

Multipole Modeling for Magnetic Microscopy Constrained by Euler Deconvolution

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Open research: The source code used to generate all of the results presented in this research can be freely accessed and reused under the terms of an open license. You can find it at <https://doi.org/10.5281/zenodo.20446907> and <https://github.com/compgeolab/micromag-spherical-harmonic>.

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Plain language summary

Magnetic microscopy allows scientists to measure tiny magnetic signals from individual mineral grains and small geological samples. These measurements can provide valuable information about the Earth's ancient magnetic field and the magnetic properties of rocks. As magnetic microscopes continue improving, they can now measure signals at higher resolutions and closer distances to the sample surface. However, these advances also reveal more complex magnetic patterns that are often too complicated to be accurately represented by traditional dipole-based methods. In this study, we developed a new approach that combines spherical harmonic models with an automatic estimate of the magnetic source location using Euler deconvolution. The method was tested using both computer-generated (fake) magnetic models and real magnetic microscopy data. The results show that the new approach more accurately reproduces complex magnetic signals, especially when measurements are acquired close to the sample surface where non-dipolar magnetic effects become important. This work provides a practical way to extract more information from modern magnetic microscopy measurements and may improve future studies of magnetic minerals, paleomagnetism, and rare geological materials.

Abstract

Magnetic microscopy is rapidly advancing toward higher spatial resolutions and smaller sensor-to-sample distances. Under these conditions, dipolar approaches become increasingly limited because higher-order magnetic contributions can no longer be neglected when retrieving reliable full-vector information for paleomagnetic and rock magnetic applications. However, multipolar representations often require prior knowledge of source geometry and expansion center location, which can involve costly auxiliary measurements and additional processing complexity. This challenge becomes particularly relevant for planar magnetic microscopy observations, where the incomplete sampling geometry violates the ideal assumptions of spherical harmonic (SH) expansions. Here, we present an Euler-based approach that provides initial estimates of the expansion center for SH inversions, with optional nonlinear refinement through simplex optimization. The proposed methodology was evaluated using both synthetic micromagnetic models and real magnetic microscopy data, and its performance was compared with conventional dipolar approaches. Synthetic experiments with known ground-truth models demonstrated improved accuracy of SH-based inversions, particularly for near-surface measurements where higher-order magnetic signals become significant, while maintaining comparable performance at larger sensor heights. Application to real data acquired at approximately 5 μm sensor height showed broadly similar results for both approaches in most cases, although the SH framework produced more coherent solutions for anomalies affected by external

magnetic contributions. These results represent an important step toward retrieving complex magnetic source information directly from planar magnetic anomaly measurements.

1 Introduction

Recent advances in magnetic microscopy have expanded its potential for quantitative paleomagnetic applications by enabling the investigation of increasingly complex magnetic sources. Magnetic microscopy will continue to push toward smaller sensor-sample standoff and higher spatial resolution, revealing complex magnetic behavior beyond dipole-dominated signatures. Under these conditions, higher-order contributions (quadrupole, octupole, and beyond) become non-negligible (e.g., [Bellon et al., 2026](#); [Cortés-Ortuño et al., 2021, 2022](#)), particularly for larger grains, clustered sources, and measurements close to the sample. Although the dipolar contribution can alone be used to efficiently describe the full magnetic moment of a source ([Bellon et al., 2026](#)), it is the non-dipolar components of a stray field that will carry a particle's domain-state signature ([Cortés-Ortuño et al., 2022](#); [Fabian and de Groot, 2019](#)). Spherical-harmonic (SH) expansions ([Cortés-Ortuño et al., 2022](#); [Lima et al., 2023](#)) provide a compact representation of fields by expressing the external magnetic potential on a virtual spherical surface enclosing the source.

In magnetic microscopy, [Lima et al. \(2023\)](#) showed that it is possible to use surface-plane measurements — e.g., the vertical component of the magnetic field of a thin section acquired with a Quantum Diamond Microscope (QDM) — to invert the magnetic moment of a given source using SH expansions. It is well known that the degree 1 SH coefficients under Schmidt normalization are proportional to the three components of a magnetic dipole located at the expansion center ([Blakely, 1996](#)). On planar data, however, SH performance is fundamentally limited by the lack of a closed observation surface surrounding the expansion center. Planar measurements provide only a hemispherical sampling of the field, which breaks the orthogonality of the SH basis functions and leads to spectral leakage. Consequently, poor centering of the expansion around the magnetic field source and measurement noise can redistribute energy between spherical harmonic degrees, causing artificial higher-order contributions and introducing biases in the lower-order coefficients ([Lima et al., 2023](#)).

These issues could be mitigated, as shown by [Out et al. \(2026\)](#), by using double-sided magnetic microscopy combined with three-dimensional tomographic experiments to determine the positions of magnetic sources within the sample's subsurface. However, incorporating tomography into the experimental workflow introduces additional complexity and cost. Conventional computerized CT scans often lack the resolution to detect the smallest single-domain (SD) and single-vortex (SV) particles. Even when high-resolution techniques do succeed, they are often chemically or structurally unresolved (e.g., Focused Ion Beam tomography, [Gergov et al. \(2025\)](#); [Nikolaisen et al. \(2020\)](#)), or they must be restricted to very small areas (typically on the order of $\sim 20 \mu\text{m}^2$), as is the case for ptychographic X-ray computed nanotomography ([Bellon et al., 2024](#)).

An alternative approach is to rely solely on the observed magnetic data and invert SH coefficients only for the detected anomalies, which represents a virtual source enclosed within a single shell centered on that anomaly. Unlike gravitational potential data, for which the center of mass provides a natural reference point for spherical expansions, magnetic data lack an equivalent and uniquely defined “magnetic center” ([Lima et al., 2023](#)). This creates a major practical challenge: under the planar geometry of magnetic microscopy, even small source-centering errors may induce spectral leakage and bias the recovered SH coefficients and dipole moment.

In this context, we pursue a center-of-expansion seeding technique based on Euler deconvolution

(ED) (Reid et al., 1990; Souza-Junior et al., 2026a, 2024). The method requires only the observed magnetic field and its derivatives, avoids explicit size and shape assumptions, and is well conditioned for local planar problems. In parallel, we implement the SH formulation of Lima et al. (2023), enabling a spectrally complete representation of the magnetic field that can be propagated to arbitrary observation planes once the expansion center and truncation degree are properly constrained. We apply this methodology to both realistic synthetic and real QDM datasets and compare it against dipole-based models from Souza-Junior et al. (2026a). The synthetic dataset consists of idealized micromagnetic particle models, which allow us to delineate the conditions under which the method performs best as a function of sensor height, grain size and shape, and noise level. The real data enable an assessment of robustness under practical measurement conditions.

2 Methods

The general methodology for inversion of magnetic microscopy data follows Souza-Junior et al. (2024) and Souza-Junior et al. (2026a). A summary of the workflow is as follows:

1. Calculate the x -, y -, and z -derivatives of the data.
2. Calculate the total-gradient amplitude (Blakely, 1996) and stretch its contrast to highlight weaker sources.
3. Detect the individual magnetic signals using a Laplacian of Gaussian method (Kong et al., 2013).
4. For each detected signal:
 - (a) Isolate the data for that source.
 - (b) Estimate the position of the source using Euler Deconvolution (Reid et al., 1990).
 - (c) Estimate the magnetic moment through a linear inversion.
 - (d) Perform a non-linear inversion to refine the source position and magnetic moment estimates.
 - (e) Forward model the signal and remove it from the entire dataset.

The main change introduced here to the methodology above is the replacement of the linear and non-linear dipole inversions with spherical harmonic counterparts to account for multipolar components in the data.

2.1 Spherical-harmonic inversion of b_z

Based on the conventional spherical-harmonic (SH) representation (Lima et al., 2023), we construct an SH expansion that reproduces the measured vertical field component b_z on the observation surface. An SH expansion is valid provided that all magnetic sources lie inside a reference sphere of radius a , regardless of their geometry or exact location. However, the choice of expansion center strongly influences the conditioning and accuracy of the fit. Here we propose an alternative and more flexible strategy to choose the expansion center. Following Souza-Junior et al. (2024) and Souza-Junior et al. (2026a), we use Euler Deconvolution (Reid et al., 1990) to estimate the location of the source under a dipolar approximation, which is then used as the expansion center $\mathbf{p} = (x_c, y_c, z_c)$.

We first define a Cartesian system (x, y, z) that is aligned with the magnetic microscopy image, with the z -axis pointing upward. The origin of the system is the lower left corner of the sample. Given

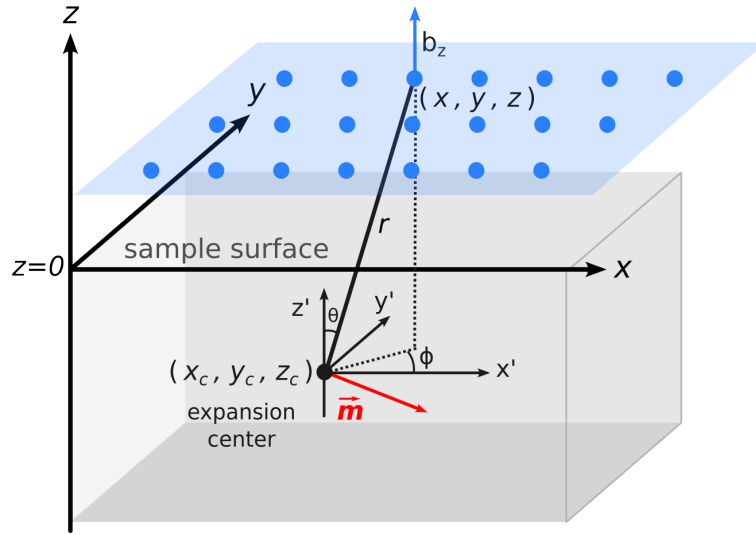


Figure 1: Schematic diagram of the measurement geometry and coordinate systems. The Cartesian coordinate system (x, y, z) is established with its origin at the sample surface ($z = 0$). Measurement data points (blue dots) are acquired on a planar grid at a constant elevation ($z > 0$) above the surface. An expansion center is located at depth within the sample volume, defining a translated local coordinate system (x', y', z') . The red vector $\vec{m}$ represents the source vector (e.g., dipole moment) at this center. The spatial relationship between the expansion center and an arbitrary data point is described by the position vector $\mathbf{r}$ and the spherical angles θ (colatitude) and ϕ (longitude).

an expansion center at (x_c, y_c, z_c) , we can define a spherical coordinate system (r, θ, ϕ) with origin on the expansion center (Figure 1). Here, r is the radial coordinate, θ is the colatitude (angular distance from the z -axis), and ϕ is the longitude or azimuth (angular distance from the x -axis). The spherical harmonic expansion of the vertical component of the magnetic field b_z is

$$b_z(r, \theta, \phi) = \mu_0 \sum_{n=1}^L \left(\frac{a}{r}\right)^{n+2} \sum_{m=0}^n \left[(n+1)P_n^m(\cos \theta) \cos \theta + \frac{\partial P_n^m}{\partial \theta}(\cos \theta) \sin \theta \right] (g_n^m \cos m\phi + h_n^m \sin m\phi), \quad (1)$$

in which μ_0 is the vacuum magnetic permeability, n is the degree, m is the order, L is the maximum degree of the expansion, a is the reference radius, P_n^m are the Schmidt-normalized associated Legendre functions, and g_n^m and h_n^m are the Gauss coefficients of the expansion. The P_n^m and their derivatives are calculated with recursion relations (Holmes and Featherstone, 2002), as implemented in the *Harmonica* library (Fatiando a Terra Project et al., 2024).

Defining a observed data vector $\mathbf{d}^o$ containing the N observations of b_z and a parameter vector $\mathbf{c}$ as

$$\mathbf{c} = \begin{bmatrix} g_1^0 & g_1^1 & g_2^0 & g_2^1 & \cdots & g_L^L & h_1^1 & h_2^1 & h_2^2 & h_3^1 & \cdots & h_L^L \end{bmatrix}_{M \times 1}^T, \quad (2)$$

we can formulate the forward problem in Equation 1 as a linear system $\mathbf{d} = \mathbf{G}\mathbf{c}$. The Jacobian matrix $\mathbf{G}$ has columns corresponding to the cosine- and sine-type basis functions associated with the Gauss coefficients g_n^m and h_n^m and is defined as

$$\mathbf{G} = \begin{bmatrix} K_{1,1}^0 & K_{1,1}^1 & K_{2,1}^0 & K_{2,1}^1 & \cdots & K_{L,1}^L & J_{1,1}^1 & J_{2,1}^1 & J_{2,1}^2 & \cdots & J_{L,1}^L \\ K_{1,2}^0 & K_{1,2}^1 & K_{2,2}^0 & K_{2,2}^1 & \cdots & K_{L,2}^L & J_{1,2}^1 & J_{2,2}^1 & J_{2,2}^2 & \cdots & J_{L,2}^L \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ K_{1,N}^0 & K_{1,N}^1 & K_{2,N}^0 & K_{2,N}^1 & \cdots & K_{L,N}^L & J_{1,N}^1 & J_{2,N}^1 & J_{2,N}^2 & \cdots & J_{L,N}^L \end{bmatrix}_{N \times M}, \quad (3)$$

in which $K_{n,i}^m$ and $J_{n,i}^m$ are

$$K_{n,i}^m = \left(\frac{a}{r_i} \right)^{n+2} \left[(n+1)P_n^m(\cos \theta_i) \cos \theta_i + \frac{\partial P_n^m}{\partial \theta}(\cos \theta_i) \sin \theta_i \right] \cos m\phi_i, \quad (4)$$

$$J_{n,i}^m = \left(\frac{a}{r_i} \right)^{n+2} \left[(n+1)P_n^m(\cos \theta_i) \cos \theta_i + \frac{\partial P_n^m}{\partial \theta}(\cos \theta_i) \sin \theta_i \right] \sin m\phi_i. \quad (5)$$

The parameter vector $\mathbf{c}$ is then obtained through a damped linear least-squares inversion (zero-order Tikhonov regularization). This approach estimates $\mathbf{c}$ by minimizing the objective function

$$\Phi(\mathbf{c}) = \|\mathbf{d}^o - \mathbf{G}\mathbf{c}\|_2^2 + \lambda \|\mathbf{c}\|_2^2 \quad (6)$$

in which λ is the positive regularization parameter. However, the columns of the Jacobian $\mathbf{G}$ can have very different scales as the degree n increases, leading to instability of the inverse problem. To counteract this, we normalize the Jacobian matrix using a diagonal matrix $\mathbf{S}$, whose diagonal elements $S_{ii} = \sigma_i$ are the standard deviation of the i -th column of $\mathbf{G}$ (Soler and Uieda, 2021). The objective function is transformed to

$$\Phi(\mathbf{w}) = \|\mathbf{d}^o - \underbrace{\mathbf{G}\mathbf{S}^{-1}}_{\mathbf{D}} \underbrace{\mathbf{S}\mathbf{c}}_{\mathbf{w}}\|_2^2 + \lambda \|\mathbf{w}\|_2^2 \quad (7)$$

The optimal normalized coefficient vector $\mathbf{w}$ is obtained by solving the corresponding normal equations

$$(\mathbf{D}^T \mathbf{D} + \lambda \mathbf{I})\mathbf{w} = \mathbf{D}^T \mathbf{d}^o \quad (8)$$

where $\mathbf{I}$ is the identity matrix. Once an estimate of $\mathbf{w}$ is obtained, we calculate the spherical harmonic Gauss coefficients as $\mathbf{c} = \mathbf{S}^{-1}\mathbf{w}$.

Another advantage of the normalization of the Jacobian matrix is that the range of acceptable values for the regularization parameter λ will be more predictable since it won't depend on the model and data dimensions as strongly. Here, we use a regularization parameter of $\lambda = 10^{-1}$, which was empirically selected as the value that best preserves the benchmark results obtained from the micromagnetic particle simulations (see Section 3).

Once the Gauss coefficients have been estimated, the Cartesian components of the magnetic moment (in Am^2) of a dipolar source located at the expansion center (x_c, y_c, z_c) are calculated directly from the first-degree coefficients. Following Blakely (1996),

$$\mathbf{m} = \begin{bmatrix} m_x \\ m_y \\ m_z \end{bmatrix} = 4\pi\mu_0 a^3 \begin{bmatrix} g_1^1 \\ h_1^1 \\ g_1^0 \end{bmatrix} \quad (9)$$

in which a is the reference radius that contains the magnetic source. For our purposes, it is safe to assume that the source is contained in a circle of radius equal to the minimum radial coordinate r_i of the data.

2.1.1 Nonlinear Position Refinement

The estimated Gauss coefficients and the corresponding magnetic moment $\mathbf{m}$ are strongly dependent on the choice of expansion center (x_c, y_c, z_c) . It is also known that Euler Deconvolution is sensitive to random noise in the data and the presence of interfering sources in the data window (Silva et al., 2001; Uieda et al., 2025). Thus, the estimated magnetic moment based solely on the Euler Deconvolution and spherical harmonics results will likely be biased.

To refine the estimate, we employ a non-linear inversion method similar to what Souza-Junior et al. (2026a) did for dipole inversion. We refine the expansion center estimate by minimizing the mean-squared error of the b_z prediction using the Nelder-Mead derivative-free optimization method (Nelder and Mead, 1965; Virtanen et al., 2020). At each candidate center, the spherical coordinates (r, θ, ϕ) of all data points are recomputed, the design matrix G is rebuilt, and the coefficients are re-estimated. This refinement is more computationally demanding, but it substantially improves stability and accuracy when the sources are clustered in the microscopy image or are asymmetric. The Euler Deconvolution and linear spherical harmonic inversion results are used as initial estimates for the non-linear Nelder-Mead optimization.

2.2 Performance metrics

To enable a systematic comparison between the tested inversion strategies for the synthetic models, a set of complementary misfit metrics was adopted to evaluate the recovery of the main source parameters. These metrics quantify the accuracy of the estimated magnetic moment direction and intensity and the source position relative to their known true values. The directional accuracy was measured by the angular misfit:

$$\Delta\alpha [^\circ] = \arccos\left(\frac{\hat{\mathbf{m}} \cdot \mathbf{m}^{\text{true}}}{\|\hat{\mathbf{m}}\| \|\mathbf{m}^{\text{true}}\|}\right), \quad (10)$$

where $\hat{\mathbf{m}}$ denotes the recovered dipole-moment vector, $\mathbf{m}^{\text{true}}$ the corresponding true dipole-moment vector. The relative dipole-moment misfit quantified the intensity accuracy:

$$\Delta m [\%] = 100 \frac{|\|\hat{\mathbf{m}}\| - \|\mathbf{m}^{\text{true}}\||}{\|\mathbf{m}^{\text{true}}\|}, \quad (11)$$

Finally, we assessed the position misfit, defined as the Euclidean distance between the true source centroid $\mathbf{p}^{\text{true}}$ and recovered expansion center coordinates $\hat{\mathbf{p}} = (x_c, y_c, z_c)$:

$$\Delta r [\mu m] = \|\hat{\mathbf{p}} - \mathbf{p}^{\text{true}}\|. \quad (12)$$

3 Synthetic Model

In this section, we evaluate the effectiveness of the SH method using a simulated realistic dataset of idealized shaped sub-micron magnetite (Fe_3O_2) particles. A total of 20 particle models were considered, comprising four shape classes: spheres, prolate and oblate ellipsoids, and euهدral cuboctahedral particles. For the spherical particles, the radius ranged from 0.025 to 0.5 μm . Prolate ellipsoids semi-axes were defined with $a > b = c$, while oblate ellipsoids follow $a = b > c$, with semi-axis ratios of 2:1 between the major and minor axes. In both cases, the characteristic size was varied over the same range as for the spheres radii. For the euهدral cuboctahedral particles, geometric parameters are controlled by a maximum elongation parameter E_{max} , which varies from 0.05 to 1.

To compute the stray magnetic fields, we first determined the Local Energy Minimum (LEM) domain states for each particle using the open-source micromagnetic solver MERRILL (Micromagnetic Earth Related Robust Interpreted Language Laboratory, Ó Conbhuí et al., 2018). Each particle geometry was discretized into linear tetrahedral finite elements, and the total magnetic free energy was minimized. For every particle, 100 independent minimizations were performed, and the thermodynamically most probable configuration (i.e., the lowest-energy state) was selected as the representative LEM. We obtained a variety of domain-states, in a way that we observe a transition from single-domain (SD), to single-vortex (SV), up to multivortex (MV) as the particle sizes increase. The forward stray field was then computed using a tetrahedral integration approach (Bellon et al., 2026), adapted from Fabbri (2008). This method exploits the tetrahedral discretization used in MERRILL by evaluating the magnetic field contribution of each uniformly magnetized element based solely on its geometry and magnetization direction.

Before applying the SH framework to planar observations, we evaluated its performance under ideal conditions by fitting the micromagnetically predicted stray field on a spherical shell enclosing each particle. The shell was defined at a constant radial offset from the particle, corresponding to 5% of its largest semi-axis, and only the vertical component b_z was considered to remain consistent with QDM measurements. In this configuration, the field is sampled over a closed surface, satisfying the orthogonality conditions required for spherical harmonic expansions.

SH fits up to degree 10 reproduce the synthetic stray field with negligible residuals (Figure 2), with no significant improvement observed for higher degrees. A damping factor of 10^{-1} was adopted, as it provides a stable solution while preserving the fidelity of the recovered coefficients. Together, this choice of maximum degree and damping was found to yield consistent results in benchmark tests, in which the dipole moment estimated from the SH coefficients agrees with the total magnetic moment computed directly from the micromagnetic models (i.e., the sum of the m_x , m_y , and m_z contributions over all tetrahedral elements from MERRILL). This demonstrates that the resulting coefficients accurately capture the natural spatial decay of the field and provides a reference benchmark for subsequent analyses on planar data, where these ideal conditions are not met.

3.1 Micromagnetic benchmarking: dipolar vs multipolar approaches

To compare the effectiveness of the SH approach in a magnetic microscope setting, the stray field of all the 20 particles was predicted on a two-dimensional grid with sensor–source distances ranging from 0.05 to 10.0 μm , sampled at discrete heights. We evaluate the performance of dipolar and spherical harmonic inversion approaches, considering both linear and nonlinear formulations, across a range of particle geometries and characteristic scales. The dipolar models followed the approaches presented by Souza-Junior et al. (2024) and Souza-Junior et al. (2026a), respectively, for linear and nonlinear

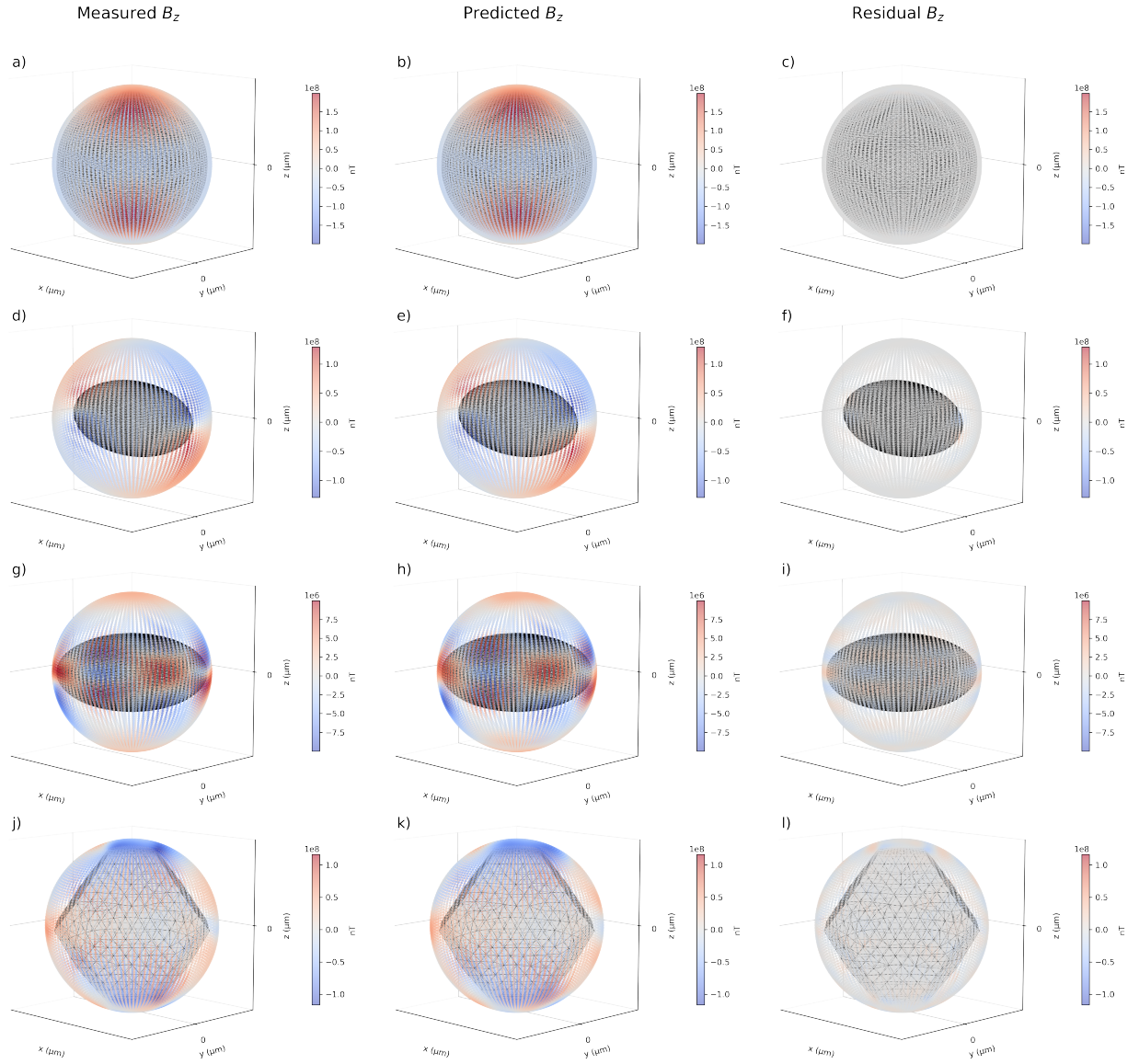


Figure 2: Micromagnetic forward models used as benchmark datasets for evaluating the inversion approaches across all tested particle geometries. For each case, MERRILL simulations produced B_z on a spherical shell (left), the shell data were fit with real SH coefficients (middle), and residuals are shown on the right.

cases.

The analysis includes the aforementioned geometries with variations in effective size, allowing a systematic assessment of model robustness under increasing geometric complexity, although without noise. To quantify the inversion accuracy by benchmarking against the reference micromagnetic model, we evaluated the angular, magnetic moment, and spatial position misfits (see Section 2.2).

Figure 3 summarizes the results. Each column corresponds to a different particle geometry or scaling condition, while rows represent distinct metrics used. A consistent pattern across all configurations is the improved performance of nonlinear inversions relative to their linear counterparts. Nonlinear methods yield results closer to the reference values, indicating a more accurate recovery of source properties. This behavior is particularly evident in the directional and intensity misfits, where linear solutions exhibit larger variability and systematic deviations. Comparing model parameterizations, spherical harmonic approaches exhibit greater stability than dipolar approximations, particularly

for sensor heights near the particles. While the performance of dipolar models degrades significantly for near-superficial, elongated, and/or flattened geometries. Both approaches provide reasonable estimates for distal sources, considering the attenuation of high-order multipole effects.

The influence of particle geometry is clearly observed across all panels. As the aspect ratio increases, both dipolar and SH solutions exhibit larger misfits. Although the effect is substantially more pronounced for dipolar models. In contrast, SH inversions maintain relatively stable behavior, reflecting their greater flexibility in representing more complex magnetic field distributions. Especially in the SH nonlinear, which yielded lower values for angular and magnetic moment misfits, where most results for all geometries and grain sizes fall below the $1^\circ/1\%$ threshold (dashed black line). The lower panels (Figure 3) reveal systematic trends in the position misfit that further highlight differences between methods. These results mostly remained under the $1.0 \mu m$ across all approaches, except for the two largest grain sizes, for which the position error increased proportionally with sensor height. On the other hand, for the SH nonlinear approach these higher misfits in position showed no significant negative influence in both directional and intensity estimations.

Overall, the results demonstrate that the inversion strategy play a critical role in accurately recovering magnetic source properties. Nonlinear approaches are more consistent. However, only the SH counterpart showed reliable estimates across all tested scenarios, while nonlinear dipolar models are prone to significant systematic errors at near particles surface distances, and when applied bigger and/or non-idealized source geometries (prolates and oblates).

3.2 Evaluation of noise level effects

Simulating the noise level is one of the key pieces needed to complete the puzzle of a “realistic scenario”. Since the nonlinear approaches yielded more stable solutions (see Section 3.1), these methods were applied using two distinct fixed noise levels (20 and 100 nT, both values reported in Glenn et al., 2017). Instead of the realistic QDM noise described by Bellon et al. (2026), high-frequency zero-mean Gaussian noise was adopted, as the window size of the data is much smaller than a typical QDM field of view. Then, the fixed-noise forward models were calculated for the intermediate particle size of each modeled geometry: sphere ($R = 0.250 \mu m$), prolate ($a = 0.250 \neq b = c = 0.125 \mu m$), oblate ($a = b = 0.250 \neq c = 0.125 \mu m$), and euhedral ($E_{max} = 0.5 \mu m$).

The results are summarized in Figure 4. Each column corresponds to a different particle geometry, while the rows represent distinct misfit metrics. The simulations show consistent behavior for both directional and intensity misfits: for sensor heights $\leq 1 \mu m$, the SH approach performs significantly better than the dipolar model. In some cases, SH results at higher noise levels (100 nT, solid red line) are more accurate than those for the dipolar approach at lower noise levels (20 nT, dashed blue line). This discrepancy decreases markedly for heights $\geq 1 \mu m$ as the higher-order signals are attenuated, but only over a limited range (around $2 \mu m$). As the distance increases further, magnetic field attenuation becomes dominant, causing the fixed noise level to be too high for reliable results. The dotted vertical lines indicate the height at which the noise exceeds 10% of the maximum magnetic anomaly amplitude. This threshold also marks the point beyond which both approaches yield unreliable results due to a lower signal-to-noise ratio. The position misfits remained nearly identical for both approaches across the full range of heights.

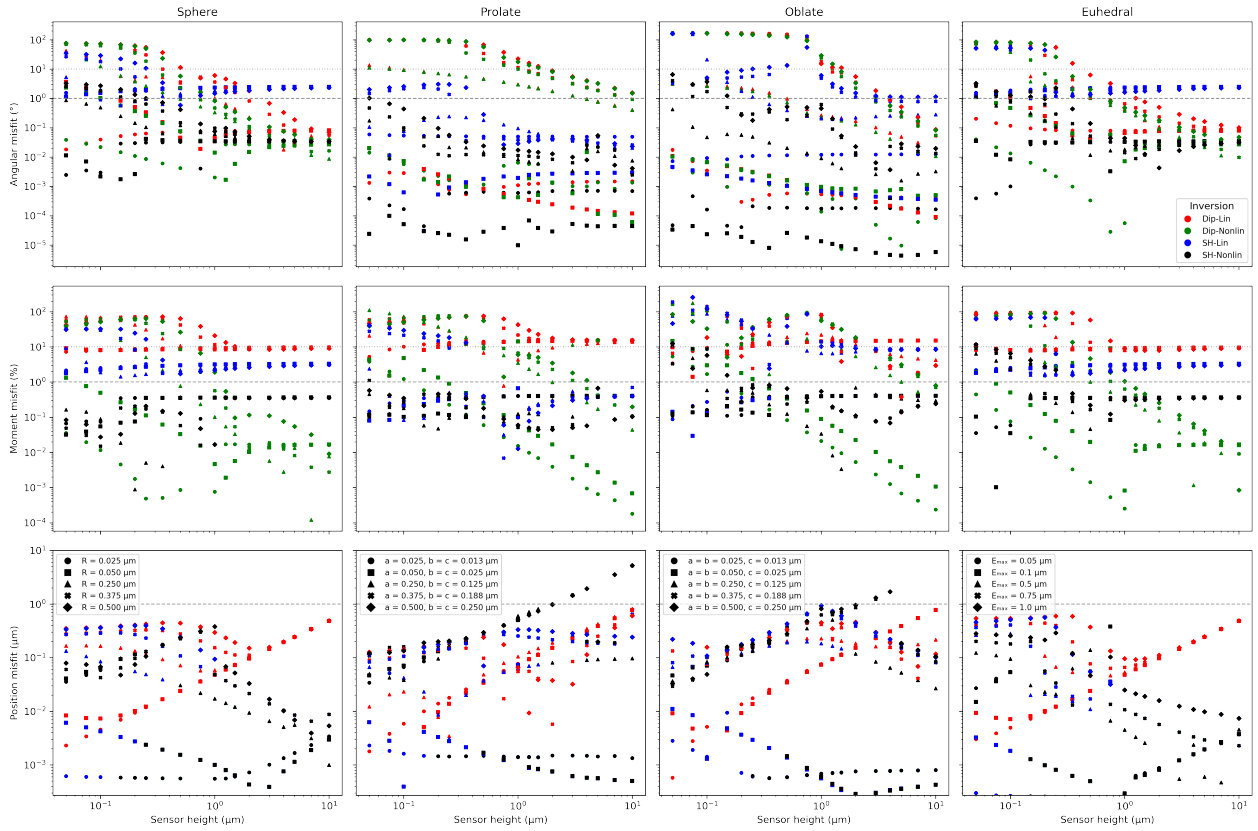


Figure 3: Comparison of inversion performance metrics obtained from synthetic micromagnetic models for different source geometries, particle sizes, and inversion strategies as a function of sensor-to-sample distance. Columns correspond to spherical, prolate, oblate, and irregular particle geometries (left to right), while rows show angular, magnetic moment, and position misfits (top to bottom). Colors indicate the inversion method: linear dipole (Dip-Lin, red), nonlinear dipole (Dip-Nonlin, green), linear SH (SH-Lin, blue), and nonlinear SH (SH-Nonlin, black). Marker symbols represent different particle sizes or characteristic dimensions for each geometry, as indicated in the panel legends. Horizontal dashed lines indicate reference thresholds used for performance assessment. All metrics are shown as a function of increasing sensor height.

4 Real Data Application

After assessing the reliability of the SH approach with the realistic synthetic scenario, the next step involves the application in real magnetic microscopy measurements. The dataset used in this section was obtained using the Ensaio software (Fatiando a Terra Project et al., 2025). It consists of a high-resolution magnetic map of a stalagmite sample from Wintimdouine Cave, Morocco (Carmo et al., 2019). The data were collected to investigate the magnetic remanence properties of hematite and magnetite within the sample. The isothermal remanent magnetization (IRM) of the sample was mapped using a QDM at the Harvard Paleomagnetism Lab, Department of Earth and Planetary Sciences, Harvard University. The IRM was imparted by applying magnetic fields of 2.7 T in the +y direction and 0.3 T in the −y direction. The QDM data were measured with a bias magnetic field of 0.9 mT, which was applied during the measurements and periodically reversed, and a background field of approximately 400 nT, as described by Hess et al. (2024). The data were acquired in projected magnetic microscopy (PMM) mode and converted to the vertical component of the magnetic field (b_z) using a spectral approach (Fu et al., 2020; Glenn et al., 2017; Lima and Weiss, 2009), which was processed with the aid of the QDMLab software (Volk et al., 2022). The imaged area covers $1410 \mu\text{m} \times 2256 \mu\text{m}$, with a grid spacing of $2.35 \mu\text{m}$,

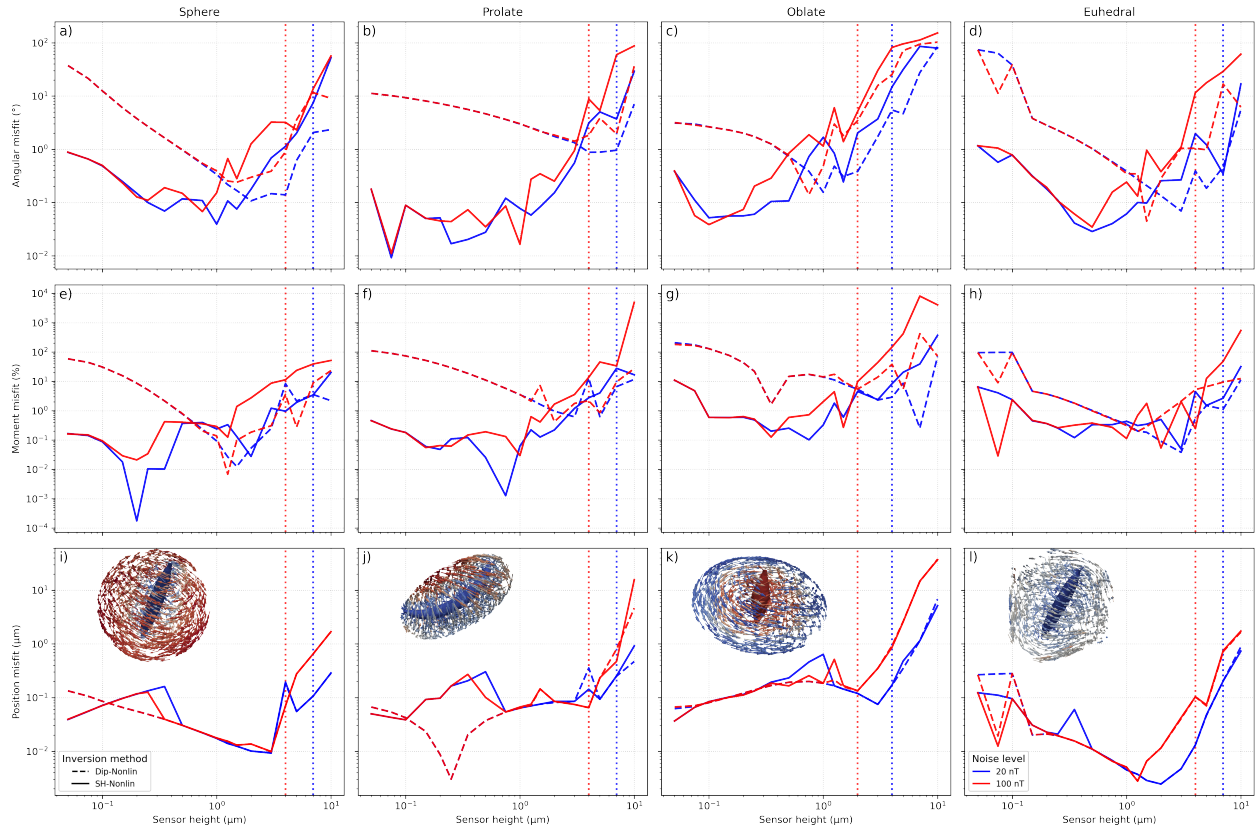


Figure 4: Comparison of inversion performance metrics obtained from synthetic micromagnetic models with fixed noise levels for different source geometries and inversion strategies as a function of sensor-to-sample distance. Columns correspond to spherical, prolate, oblate, and irregular particle geometries (left to right), while rows show angular, magnetic moment, and position misfits (top to bottom). Line colors indicate the noise level: 20 nT (blue) and 100 nT (red), while line styles represent the inversion method: dipolar nonlinear (Dip-Nonlin, dashed) and SH nonlinear (SH-Nonlin, solid). Micromagnetic structures corresponding to the intermediate particle size used for each geometry are shown as insets in the bottom row. Vertical dotted lines indicate the sensor height at which the noise level exceeds 10% of the maximum magnetic anomaly amplitude for each noise scenario. All metrics are shown as a function of increasing sensor height.

corresponding to approximately 5.76×10^5 measurement points. Data acquisition was conducted at a constant sensor-sample distance of $5 \mu\text{m}$, enabling high-resolution mapping of the magnetic field distribution.

A slice of the QDM data was selected for comparison between both nonlinear approaches, as shown in Figure 5. The data processing used to estimate the magnetic moment vector of each identified anomaly followed the workflow presented by Souza-Junior et al. (2026a), which consists of semi-automatic anomaly window selection, nonlinear inversion of the anomaly source, and forward modeling of its signal to subtract its influence from less pronounced neighboring anomalies. The same workflow was subsequently applied to the nonlinear SH approach.

Since the data were acquired using QDM at a sensor height of $5 \mu\text{m}$ (Figure 5a), substantial differences between the approaches are not expected. At this distance, higher-order signal contributions are already strongly attenuated for most particles, so the measured field is predominantly controlled by the lower-order terms, except in the case of larger grains (Bellon et al., 2026). This expectation is consistent with the results, as both approaches produced similar estimates and residual maps (Figure 5c and f) and magnetic moment (Figure 5d). However, minor discrepancies were observed, either for the larger anomaly, where residual higher-order contributions occurred, or for weaker anomalies that

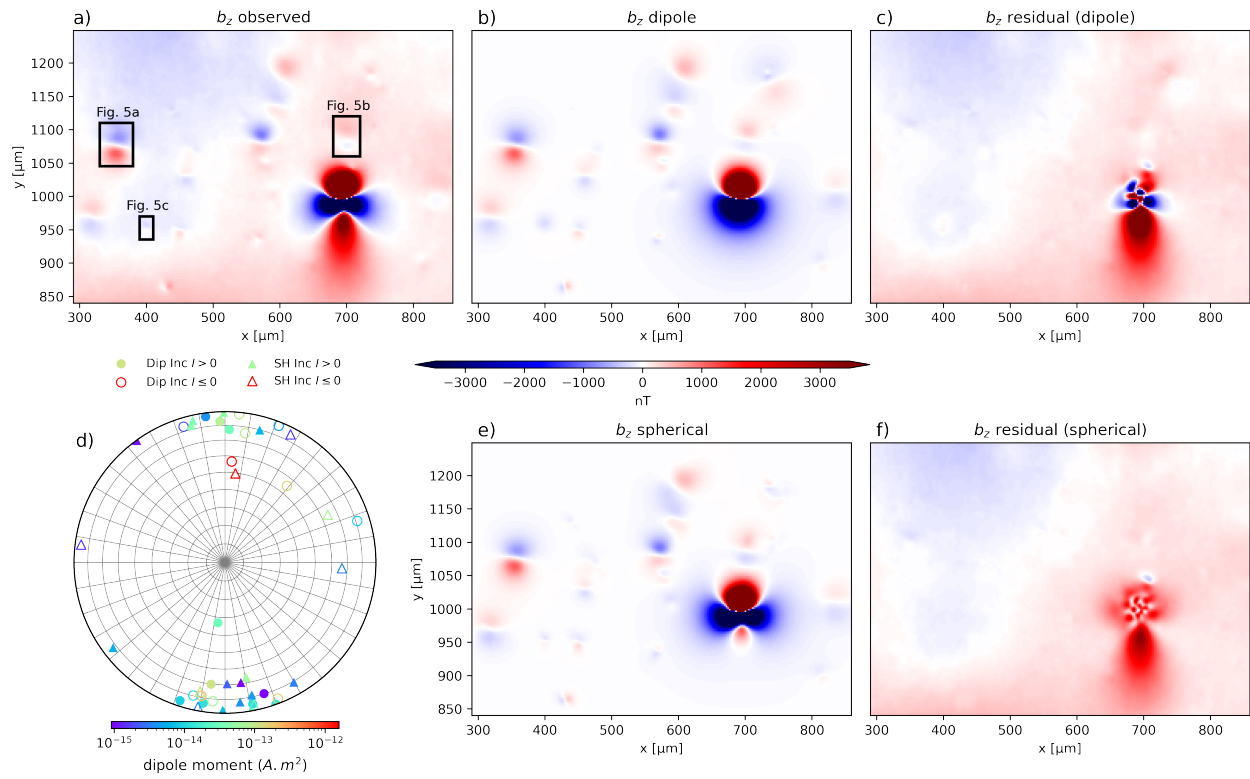


Figure 5: Comparison between nonlinear dipolar and spherical harmonic (SH) inversion results for real QDM data. Panel (a) shows the observed magnetic field and the selected zoomed-in anomaly windows (Figure 6). Panels (b) and (c) correspond to the nonlinear dipolar inversion result and its residuals. Panel (d) shows the stereographic projection with the recovered magnetic moment directions for both approaches, while panels (e) and (f) show the predicted b_z field and residuals for the nonlinear SH approach.

were highly affected by background magnetic shifts, noise levels, or influenced by the stringer sources. These conditions may still influence the inversion.

To further illustrate the behavior of both inversion strategies when applied to real QDM data, three selected anomalies were examined in greater detail through zoomed-in views of representative cases (Figure 6). These examples allow a direct visual comparison between the observed magnetic field patterns and the corresponding forward responses predicted by the nonlinear dipole and spherical harmonic approaches, as well as their recovered magnetic moment directions.

The first case (Figure 6a) corresponds to a strongly dipolar and well-isolated anomaly, used as a reference example for which both methods are expected to yield similar results. Consistent with this expectation, both approaches reproduced the observed dipolar pattern with high fidelity, yielding comparable source locations and magnetic moment directions. The second case (Figure 6b) represents an anomaly affected by strong external interference, located within a region of positive field offset and superimposed on the tail of the strongest anomaly in the selected slice. In this more complex setting, differences between the modeled responses become more evident, although the SH approach is more consistent with the direction of the imparted IRM. The third case (Figure 6c) corresponds to a weak and small-amplitude anomaly, providing a more challenging low-signal scenario. In this case, the dipolar approach struggled due to the weaker signal and the limited spatial extent of the anomaly, while the SH provides coherent estimates for both position and magnetic moment, as evidenced by the similar observed and predicted maps. Taken together, these examples indicate that both approaches perform similarly for well-behaved isolated anomalies, whereas the observed discrepancies suggest a

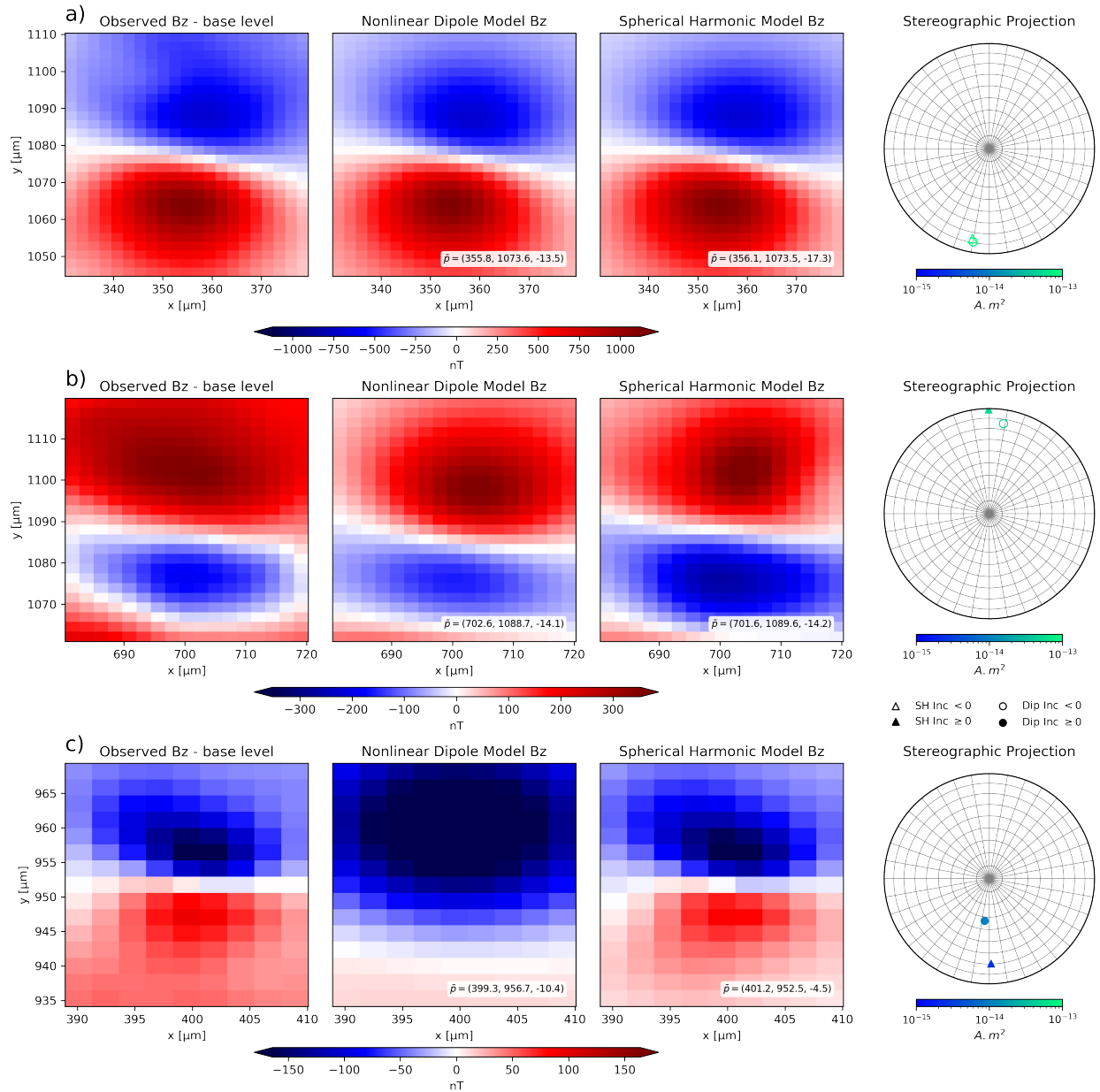


Figure 6: Detailed comparison between nonlinear dipolar (Dip) and spherical harmonic (SH) inversion results for three representative anomalies extracted from the real QDM dataset. Rows correspond to three selected cases representing distinct signal characteristics: (a) an isolated dipolar anomaly, (b) an anomaly influenced by external magnetic contributions, and (c) a weak low-amplitude anomaly. Columns show the observed magnetic field, the nonlinear dipolar predicted b_z field, the nonlinear SH predicted b_z field, and the corresponding stereographic projections with the recovered magnetic moment directions for both approaches. Estimated source positions are indicated within the predicted field maps.

slightly more refined estimate from the SH approach, which could become increasingly significant as the sensor height decreases.

5 Discussions

By incorporating higher-order magnetic moments, multipole fitting provides additional degrees of freedom to capture the magnetic complexity of remanence carriers. This approach is particularly

valuable in high-resolution magnetic maps that resolve quadrupolar and higher-order features, or when modeling non-uniform and clustered magnetic sources.

For the ground-truth models, the superior performance of SH models observed in our experiments is a direct consequence of their ability to capture the stray fields generated by non-spherical geometries and complex internal magnetic structures. The near-surface stray fields of these models reveal that for grains with significant shape anisotropy or those exhibiting vortex state domains, the dipolar term accounts for only a fraction of the total magnetic signal. The SH inversion is more capable of fitting the higher-order magnetic signals without biasing the degree one coefficients, leading to the significantly lower residuals reported in our benchmarks. Thus, this performance of the tested inversion strategies depends primarily on sensor height and source geometry. For most simulated cases, both nonlinear approaches converged to similar solutions at larger sensor-to-sample distances, whereas discrepancies became more evident for larger grains and at lower heights. This behavior is consistent with the rapid distance-dependent attenuation of higher-order magnetic field components (Bellon et al., 2026; Cortés-Ortuño et al., 2022). As the observation height increases, the external field progressively approaches a dipolar signature, reducing the practical advantage of more flexible source representations.

However, this behavior should not be generalized to lower sensor heights, where higher-order terms become increasingly relevant. The comparison using real QDM data broadly supports this interpretation. For isolated and well-defined anomalies, both approaches yielded similar solutions, whereas more noticeable differences emerged for weak or externally perturbed signals. Therefore, under standard QDM acquisition conditions, the choice between the two nonlinear approaches may have a limited practical impact for most particle sizes. Although experimental data introduce additional sources of uncertainty not present in the synthetic tests, the same general trends were observed.

5.1 Euler Deconvolution as a Strategic Seed for Expansion Centers

The efficacy of SH expansions in magnetic microscopy strongly depends on the accurate definition of the source expansion center. The ED solution provides first-order estimates of the source coordinates (x_c, y_c, z_c) without requiring prior information regarding particle shape or size. By independently constraining source location before coefficient estimation, the original highly nonlinear optimization problem is transformed into a substantially better-conditioned inversion problem. This separation reduces the number of free parameters during coefficient estimation and improves convergence stability for local planar inversions.

In addition, the Euler-derived estimate provides an efficient starting point for subsequent nonlinear refinements. When required, the initial location can be further optimized using derivative-free search algorithms such as the Nelder-Mead method (Nelder and Mead, 1965), minimizing local residual RMS values. As demonstrated by the micromagnetic models, this refinement significantly improves solution accuracy and stability, particularly for non-ideal source geometries. Together, these steps create a flexible framework capable of extracting complex magnetic signatures directly from measured anomalies while relying only on a single measurement plane. Consequently, higher-order magnetic characterization can be performed without dependence on auxiliary imaging techniques.

Despite these advantages, some limitations remain. Higher-order spherical harmonic terms become progressively more sensitive to high-frequency noise and positioning uncertainties, particularly at very low sensor-to-sample distances where short-wavelength magnetic components dominate the observed field. As noted by Lima et al. (2023), increasing the expansion degree beyond the information content of the data can lead to overfitting, where higher-order coefficients begin fitting measurement

noise rather than physically meaningful signal components.

The sensitivity of source position estimation is also partially inherited from the ED formulation itself, which relies on directional derivatives of the magnetic field and tends to affect vertical coordinate estimates more strongly. This issue can be mitigated through subsequent nonlinear optimization (Section 2.1.1) or through more robust approaches for determining expansion centers (e.g. Euler inversion, Uieda et al., 2025). Additionally, the intrinsic non-uniqueness of multipolar representations implies that different coefficient combinations may generate similar external magnetic fields. Therefore, the final solution may still depend on regularization choices, local anomaly isolation, and the selected maximum expansion degree.

Nevertheless, the ED-seeded approach remains computationally attractive and scalable for large magnetic microscopy datasets. Because the source location is independently constrained prior to coefficient estimation, the dimensionality of the nonlinear optimization problem is substantially reduced. This separation improves numerical stability and makes the method well-suited for the increasingly large and high-resolution datasets generated by modern magnetic microscopy systems.

5.2 Implications for High-Resolution Paleomagnetic Microscopy

Since the findings of Berndt et al. (2016), relatively few magnetic microscopy applications have been directed toward paleomagnetic studies because the number of magnetic grains required to recover statistically reliable paleodirections was considered prohibitively large for these techniques. Nearly a decade later, however, Bellon et al. (2025) demonstrated through micromagnetic modeling of single-vortex (SV) grains that the number of grains required may be orders of magnitude lower than previously assumed, ranging from several hundred to a few thousand grains for bias fields $\geq 2 \mu T$. These results substantially changed the perspective on the role of magnetic microscopy in paleomagnetism. If reliable directional information can be obtained from much smaller grain populations, the emphasis shifts from measuring large samples toward improving the characterization of individual grains and small particle assemblages.

This shift naturally motivates the development of magnetic microscopy systems with increasingly higher spatial resolution and progressively smaller sensor-to-sample distances, as observed in recent Quantum Scanning Microscopes (QSM) developments (e.g. Ahmed et al., 2026). Such improvements are desirable because they enable the magnetic signal of individual grains and local remanence structures to be resolved with greater fidelity. However, moving closer to the source also places measurements within the near-field regime, where higher-order magnetic contributions become increasingly significant, particularly for non-uniform grains and interacting particle assemblages (Bellon et al., 2026; Cortés-Ortuño et al., 2021, 2022). Under these conditions, purely dipolar approaches may no longer provide sufficiently accurate representations of the measured magnetic field. Since multipole terms of degree m scale with $1/r^{m+2}$, the contribution of higher-order components decays progressively faster with distance from the source. Consequently, lower-order terms dominate the far-field behavior, whereas near-field measurements retain substantial contributions from higher-order harmonics. These additional degrees of freedom are required to capture the short-wavelength spatial variations and complex stray-field structures associated with surface-proximal sources.

The framework proposed here directly addresses this emerging regime. By combining spherical harmonic expansions with Euler deconvolution-based estimates of the expansion center, complex magnetic signatures can be modeled with substantially greater stability and physical realism. This approach enables high-resolution magnetic microscopy data to be interpreted beyond the dipolar approximation while remaining computationally practical for large datasets. Consequently, these

developments may help bridge the gap between modern magnetic microscopy capabilities and quantitative paleomagnetic applications, particularly for the analysis of small magnetic carriers, complex remanence structures, and weakly magnetized geological materials.

6 Conclusions

This study evaluated a spherical harmonic expansion approach to retrieve full-vector information from bi-dimensional magnetic microscopy observations using Euler deconvolution as a strategy for constraining expansion centers. Synthetic experiments based on micromagnetic models demonstrated that the relative performance of the inversion approaches depends strongly on sensor-to-sample distance and source geometry. At larger distances, solutions progressively converge toward dipolar behavior due to the attenuation of higher-order magnetic contributions, whereas lower sensor heights emphasize complex magnetic structures that require richer source representations. The use of Euler deconvolution as an initial estimate for the expansion center substantially improves inversion conditioning and reduces the dimensionality. When combined with the nonlinear refinement, this approach stably retrieves multipolar source models directly from single observation plane magnetic anomalies. As magnetic microscopy systems continue moving toward higher spatial resolutions and smaller sensor-to-sample distances, measurements increasingly enter a near-field regime where higher-order magnetic signatures become significant components of the observed field. Under these conditions, approaches relying exclusively on dipolar assumptions may become insufficient. The approach presented here provides a physically consistent alternative capable of representing this increased complexity while remaining computationally tractable. These developments may contribute to expanding the application of magnetic microscopy to quantitative paleomagnetic studies, particularly for the investigation of small magnetic carriers, rare geological materials, and complex remanence structures.

7 Open research

The Python source code used to produce all results and figures presented here and the QDM magnetic microscopy data used in this study are available from Souza-Junior et al. (2026b), which can also be found on <https://github.com/compgeolab/micromag-spherical-harmonic> under the MIT (source code), CC-BY (text and figures), and CC-0 (data) licenses.

We made use of the following Python packages in this study: NumPy (Harris et al., 2020) for array manipulation and linear algebra; SciPy (Virtanen et al., 2020) for linear algebra routines and nonlinear optimization; scikit-image (Van der Walt et al., 2014) for image processing and blob detection; Matplotlib (Hunter, 2007) for generating figures and stereographic projections; Verde (Uieda, 2018) for grid generation and least-squares solutions; Harmonica (Fatiando a Terra Project et al., 2024) for upward continuation, magnetic data processing, and the computation of Schmidt semi-normalized associated Legendre functions and their derivatives; Choclo (Fatiando a Terra Project et al., 2022) for optimized forward-modeling kernel functions; Numba (Lam et al., 2015) for just-in-time compilation; and xarray (Hoyer and Hamman, 2017) for handling coordinate-aware multidimensional datasets.

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