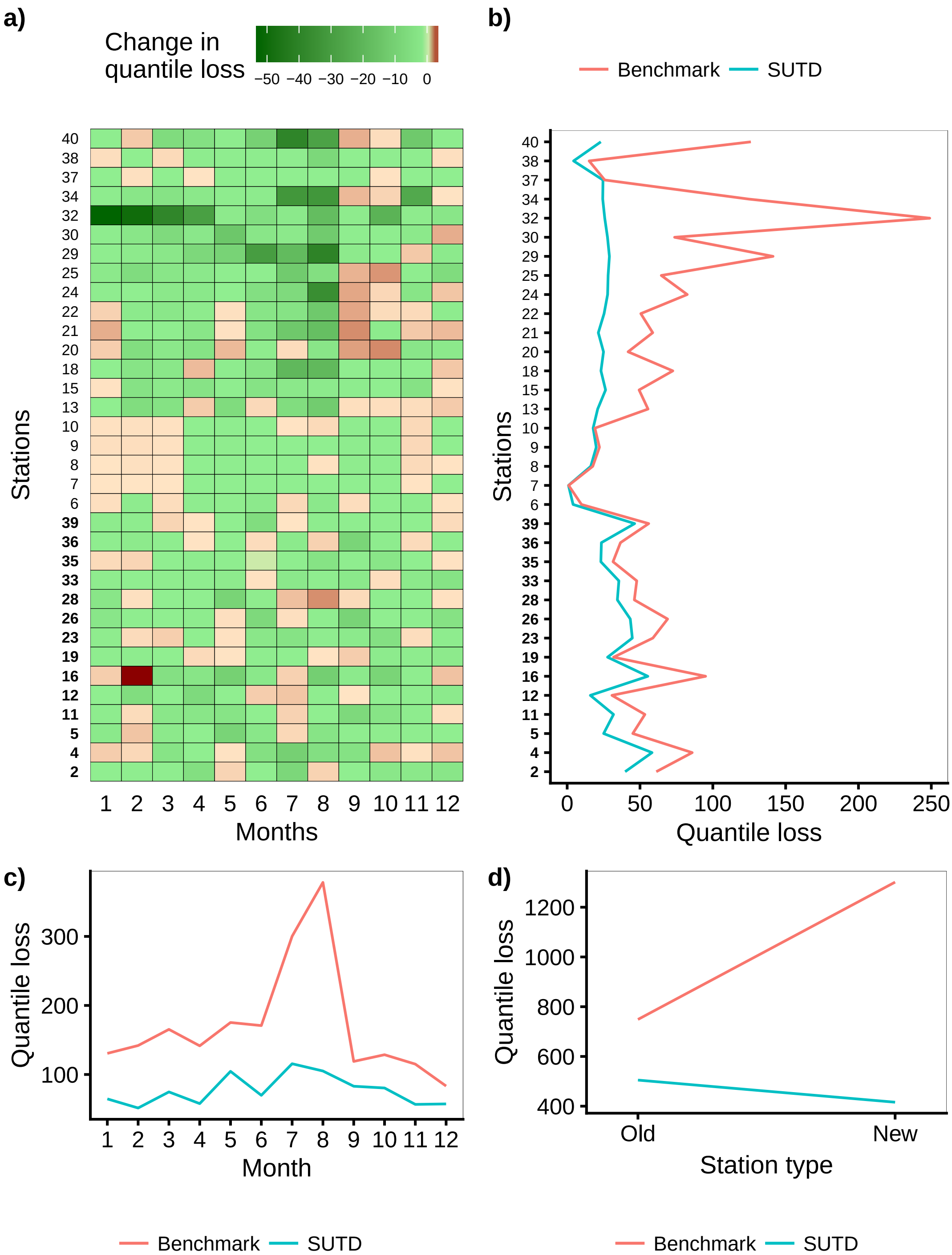


Figure 4. Performance comparison between our method and the benchmark. a) The change in quantile loss (SUTD minus benchmark) for all station-month pairs; negative change indicates that our method reduces the quantile loss. Stations in the y-axis are grouped according to their type: the first 14 stations are the old stations (bold-faced), and the remaining are new stations. b) Total quantile loss, summed over all months for each station. c) Total quantile loss, summed over all stations for each month. d) Total quantile loss by station type.

with poor data quality: only year 1995 has non-zero data. As a result, the benchmark produced zero as the quantile estimates for January to April, while its estimates for other months are close to zero. Therefore, the benchmark was penalized heavily at this station. Contrarily to the naive benchmark, we treated Station 32 as a new station, thus achieved better estimations than the benchmark did in all months, particularly January to April (Figure 4a). To quantify the impact of this station on the benchmark's performance, we compared out results with a "less naive" benchmark which treated Station 32 in the same way as Station 7, i.e, assuming that it has no data at all. While the old benchmark scored a quantile loss of 248.93 points at Station 32, the new benchmark scored 25.65 points, which is 0.16 points better than our prediction. Consequently, the overall score of the SUTD method is 49.6% when compared to the new benchmark. The significant difference between the two benchmarks emphasizes the importance of data quality in quantile estimation. As this discussion shows, 5.5 percentage point of our performance came from carefully scrutinizing the data.

The naive benchmark also performed poorly in stations 24, 25, 29, 30, 34 and 40 (Figure 4b). These are stations that only have data for 1995, the last year of the training period. Based on observations of just one year (and incomplete data in each month), the benchmark derived very low quantile estimates and was also penalized heavily. In particular, these stations account for the higher penalties incurred for the benchmark in July and August (Figure 4c), when a large number of summer storms are not captured in the training data (Figure 5). The SUTD method, on the other hand, was able to derive good quantile estimates by resampling from the old stations. A different picture is seen with stations 7, 8, 9, 10 and 37. Here, without training data, the benchmark relied on other stations by taking their average estimates; its quantile losses here are significantly smaller than those at stations 24, 25, 29, 30, 34 and 40 above (Figure 4b). This implies that in the context of extreme value estimation, having little data may be misleading, which could be worse than having no data at all. These cases show that extreme value estimation is sensitive to data availability, and that pooling information from nearby stations is a good way to overcome the lack of data.

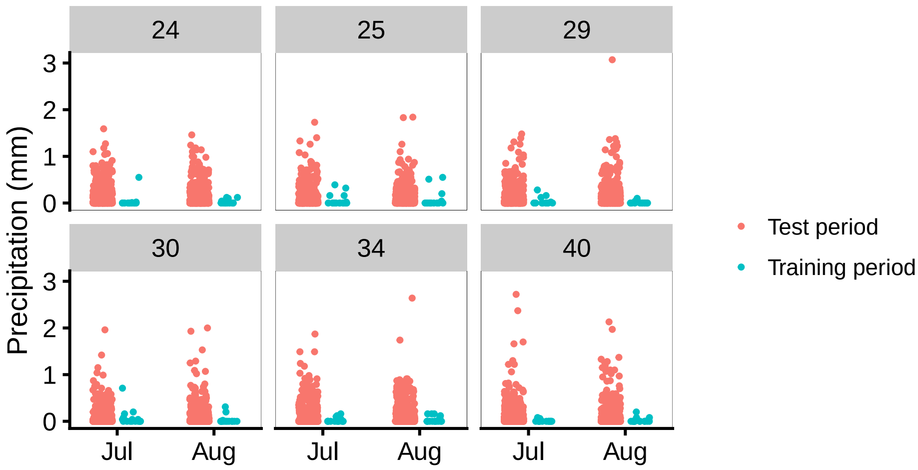


Figure 5. Jitter plots showing July and August precipitation distribution in the training and test data for stations that started collecting observations in 1995, the last year of the training period.

Interestingly, our method performed better with the new stations than with the old ones. Figure 4d shows that the total quantile loss of the new stations is less than that of the old stations, even though there are more new stations than old ones. Moreover, quantile loss is consistent across the new stations while it fluctuates more with the old stations. The consistent performance may be attributed to the fact that these stations were sampled from a common pool, and the better performance suggests that using information from nearby stations may improve quantile estimation as compared to using each station's own data alone.

Finally, we found that in this experiment, using inverse distance achieves slightly better results than using squared inverse distance. With the latter sampling scheme, our method achieved a score of 54.5%, a 0.6 percentage point reduction. This is probably due to the flatness of the Netherlands. In other countries with rougher terrains, similarity among stations may reduce more sharply over space, and the differences between the two schemes may become larger.

4 Conclusion

In this study, we experimented with estimating extreme precipitation (0.998-quantile, corresponding to monthly maximum over 20 years) with data across 40 stations in the Netherlands. The test stations belong to two distinct types: old stations whose data are available throughout the training period, and new stations whose data are limited or absent in the training period. For the former, we fitted a generalized pareto distribution (GPD) to each station and then estimated their quantiles from the fitted distribution. For the latter, we first constructed an ensemble of bootstrapped time series, then estimated the 0.998 quantile for each one with GPD, and average these quantiles across the ensemble to arrive at the the final estimate. The bootstrap resampling was carried out with inverse distance weights.

Our estimation is much better than the benchmark, which uses the monthly maximum in the training period. Overall, quantile losses is reduced by 55.1%; a range of reduction from fair to large is observed across most stations and all months. These encouraging results have some useful implications. First, good quantile estimation among the old stations suggests that GPD is a useful distribution for estimating monthly maxima; it should be considered for future extreme value studies beside the usual families of distributions discussed in Papalexiou and Koutsoyiannis (2013). Second, that our method performs equally well for the new stations suggests that bootstrapping with inverse distance weights is a fair way to make use of spatial relationship to estimate quantiles for station without data. Last but not least, we show that simple methods may work well and are a good starting points to build more sophisticated methods.

A potential improvement is to try a few different extreme value distributions and choose one that fits best for each station. Another limitation of this experiment is that the sole spatial data available is relative distances among station. We believe that other station characteristics may improve prediction. Particularly, precipitation is known to increase with elevation. Although it turned out that lacking elevation data did not hinder good prediction in this case (due to the Netherlands' natural landscape), such data may be important for future studies in other countries.

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