

A particle representation of tropical shallow water waves

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Abstract

The linearized shallow water equations on a general β -plane reduce to a single meridional structure equation for the meridional velocity. We show that this equation is exactly the Schrödinger equation for a “charged” particle minimally coupled to a four-vector potential whose analogue magnetic field, $\mathbf{B}_W = -(\beta/c_g)\hat{\mathbf{x}}$, is uniform and westward. No *ad hoc* term is required: every term is generated by the minimal coupling. The analogue magnetic length is the equatorial Rossby radius of deformation, the Hermite index of the Matsuno solution is the Landau level index, and the Matsuno dispersion relation is recovered exactly as the Landau spectrum of the constructed Hamiltonian rather than merely resembling it. The zonal wavenumber is the canonical momentum, the gauge-invariant kinematic momentum being $k + \beta/2\omega$. Because the vector potential depends on frequency, the eigenvalue problem is implicit, and this is the origin of the Rossby branch.

21 **Introduction**

22 Wave phenomena in the atmosphere and ocean play an important role in many geophysical
23 processes such as energy transport, climate processes, and cloud formation. Among the most
24 important categories for synoptic and large scale phenomena are the shallow water waves, whose
25 wavelengths are much longer than the depth of the fluid; they have been studied in great detail and
26 are summarized in many textbooks¹⁻⁴. The tropical shallow water waves have received special
27 attention because the tropics are energy source regions of many atmospheric and oceanic
28 processes. Important discoveries concerning these waves have been made by many researchers⁵⁻
29 ¹⁹, and the observational picture has since been placed on a firm spectral footing²⁰ and reviewed²¹.

30 All previous studies have been concerned with the “wave” properties of these waves. For long-
31 term climatic studies their role can be hard to recognize: wave profiles are complicated, made
32 worse by reflection, refraction, and diffraction, and coupled with complex boundaries the resultant
33 signals are difficult to analyze, let alone to interpret. One feasible way to reduce such complexities
34 is to reformulate these waves into a system in which they can be regarded as particles, or at least
35 are associated with some particle properties. Unlike waves, which are “smeared out” in space and
36 time, particles are compact and usually easier to identify.

37 The present study is motivated by the wave–particle duality established by quantum physics early
38 in the twentieth century^{22,23}: perhaps, by a suitable formulation, waves in the atmosphere and ocean
39 can also be viewed as particles, and such a view will simplify their analysis. The bound electron
40 of the hydrogen atom illustrates the gain. Its wave description requires a probability density $\Psi^*\Psi$
41 that is already a complicated function of space, and the time series of that signal at a point would
42 be almost impossible to analyze; all of this complication disappears once the signal is recognized
43 as one single particle, from which many properties of hydrogen can be deduced.

The idea that geophysical wave problems may be illuminated by the methods of condensed matter physics has since been pursued independently and from a quite different direction. Delplace et al.²⁴ showed that the Kelvin and Yanai modes owe their existence to a topological invariant of the shallow water system — the first Chern number, of value two, of the bulk Poincaré wave bands — in close analogy with the protected edge states of a topological insulator²⁵. That work and the present one single out the same two modes by different routes.

Here we ask whether geophysical shallow water waves can be formulated as particles, and if so, what rules of behavior they obey. The analysis below indicates that these particles obey a Schrödinger equation with minimal coupling to a four-vector potential, just as charged quantum mechanical particles do. This paper develops and completes an argument first presented at the Eighth Conference on Atmospheric and Oceanic Waves and Stability²⁶, from which the construction and the observation that the resulting dispersion relation resembles the Landau spectrum of a free electron in a uniform magnetic field are taken. Three matters left open there are settled here: the gauge structure of the four-vector potential, which accounts for an *ad hoc* phase shift and shows two apparently different versions of the construction to be one physical system in two gauges; the status of the correspondence with the Landau problem, shown below to be an identity rather than an analogy, together with the meaning of the constant α , which was explicitly left for future study; and the physical interpretation of the particles. The four-vector potential is written down and verified here rather than derived; it is obtained from Hamilton's principle, by linearizing the Eulerian variational principle for rotating flow²⁷, in a companion paper²⁸.

64 Results

65 *The meridional structure equation*

66 On a general β -plane that is not necessarily centered on the equator, the linearized shallow water
67 equations reduce to a single meridional structure equation for the meridional velocity perturbation
68 v (Methods),

$$69 \quad \frac{d^2 v}{dy^2} + \left[\frac{\omega^2}{c_g^2} - k^2 - \frac{\beta k}{\omega} - \frac{(f_0 + \beta y)^2}{c_g^2} \right] v = 0 \quad (1)$$

70 where k is the zonal wavenumber, ω the angular frequency, f_0 the Coriolis parameter at the
71 reference latitude, and c_g the speed of long gravity waves. Equation (1), with the boundary
72 condition $v \rightarrow 0$ at $y \rightarrow \pm\infty$, poses an eigenvalue problem. Since $f_0 + \beta y = \beta(y - y_0)$ with $y_0 =$
73 $-f_0/\beta$, the last term in the bracket is $-(\beta/c_g)^2 (y - y_0)^2$, so the problem is that of a particle in a
74 parabolic well centered on the latitude at which the Coriolis parameter vanishes. The solution is

$$75 \quad v = H_n \left[\left(\frac{\beta}{c_g} \right)^{1/2} (y - y_0) \right] \exp \left[-\frac{\beta(y - y_0)^2}{2c_g} \right] \quad (2)$$

76 where H_n is the Hermite polynomial of order n and the normalization constant has been omitted,
77 Eq. (1) being homogeneous^{3,6}. The dispersion relation is

$$78 \quad \frac{\omega^2}{c_g^2} - k^2 - \frac{\beta k}{\omega} = \frac{(2n+1)\beta}{c_g} \quad (3)$$

79 with $n = 0, 1, 2, \dots$. On an equatorial β -plane, $f_0 = 0$ and $y_0 = 0$, and the oscillation is symmetric
80 about the equator; for a general β -plane the eigenfunctions are the same functions displaced to
81 $y = y_0$, and the dispersion relation is unchanged. The trapped solutions are of practical interest
82 chiefly in the tropics, because for a mid-latitude reference the turning latitude y_0 lies outside the
83 range over which the β -plane approximation is defensible.

84 ***Failure of a harmonic-oscillator representation***

85 Matsuno⁶ noticed the similarity between Eq. (1) and the quantum mechanical problem of a particle
86 bound in a harmonic potential, and one may be tempted to build a particle formulation directly on
87 the Schrödinger equation for a harmonic oscillator. We first show that this is unsatisfactory. If a
88 Schrödinger equation can be written down whose wave amplitude is the v of Eq. (1), it is justified
89 to declare that the corresponding particle exists, even though we do not yet know what kind of
90 particle it is; its behavior must then be interpreted consistently with quantum mechanics.

91 Consider the case in which the waves represent the amplitude of a particle moving in a simple
92 harmonic oscillator potential, which is proportional to $(\beta/c_g)^2(y - y_0)^2$. The quantum
93 mechanical operators for this particle are

$$94 \quad E = i \frac{\partial}{\partial t} , \quad \mathbf{p} = -i\nabla , \quad H = \alpha(-i\nabla)^2 + \alpha \left(\frac{\beta}{c_g} \right)^2 (y - y_0)^2 \quad (4)$$

95 where α plays the role of $\hbar^2/2m$. Since we are *not* dealing with elementary particles here, there is
96 no reason to expect $\hbar$ and m to retain their usual meanings, and it is convenient to absorb them
97 into the single constant α , which carries the dimensions of $\hbar/m$ (Methods). Taking the plane wave
98 function

$$99 \quad \Psi(\mathbf{r}, t) = v(y) \exp \left[ikx - i\alpha \left(\frac{\omega^2}{c_g^2} \right) t \right] \quad (5)$$

100 in which the energy term $E/\hbar$ has been written explicitly as $\alpha(\omega^2/c_g^2)$, the energy of a plane wave
101 of angular frequency ω , and substituting it into the Schrödinger equation with the operators of Eq.
102 (4), we obtain

$$103 \quad \frac{d^2v}{dy^2} + \left[\frac{\omega^2}{c_g^2} - k^2 - \left(\frac{\beta}{c_g} \right)^2 (y - y_0)^2 \right] v = 0 \quad (6)$$

104 Comparing Eq. (6) with Eq. (1), the term $\beta k/\omega$ is missing. To repair this we may change the wave
 105 function of Eq. (5) to

$$106 \quad \Psi(\mathbf{r}, t) = v(y) \exp \left[i \left(k + \frac{\beta}{2\omega} \right) x - i\alpha \frac{\omega^2}{c_g^2} t \right] \quad (7)$$

107 and at the same time rewrite the harmonic oscillator potential as

$$108 \quad V = \alpha \left[\left(\frac{\beta}{c_g} \right)^2 (y - y_0)^2 - \frac{\beta^2}{4\omega^2} \right] \quad (8)$$

109 With these two changes Eq. (1) is recovered exactly. But in writing Eqs. (7) and (8) we have
 110 asserted *a priori* that the zonal wavenumber must be shifted by $\beta/2\omega$ — equivalently, that the
 111 wave phase must be shifted by $\exp(i\beta x/2\omega)$ — and that the potential must be lowered by
 112 $\alpha\beta^2/4\omega^2$. A potential may always be shifted by an arbitrary constant without changing the
 113 physics, but there is no obvious justification for the phase shift; as it stands it is no more than an
 114 *ad hoc* manipulation. It would be far more satisfactory to have these quantities specified by the
 115 operators, that is, by the external conditions in which the particle moves, rather than inserted by
 116 hand into the wave function.

117 ***A charged particle in a four-vector potential***

118 We now derive Eq. (1) by a more satisfactory procedure²⁶. The shallow water waves are assumed
 119 to be associated with the motion of a “charged” particle moving in a four-vector potential field,
 120 i.e., in a field similar to the electromagnetic potential. The charge here does not mean electric
 121 charge; it merely indicates that the particle responds to this field as an electrically charged particle
 122 responds to an external electromagnetic field, and may equally be called the coupling constant²⁹.
 123 Units are used in which $c = \hbar = e = 1$, and the subscript W marks the analogue potentials
 124 throughout, so that they are never confused with the genuine electromagnetic **A** and **B** (Methods).

125 Since a charged particle in such a field may in general perform three-dimensional motion, we
 126 assume a wave function periodic in both the x - and the z -direction,

$$127 \quad \Psi(\mathbf{r}, t) = v(y) \exp \left[ikx + ik_z z - i\alpha \frac{\omega^2}{c_g^2} t \right] \quad (9)$$

128 There are two ways of choosing the potential so as to obtain the correct equation of motion.

129 **(i)** Let $k_z = f_0/c_g$ and let the four-vector potential $A_{W\mu}$ be

$$130 \quad A_{W\mu} = \left(-\frac{\alpha\beta^2}{4\omega^2}; -\frac{\beta}{2\omega}, 0, -\frac{\beta y}{c_g} \right) \quad (10)$$

131 corresponding to a constant scalar potential $\phi = -\alpha\beta^2/4\omega^2$, which exerts no force, and a three-
 132 dimensional vector potential $\mathbf{A}_W = -(\beta/2\omega)\hat{\mathbf{x}} - (\beta y/c_g)\hat{\mathbf{z}}$, where $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$ are unit vectors
 133 pointing east, north, and upward. This vector potential produces a force field analogous to a
 134 magnetic field,

$$135 \quad \mathbf{B}_W = \nabla \times \mathbf{A}_W = -\frac{\beta}{c_g} \hat{\mathbf{x}} \quad (11)$$

136 which is uniform and points to the west. The Hamiltonian is then

$$137 \quad H = \alpha(-i\nabla - \mathbf{A}_W)^2 + \phi \quad (12)$$

138 with the charge set to unity. Substituting the wave function (9) into the Schrödinger equation $H\Psi =$
 139 $i \partial\Psi / \partial t$ and dividing throughout by α returns precisely Eq. (1), the wave equation sought. No
 140 term has been inserted by hand: the $\beta k/\omega$ term is generated by the cross term between $-i \partial / \partial x$
 141 and A_{Wx} , and the residual $\beta^2/4\omega^2$ left over from that square is removed by the constant scalar
 142 potential, which is physically inert.

143 **(ii)** A second choice that yields the same equation of motion is to set

$$k_z = 0, \quad A_{Wz} = -\frac{f_0 + \beta y}{c_g} \quad (13)$$

leaving A_{Wx} and ϕ as before. The resulting $\mathbf{B}_W$ -field is again given by Eq. (11). This version requires no vertical dependence of the wave function, which is more in line with the original problem of purely horizontal basic-state motion. The potential is assumed here; the companion paper²⁸ derives it. As shown below, models (i) and (ii) are not two different models at all, but the same physical system written in two gauges.

Gauge structure of the representation

The four-vector potential of Eq. (10) is not unique. In units with $c = \hbar = e = 1$, the transformation

$$\mathbf{A}_W \rightarrow \mathbf{A}_W + \nabla\chi, \quad \phi \rightarrow \phi - \frac{\partial\chi}{\partial t}, \quad \Psi \rightarrow \Psi \exp(i\chi) \quad (14)$$

for any single-valued function $\chi(\mathbf{r}, t)$ leaves both the force field $\mathbf{B}_W = \nabla \times \mathbf{A}_W$ and the Schrödinger equation invariant. The physically meaningful quantity is therefore not the canonical momentum operator $\mathbf{p} = -i\nabla$, which is gauge dependent, but the gauge-invariant kinematic momentum operator $\boldsymbol{\pi} = -i\nabla - \mathbf{A}_W$. This distinction, a formality in many textbook problems, is the key to the difficulty encountered above.

The x -component of the vector potential, $A_{Wx} = -\beta/2\omega$, is a constant and therefore contributes nothing to $\mathbf{B}_W$: it is pure gauge. It can be removed entirely by choosing $\chi_1 = (\beta/2\omega)x$ in Eq. (14), which gives $A_{Wx} \rightarrow 0$ and, simultaneously,

$$\Psi = v(y)e^{ikx} \rightarrow v(y)\exp\left[i\left(k + \frac{\beta}{2\omega}\right)x\right] \quad (15)$$

Equation (15) is exactly the phase shift that had to be imposed by hand in Eq. (7). In the gauge $A_{Wx} = 0$ the term π_x^2 of the Hamiltonian becomes a pure multiplication by $(k + \beta/2\omega)^2$, and π_z^2 becomes $(\beta/c_g)^2(y - y_0)^2$; the Hamiltonian is then precisely the harmonic oscillator

Hamiltonian with the potential of Eq. (8) and the wave function of Eq. (7). In other words, **the harmonic oscillator representation is the four-vector potential representation written in a particular gauge**. Its two apparently arbitrary ingredients are not arbitrary at all: the phase shift is the gauge function χ_1 , and the constant $-\alpha\beta^2/4\omega^2$ subtracted from the potential is the scalar potential of Eq. (10).

The physical content is the following. The zonal wavenumber k of the shallow water solution is the *canonical* momentum of the particle, while the gauge-invariant *kinematic* momentum, the one that enters the energy, is

$$\pi_x = k + \frac{\beta}{2\omega} \quad (16)$$

The term $\beta k/\omega$, which the harmonic oscillator model could not produce, is neither spurious nor a gauge artifact: it is the cross term that arises when the energy is expressed in terms of the canonical rather than the kinematic momentum, the difference between the two, $\beta/2\omega$, being the analogue of eA_x/c in the electromagnetic problem. This was anticipated in the earlier work²⁶, where the first-order term in k was attributed to the gauge used in deriving the shallow water equations, although the gauge function itself was not identified.

Models (i) and (ii) differ only in how the kinematic momentum along z is apportioned between the wavenumber and the potential: in model (i), $\pi_z = k_z - A_{Wz} = f_0/c_g + \beta y/c_g$; in model (ii), $\pi_z = 0 + (f_0 + \beta y)/c_g$. These are identical, and the two descriptions are connected by the gauge function $\chi_2 = -f_0 z/c_g$, which converts $A_{Wz} = -\beta y/c_g$ into $A_{Wz} = -(f_0 + \beta y)/c_g$ while simultaneously removing the factor $\exp(ik_z z)$ from the wave function (9). The two are therefore the same physical system, and the question of which is better is one of convenience rather than of physics: model (ii) is preferable because it requires no vertical wavenumber and hence no fictitious

187 third dimension, the auxiliary z -dependence of model (i) being entirely a gauge artifact. The
 188 vertical wavenumber cannot be chosen freely: taking $k_z \neq 0$ in model (ii) replaces f_0 by $f_0 + k_z c_g$
 189 in Eq. (1), which amounts to relocating the reference latitude, so that reproducing Eq. (1) fixes
 190 $\pi_z = (f_0 + \beta y)/c_g$ uniquely, whatever gauge is used to express it.

191 Because the Schrödinger equation is invariant under Eq. (14), the physics depends on $\mathbf{A}_W$ only
 192 through $\mathbf{B}_W$: two four-vector potentials producing the same $\mathbf{B}_W$ describe the same particle, and
 193 hence the same shallow water waves, however different they may look. The one exception in the
 194 electromagnetic case is the Aharonov–Bohm effect, in which the wave function acquires an
 195 observable phase from a region where $\mathbf{B}$ vanishes but $\mathbf{A}$ does not. Since $\mathbf{B}_W$ here is uniform and
 196 unbounded, no such effect arises, but the possibility is worth keeping in mind for geometries in
 197 which β varies or the domain is not simply connected.

198 *The dispersion relation is a Landau spectrum*

199 The particle constructed above is a charged particle moving in a uniform magnetic-like field. The
 200 corresponding electromagnetic problem — a free electron in a uniform magnetic field — was
 201 solved by Landau³⁰ in order to explain the diamagnetism of metals; for more recent summaries see
 202 refs.^{31,32}. The energy levels are

$$203 \quad E = \frac{\hbar^2 k_{\parallel}^2}{2m} + \left(n + \frac{1}{2}\right) \hbar \omega_c \quad (17)$$

204 where $k_{\parallel}$ is the kinematic wavenumber along $\mathbf{B}$, $n = 0, 1, 2, \dots$ is the Landau level index, and $\omega_c =$
 205 $|e|B/mc$ is the cyclotron frequency (not to be confused with the Larmor precession frequency,
 206 which is half of it). The dispersion relation between E and $k_{\parallel}$ is a family of parabolas, one for each
 207 Landau level.

208 The earlier work²⁶ noted that Eq. (17) and the shallow water dispersion relation have the same
 209 form, drew an analogy between the two, and observed that the units of the two systems are not the
 210 same. The correspondence is in fact stronger than an analogy. It is exact, and Eq. (3) can be *derived*
 211 from Eq. (17) rather than compared with it.

212 Take the Hamiltonian of Eq. (12). The field $\mathbf{B}_W$ is uniform and directed along x , so the motion
 213 separates into free motion along $\mathbf{B}_W$, with the kinematic momentum π_x of Eq. (16), and Landau-
 214 quantized motion in the perpendicular (y, z) plane. In the units adopted here α plays the role of
 215 $\hbar^2/2m$, so that $m = 1/2\alpha$, while $e = c = \hbar = 1$ and $B_W = \beta/c_g$; the cyclotron frequency is
 216 therefore $\hbar\omega_c = |e|B/m = 2\alpha\beta/c_g$. The total energy, including the constant scalar potential of
 217 Eq. (10), is

$$218 \quad E = \alpha \left(k + \frac{\beta}{2\omega} \right)^2 + (2n + 1) \frac{\alpha\beta}{c_g} - \frac{\alpha\beta^2}{4\omega^2} \quad (18)$$

219 Setting $E = \alpha\omega^2/c_g^2$, as required by the wave function (9), and dividing by α , the square expands
 220 to give $k^2 + \beta k/\omega + \beta^2/4\omega^2$, the last term of which is cancelled identically by the scalar
 221 potential. What remains is exactly Eq. (3). The dispersion relation of tropical shallow water waves
 222 is thus the Landau spectrum of the Hamiltonian of Eq. (12); the Hermite index n of the Matsuno
 223 solution, Eq. (2), is the Landau level index of Eq. (17); and the Hermite functions of Eq. (2) are
 224 the Landau wave functions in the gauge $A_{Wx} = 0$. This makes the appearance of the factor
 225 $(2n + 1)$ in Eq. (3) — otherwise simply a property of the Hermite equation — a physical rather
 226 than a mathematical fact: it is the quantization of the particle's gyration about the analogue
 227 magnetic field.

The analogue magnetic field, magnetic length and guiding centre

The only force acting on the particle is the analogue magnetic field of Eq. (11): uniform, westward, and produced solely by the meridional gradient of the planetary vorticity. The “charge” of the particle is a coupling constant that measures how strongly the wave field responds to that gradient, and has been set to unity. The scalar potential is constant and exerts no force at all; it is present only to keep the energy zero point consistent with the definition of the canonical momentum.

The physical statement contained in this is that the β -effect acts on a shallow water wave packet in the same way that a uniform magnetic field acts on a charged particle. The meridional trapping of equatorial waves — the equatorial waveguide — is thereby identified as cyclotron gyration, and its quantization as Landau quantization. For the Earth’s atmosphere with $\beta \approx 2.3 \times 10^{-11} \text{ m}^{-1} \text{ s}^{-1}$ and a first baroclinic mode speed $c_g \approx 30 \text{ m s}^{-1}$, the analogue field strength is $B_W = \beta/c_g \approx 7.6 \times 10^{-13} \text{ m}^{-2}$.

For an electron in a magnetic field, the natural length scale is the magnetic length $\ell_B = (\hbar c/|e|B)^{1/2}$, the characteristic radius of the lowest Landau orbit. For the analogue field of Eq. (11), in the present units, this is simply $B_W^{-1/2}$, so that

$$\ell_B = \left(\frac{c_g}{\beta}\right)^{1/2} \equiv L_{eq} \quad (19)$$

which is the equatorial Rossby radius of deformation. This is not a coincidence but an identity: the Gaussian factor in Eq. (2) is $\exp[-(y - y_0)^2/2\ell_B^2]$, which is the lowest Landau wave function. The meridional width of an equatorially trapped wave and the cyclotron radius of the analogue particle are the same quantity. For the parameters quoted above, $\ell_B \approx 1150 \text{ km}$; for a shallower equivalent depth, say $c_g \approx 2.8 \text{ m s}^{-1}$, it is about 350 km. Correspondingly, the Landau level

spacing measured in the units of Eq. (3) is $\beta/c_g = \ell_B^{-2}$, so that the spectrum of meridional modes is fixed by the deformation radius alone.

In the electromagnetic Landau problem each level is enormously degenerate: for a given n , the wave function may be centered on any guiding-center coordinate. Here the guiding center is the latitude at which the kinematic momentum $\pi_z = (f_0 + \beta y)/c_g$ vanishes, that is, $y = y_0 = -f_0/\beta$, the latitude at which the Coriolis parameter itself vanishes. The freedom to displace the guiding center is exactly the freedom to choose k_z , and reproducing Eq. (1) forces k_z to a single value. The two-dimensionality of the system therefore removes the Landau degeneracy and pins the guiding center to the zero- f latitude — the particle-representation statement of the familiar fact that equatorial waves are trapped about the equator and not about some arbitrary latitude.

Finally, since Ψ is proportional to $v(y)$, the probability density of the particle is $\Psi^*\Psi \propto |v(y)|^2$, which is proportional to the meridional kinetic energy density of the wave, $\frac{1}{2}\rho H v^2$. The particle is therefore most likely to be found where the meridional motion of the fluid is most vigorous. This gives the quantum mechanical formalism a direct fluid dynamical reading: the “position” of the particle is the latitude at which the wave carries its meridional energy, and the spreading of $|\Psi|^2$ over the deformation radius is the meridional extent of the waveguide.

A Hamiltonian that depends on its own eigenvalue

An unusual feature of the present construction is that the potential depends on the frequency: $A_{Wx} = -\beta/2\omega$ and $\phi = -\alpha\beta^2/4\omega^2$ both contain ω , while the energy is $E = \alpha\omega^2/c_g^2$. The Hamiltonian is therefore a function of its own eigenvalue, $H = H(E)$, and the eigenvalue problem is implicit rather than linear. This is the sharpest disanalogy with the electromagnetic Landau problem, in which H is independent of E .

Far from being a defect, this is the feature that produces the most characteristic property of the tropical wave spectrum. Multiplying Eq. (3) by ω gives

$$\frac{\omega^3}{c_g^2} - \omega \left[k^2 + \frac{(2n+1)\beta}{c_g} \right] - \beta k = 0 \quad (20)$$

a cubic with three roots for each n : two of high frequency, corresponding to eastward- and westward-propagating inertia-gravity waves, and one of low frequency, corresponding to the westward-propagating Rossby wave. Had A_{Wx} been independent of ω — that is, had the vector potential not depended on the particle's own energy — the term $\beta k/\omega$ would be absent, Eq. (3) would be quadratic in ω , and only the two gravity wave branches would survive. **The Rossby wave exists, in this representation, precisely because the particle moves in a field that depends on its own energy.** The usual care must of course be taken with the $n = 0$ case, for which one of the three roots is spurious⁶.

It would be natural to read this as indicating that the four-vector potential is a self-consistent field, determined together with the state of the particle rather than imposed from outside. That reading is not correct: the frequency enters through the elimination of u and Φ by which Eq. (1) was obtained, and the companion paper²⁸ shows that the parent Lagrangian contains no reference to ω at all. The self-referential character of H is the price of reducing three coupled fields to one.

The Yanai wave as the lowest Landau level

The particle picture is at its most informative for the gravest meridional mode, $n = 0$, whose solution is the mixed Rossby-gravity or Yanai wave^{5,6}. Because the field $\mathbf{B}_W$ of Eq. (11) is uniform, the perpendicular part of the Hamiltonian (12) can be factorized in the standard way. Defining

$$a = \frac{\pi_y - i\pi_z}{\sqrt{2\beta/c_g}}, \quad a^\dagger = \frac{\pi_y + i\pi_z}{\sqrt{2\beta/c_g}}, \quad [a, a^\dagger] = 1 \quad (21)$$

where $\pi_y = -i \partial / \partial y$ and $\pi_z = \beta(y - y_0)/c_g$ are the gauge-invariant kinematic momenta perpendicular to $\mathbf{B}_W$, the perpendicular kinetic energy becomes $\pi_y^2 + \pi_z^2 = (2\beta/c_g) \left(a^\dagger a + \frac{1}{2} \right)$, whose eigenvalues are $(2n + 1)\beta/c_g$, exactly the term appearing on the right of Eq. (3). The Yanai wave is the state annihilated by a , and it therefore satisfies a **first-order** equation rather than the second-order structure equation (1):

$$a v = 0 \Rightarrow \frac{dv}{dy} + \frac{\beta(y-y_0)}{c_g} v = 0 \Rightarrow v \propto \exp \left[-\frac{\beta(y-y_0)^2}{2c_g} \right] \quad (22)$$

which reproduces Eq. (2) at $n = 0$. Three consequences follow.

First, the Yanai wave is the cyclotron ground state. Its meridional structure is a nodeless Gaussian of width exactly the analogue magnetic length of Eq. (19), and its contribution $\beta/c_g = \ell_B^{-2}$ to the dispersion relation is pure zero-point energy of the gyration, with no excitation of the cyclotron motion at all. Every other equatorial mode follows from it by repeated application of $a^\dagger$, which is why n counts nodes. That index cannot be altered by a gauge transformation: gauge transformations leave $\mathbf{B}_W$, and hence the whole Landau spectrum, invariant, and since $\Psi^* \Psi$ is gauge invariant, so is the nodal structure of v . Moving between meridional modes requires the ladder operator, not the gauge function.

Second, the symmetry of the mode follows immediately. With $H_0 = 1$, the meridional velocity v is symmetric about y_0 , while u and Φ , obtained from v through a derivative and multiplication by $f_0 + \beta y$, are antisymmetric. The Yanai wave therefore has maximum cross-equatorial flow precisely where the Coriolis parameter vanishes, with a node in geopotential and zonal wind there — the opposite symmetry to the Kelvin and $n = 1$ Rossby modes. In the particle language, the

guiding center of the lowest Landau orbit sits on the zero- f latitude, and the gyration is what carries fluid across it: the structural reason why the Yanai wave is the most effective equatorial mode for cross-equatorial momentum transport.

Third, $n = 0$ is the only level at which the Rossby and gravity branches do not separate. Setting $n = 0$ in Eq. (20) and clearing denominators, the cubic factorizes as $(\omega + kc_g)(\omega^2 - kc_g\omega - \beta c_g) = 0$. The root $\omega = -kc_g$ must be discarded: the reduction of the shallow water equations to Eq. (1) requires division by $c_g^2 k^2 - \omega^2$, which vanishes there⁶. What survives is a quadratic, not a cubic, and taking $\omega > 0$,

$$\omega = \frac{1}{2} [kc_g + \sqrt{k^2 c_g^2 + 4\beta c_g}] \quad (23)$$

a single continuous curve with $\omega \rightarrow kc_g + \beta/k$ as $k \rightarrow +\infty$ and $\omega \rightarrow -\beta/k$ as $k \rightarrow -\infty$. The mode is thus inertia-gravity-like at the eastward end and Rossby-like at the westward end, which is the content of the name “mixed Rossby-gravity wave.” For every $n \geq 1$ the cubic does not factorize, the gravity branches retain a frequency floor, and a separate low-frequency Rossby branch exists; only the lowest Landau level bridges the two regimes. In the particle picture the reason is that the missing third root is the one associated with $v \equiv 0$, and the lowest Landau level is the last state before that limit.

These structural properties account for the prominence of the Yanai wave in observations. It was the first of Matsuno’s modes identified in the atmosphere, in equatorial lower-stratospheric radiosonde data^{5,11}, and its antisymmetric geopotential signature makes it separable as a distinct peak in wavenumber–frequency spectra of satellite-observed convection^{20,21}. Because it transports easterly momentum upward, it was assigned a central role, alongside the Kelvin wave, in the classical wave-driven theory of the quasi-biennial oscillation^{33–35}, although current assessments

attribute the larger share of the momentum flux to a broad spectrum of smaller-scale gravity waves³⁶. In the troposphere, its convective counterpart over the Pacific is associated with tropical-depression-type disturbances and thus with one recognized pathway to tropical cyclogenesis³⁷.

Discussion

The linearized shallow water equations on a general β -plane have been reformulated as a quantum mechanical particle problem. The meridional structure equation is exactly the Schrödinger equation for a particle of unit charge minimally coupled to a four-vector potential whose analogue magnetic field is uniform and westward, and no *ad hoc* term is required. A representation based on a simple harmonic oscillator reproduces the structure equation only if a phase shift and a constant potential are inserted by hand, and both are explained by the four-vector representation: the phase shift is the gauge function $\chi_1 = (\beta/2\omega)x$ that removes the constant part of A_{Wx} , and the constant potential is the scalar potential. The zonal wavenumber is the canonical momentum, the gauge-invariant kinematic momentum is $\pi_x = k + \beta/2\omega$, and the $\beta k/\omega$ term of the dispersion relation is the cross term generated when the energy is expressed through the canonical rather than the kinematic momentum. The two versions of the vector potential are the same physical system in two gauges, related by $\chi_2 = -f_0 z/c_g$. The Matsuno dispersion relation is the Landau spectrum of the constructed Hamiltonian, the Hermite index is the Landau level index, the equatorial radius of deformation is the analogue magnetic length, and the meridional trapping of equatorial waves is cyclotron gyration. Because the vector potential depends on the frequency the eigenvalue problem is implicit, and that dependence is the origin of the Rossby branch. The Yanai wave is the lowest Landau level, annihilated by the lowering operator and therefore satisfying a first-order equation, and its mixed Rossby–gravity character follows from the factorization of the $n = 0$ cubic, which occurs at no other meridional mode. The constant α cancels identically from the

358 dispersion relation and is therefore not determined by the wave problem; it fixes only the map
359 between wave frequency and particle energy.

360 Three qualifications should be stated plainly. The correspondence is a mathematical identity
361 between two eigenvalue problems, not a claim that a material particle exists. What has been
362 established is that the shallow water wave field of a given zonal wavenumber and meridional mode
363 number can be labeled by, and computed from, a single particle state, the labels behaving exactly
364 as the quantum numbers of a charged particle in a uniform magnetic field; the economy claimed
365 is an economy of description. The units of the particle problem are likewise those of the wave
366 problem, not those of quantum mechanics: the particle's "momentum" is a wavenumber, its
367 "energy" a frequency, and its "mass" is fixed by α , whose value is a matter of normalization. No
368 dimensional bridge to elementary particle physics is intended or implied.

369 The third qualification is more interesting, because it can be stated precisely. The representation is
370 built on the structure equation for v and therefore does not describe modes for which $v \equiv 0$. The
371 Kelvin wave is the important case. The reduction of the shallow water equations to Eq. (1) requires
372 division by $c_g^2 k^2 - \omega^2$, which vanishes identically on the Kelvin branch $\omega = kc_g$; the mode is
373 annihilated by the elimination of u and Φ , before any of the present construction is applied. It is
374 the second casualty of that division, the first being the spurious root $\omega = -kc_g$ discarded at $n =$
375 0 . In the particle language the Kelvin wave is formally the state $n = -1$: substituting $n = -1$ in
376 Eq. (3) gives $\omega = kc_g$ identically. But the Landau ladder terminates, since a annihilates the ground
377 state, and there is no level below $n = 0$. The Yanai wave is the gravest mode of Eq. (1) precisely
378 because the state below it is the one with $v \equiv 0$, which is the Kelvin wave itself and not a state of
379 the particle at all. Suggestively, the lowest Landau structure is nevertheless present in the Kelvin
380 mode: on an equatorial β -plane its zonal velocity is $u \propto \exp(-\beta y^2/2c_g)$, the same Gaussian, of

the same width ℓ_B , as the $n = 0$ wave function of Eq. (2). The structure lives in u and Φ rather than in v . A representation capturing both modes would therefore have to be built on a multi-component wave function carrying (u, v, Φ) together, rather than on a scalar amplitude obtained by reducing to one of them. This is presumably why the topological treatment of Delplace et al.²⁴, which never performs such a reduction, captures the Kelvin wave on the same footing as the Yanai wave, and it indicates that the limitation belongs to the present construction rather than to the condensed-matter analogy in general.

Three directions seem worth pursuing. The first is a slowly varying medium, in which β , c_g , or the depth vary in space and the analogue magnetic field becomes non-uniform; the semiclassical machinery developed for electrons in inhomogeneous fields would then apply directly, and the drift of the guiding center would give a compact description of wave propagation through a varying basic state. The second is the incorporation of the Kelvin mode along the lines just indicated. The third, and perhaps the most interesting, is the relation between the Landau description given here and the topological description²⁴, which identifies the same two modes as exceptional by way of a Chern number rather than a level index. Landau levels and Chern numbers are intimately connected in the condensed matter setting, and it would be surprising if the connection did not survive the translation to the equatorial waveguide.

Methods

Linearized shallow water equations on a general β -plane

Shallow water waves on a β -plane have been considered by various investigators^{6,7,9}. We consider a general β -plane that is not necessarily centered on the equator, so that the local Coriolis parameter is approximated by

$$f \simeq f_0 + \beta y \quad (24)$$

where f_0 is the Coriolis parameter at the reference latitude, y is the meridional coordinate, positive to the north, and $\beta = df/dy$ is assumed constant. Under this assumption, the linearized equations governing the perturbation zonal velocity u , meridional velocity v , and geopotential Φ are

$$\frac{\partial u}{\partial t} - (f_0 + \beta y)v + \frac{\partial \Phi}{\partial x} = 0 \quad (25)$$

$$\frac{\partial v}{\partial t} + (f_0 + \beta y)u + \frac{\partial \Phi}{\partial y} = 0 \quad (26)$$

$$\frac{\partial \Phi}{\partial t} + c_g^2 \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = 0 \quad (27)$$

where c_g is the speed of long gravity waves, defined by $c_g^2 = gH$, with g the gravitational acceleration and H the mean depth of the fluid. The Coriolis terms couple u and v to each other and do no net work on the flow, as they must. Assuming periodic solutions of the form

$$(u, v, \Phi) = (u'(y), v'(y), \Phi'(y)) \exp(ikx - i\omega t) \quad (28)$$

and eliminating u and Φ , which requires division by $c_g^2 k^2 - \omega^2$, reduces Eqs. (25)–(27) to Eq. (1) for v' , with the prime dropped.

Minimal coupling in classical and quantum mechanics

Only the elements needed above are recorded here; for the physical background see refs.^{22,38}. Gaussian units are used, c denotes the speed of light, and the speed of long gravity waves is written c_g throughout, so that no confusion between the two can arise. For a particle of mass m and charge e moving in an electromagnetic field, the classical Hamiltonian is

$$H = \frac{1}{2m} \left(\mathbf{p} - \frac{e\mathbf{A}}{c} \right)^2 + e\phi \quad (29)$$

422 where ϕ is the scalar potential and $\mathbf{A}$ the vector potential of the magnetic field, $\mathbf{B} = \nabla \times \mathbf{A}$. The
 423 Hamiltonian is still the total energy, since the magnetic field does no work. The mechanical
 424 (kinematic) momentum, however, is now $\mathbf{p} - e\mathbf{A}/c$ rather than $\mathbf{p}$, while $\mathbf{p}$ itself is the canonical
 425 momentum, and it is the canonical momentum that is carried over to the quantum mechanical
 426 formulation. Replacing $\mathbf{p} \rightarrow -i\hbar\nabla$ and $E \rightarrow i\hbar \partial/\partial t$ in $H\Psi = E\Psi$, and taking $\Psi(\mathbf{r}, t) =$
 427 $\psi(\mathbf{r})\exp(-iEt/\hbar)$, gives the time-independent Schrödinger equation with minimal coupling,

$$428 \quad \frac{1}{2m} \left(-i\hbar\nabla - \frac{e\mathbf{A}}{c} \right)^2 \psi + e\phi\psi = E\psi \quad (30)$$

429 with E the total energy of the particle. Note that the coupling must be written in terms of the
 430 momentum operator, as in Eq. (30), and not as $(\nabla - e\mathbf{A}/c)^2$, whose two terms differ dimensionally
 431 by a factor $\hbar$; writing it as $(\mathbf{p} - e\mathbf{A}/c)$ throughout is the convention adopted above.

432 *The role of the constant α*

433 The earlier work²⁶ observed that the formulation goes through with $\alpha = 1$, remarked that the
 434 physical meaning of α was at that time unclear, and suggested that it might prove to be an important
 435 characteristic number of the geophysical shallow water problem. The derivation given above
 436 settles the question. α cancels identically between the two sides of Eq. (18) and does not appear in
 437 Eq. (3), so the dispersion relation places no constraint on it whatever, and it is not a characteristic
 438 number of the wave problem in the sense anticipated. What it does fix is the relation between wave
 439 frequency and particle energy, $E = \alpha\omega^2/c_g^2$, and hence the rate at which the particle gyrates in
 440 real time; carrying the dimensions of $\hbar/m$, it can be given a numerical value only after a system
 441 of units has been chosen. One choice deserves mention. Written in the form

$$442 \quad \frac{1}{2} \frac{\omega^2}{c_g^2} = \frac{1}{2} \left(k^2 + \frac{\beta k}{\omega} \right) + \left(n + \frac{1}{2} \right) \frac{\beta}{c_g} \quad (31)$$

Eq. (3) matches Eq. (17) term by term: $\frac{1}{2}\omega^2/c_g^2$ corresponds to $E, \frac{1}{2}(k^2 + \beta k/\omega)$ to $\hbar^2 k_{\parallel}^2/2m$, and β/c_g to $\hbar\omega_c$. This reading amounts to $\alpha = 1/2$, that is, $m = 1$ in units with $\hbar = 1$ — a convenient normalization, making the particle’s energy exactly half its squared kinematic momentum as for an ordinary free particle of unit mass, but a choice of units and not a requirement of the physics.

Data availability

No datasets were generated or analysed during the current study. All results reported here are analytic and are fully reproducible from the equations given in the text.

Code availability

No custom code was used in this study.

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520 **Author contributions**

521 P.K.W. conceived the study, carried out the analysis, and wrote the manuscript.

522 **Competing interests**

523 The author declares no competing interests.