A particle representation of tropical shallow water waves

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(This preprint has not been peer-reviewed)

Abstract

The linearized shallow water equations on a general β-plane reduce to a single meridional structure equation for the meridional velocity. We show that this equation is exactly the Schrödinger equation for a "charged" particle minimally coupled to a four-vector potential whose analogue magnetic field, $\mathbf{B}_W = -(\beta/c_g)\hat{\mathbf{x}}$, is uniform and westward. No *ad hoc* term is required: every term is generated by the minimal coupling. The analogue magnetic length is the equatorial Rossby radius of deformation, the Hermite index of the Matsuno solution is the Landau level index, and the Matsuno dispersion relation is recovered exactly as the Landau spectrum of the constructed Hamiltonian rather than merely resembling it. The zonal wavenumber is the canonical momentum, the gauge-invariant kinematic momentum being $k + \beta/2\omega$. Because the vector potential depends on frequency, the eigenvalue problem is implicit, and this is the origin of the Rossby branch.

Introduction

Wave phenomena in the atmosphere and ocean play an important role in many geophysical processes such as energy transport, climate processes, and cloud formation. Among the most important categories for synoptic and large scale phenomena are the shallow water waves, whose wavelengths are much longer than the depth of the fluid; they have been studied in great detail and are summarized in many textbooks[1–4]. The tropical shallow water waves have received special attention because the tropics are energy source regions of many atmospheric and oceanic processes. Important discoveries concerning these waves have been made by many researchers[5–19], and the observational picture has since been placed on a firm spectral footing[20] and reviewed[21].

All previous studies have been concerned with the "wave" properties of these waves. For long-term climatic studies their role can be hard to recognize: wave profiles are complicated, made worse by reflection, refraction, and diffraction, and coupled with complex boundaries the resultant signals are difficult to analyze, let alone to interpret. One feasible way to reduce such complexities is to reformulate these waves into a system in which they can be regarded as particles, or at least are associated with some particle properties. Unlike waves, which are "smeared out" in space and time, particles are compact and usually easier to identify.

The present study is motivated by the wave–particle duality established by quantum physics early in the twentieth century[22,23]: perhaps, by a suitable formulation, waves in the atmosphere and ocean can also be viewed as particles, and such a view will simplify their analysis. The bound electron of the hydrogen atom illustrates the gain. Its wave description requires a probability density $\Psi^*\Psi$ that is already a complicated function of space, and the time series of that signal at a point would be almost impossible to analyze; all of this complication disappears once the signal is recognized as one single particle, from which many properties of hydrogen can be deduced.

The idea that geophysical wave problems may be illuminated by the methods of condensed matter physics has since been pursued independently and from a quite different direction. Delplace et al.[24] showed that the Kelvin and Yanai modes owe their existence to a topological invariant of the shallow water system — the first Chern number, of value two, of the bulk Poincaré wave bands — in close analogy with the protected edge states of a topological insulator[25]. That work and the present one single out the same two modes by different routes.

Here we ask whether geophysical shallow water waves can be formulated as particles, and if so, what rules of behavior they obey. The analysis below indicates that these particles obey a Schrödinger equation with minimal coupling to a four-vector potential, just as charged quantum mechanical particles do. This paper develops and completes an argument first presented at the Eighth Conference on Atmospheric and Oceanic Waves and Stability[26], from which the construction and the observation that the resulting dispersion relation resembles the Landau spectrum of a free electron in a uniform magnetic field are taken. Three matters left open there are settled here: the gauge structure of the four-vector potential, which accounts for an *ad hoc* phase shift and shows two apparently different versions of the construction to be one physical system in two gauges; the status of the correspondence with the Landau problem, shown below to be an identity rather than an analogy, together with the meaning of the constant α, which was explicitly left for future study; and the physical interpretation of the particles. The four-vector potential is written down and verified here rather than derived; it is obtained from Hamilton's principle, by linearizing the Eulerian variational principle for rotating flow[27], in a companion paper[28].

Results

The meridional structure equation

On a general β-plane that is not necessarily centered on the equator, the linearized shallow water equations reduce to a single meridional structure equation for the meridional velocity perturbation v (Methods),

$$\frac{d^2v}{dy^2}+\left[\frac{\omega^2}{c_g^2}-k^2-\frac{\beta k}{\omega}-\frac{(f_0+\beta y)^2}{c_g^2}\right]v=0 \tag{1}$$

where k is the zonal wavenumber, ω the angular frequency, f_0 the Coriolis parameter at the reference latitude, and c_g the speed of long gravity waves. Equation (1), with the boundary condition $v\to 0$ at $y\to\pm\infty$, poses an eigenvalue problem. Since $f_0+\beta y=\beta(y-y_0)$ with $y_0=-f_0/\beta$, the last term in the bracket is $-\left(\beta/c_g\right)^2(y-y_0)^2$, so the problem is that of a particle in a parabolic well centered on the latitude at which the Coriolis parameter vanishes. The solution is

$$v=H_n\left[\left(\frac{\beta}{c_g}\right)^{1/2}(y-y_0)\right]\exp\left[-\frac{\beta(y-y_0)^2}{2c_g}\right] \tag{2}$$

where H_n is the Hermite polynomial of order n and the normalization constant has been omitted, Eq. (1) being homogeneous[3,6]. The dispersion relation is

$$\frac{\omega^2}{c_g^2}-k^2-\frac{\beta k}{\omega}=\frac{(2n+1)\beta}{c_g} \tag{3}$$

with $n=0,1,2,\ldots$ On an equatorial β-plane, $f_0=0$ and $y_0=0$, and the oscillation is symmetric about the equator; for a general β-plane the eigenfunctions are the same functions displaced to $y=y_0$, and the dispersion relation is unchanged. The trapped solutions are of practical interest chiefly in the tropics, because for a mid-latitude reference the turning latitude y_0 lies outside the range over which the β-plane approximation is defensible.

Failure of a harmonic-oscillator representation

Matsuno[6] noticed the similarity between Eq. (1) and the quantum mechanical problem of a particle bound in a harmonic potential, and one may be tempted to build a particle formulation directly on the Schrödinger equation for a harmonic oscillator. We first show that this is unsatisfactory. If a Schrödinger equation can be written down whose wave amplitude is the v of Eq. (1), it is justified to declare that the corresponding particle exists, even though we do not yet know what kind of particle it is; its behavior must then be interpreted consistently with quantum mechanics.

Consider the case in which the waves represent the amplitude of a particle moving in a simple harmonic oscillator potential, which is proportional to $\left(\beta/c_g\right)^2(y-y_0)^2$. The quantum mechanical operators for this particle are

$$E = i\frac{\partial}{\partial t}\,, \qquad \mathbf{p} = -i\boldsymbol{\nabla}\,, \qquad H = \alpha(-i\boldsymbol{\nabla})^2 + \alpha\left(\frac{\beta}{c_g}\right)^2 (y-y_0)^2 \tag{4}$$

where α plays the role of $\hbar^2/2m$. Since we are *not* dealing with elementary particles here, there is no reason to expect $\hbar$ and m to retain their usual meanings, and it is convenient to absorb them into the single constant α, which carries the dimensions of $\hbar/m$ (Methods). Taking the plane wave function

$$\Psi(\mathbf{r},t) = v(y)\exp\left[ikx - i\alpha\left(\frac{\omega^2}{c_g^2}\right)t\right] \tag{5}$$

in which the energy term $E/\hbar$ has been written explicitly as $\alpha\left(\omega^2/c_g^2\right)$, the energy of a plane wave of angular frequency ω, and substituting it into the Schrödinger equation with the operators of Eq. (4), we obtain

$$\frac{d^2v}{dy^2} + \left[\frac{\omega^2}{c_g^2} - k^2 - \left(\frac{\beta}{c_g}\right)^2 (y-y_0)^2\right] v = 0 \tag{6}$$

Comparing Eq. (6) with Eq. (1), the term $\beta k/\omega$ is missing. To repair this we may change the wave function of Eq. (5) to

$$\Psi(\mathbf{r},t) = v(y)\exp\left[i\left(k+\frac{\beta}{2\omega}\right)x - i\alpha\frac{\omega^2}{c_g^2}t\right] \tag{7}$$

and at the same time rewrite the harmonic oscillator potential as

$$V = \alpha\left[\left(\frac{\beta}{c_g}\right)^2 (y-y_0)^2 - \frac{\beta^2}{4\omega^2}\right] \tag{8}$$

With these two changes Eq. (1) is recovered exactly. But in writing Eqs. (7) and (8) we have asserted *a priori* that the zonal wavenumber must be shifted by $\beta/2\omega$ — equivalently, that the wave phase must be shifted by $\exp(i\beta x/2\omega)$ — and that the potential must be lowered by $\alpha\beta^2/4\omega^2$. A potential may always be shifted by an arbitrary constant without changing the physics, but there is no obvious justification for the phase shift; as it stands it is no more than an *ad hoc* manipulation. It would be far more satisfactory to have these quantities specified by the operators, that is, by the external conditions in which the particle moves, rather than inserted by hand into the wave function.

A charged particle in a four-vector potential

We now derive Eq. (1) by a more satisfactory procedure[26]. The shallow water waves are assumed to be associated with the motion of a "charged" particle moving in a four-vector potential field, i.e., in a field similar to the electromagnetic potential. The charge here does not mean electric charge; it merely indicates that the particle responds to this field as an electrically charged particle responds to an external electromagnetic field, and may equally be called the coupling constant[29]. Units are used in which $c = \hbar = e = 1$, and the subscript W marks the analogue potentials throughout, so that they are never confused with the genuine electromagnetic $\mathbf{A}$ and $\mathbf{B}$ (Methods).

Since a charged particle in such a field may in general perform three-dimensional motion, we assume a wave function periodic in both the x- and the z-direction,

$$\Psi(\mathbf{r},t) = v(y)\exp\left[ikx + ik_z z - i\alpha\frac{\omega^2}{c_g^2}t\right] \tag{9}$$

There are two ways of choosing the potential so as to obtain the correct equation of motion.

(i) Let $k_z = f_0/c_g$ and let the four-vector potential $A_{W\mu}$ be

$$A_{W\mu} = \left(-\frac{\alpha\beta^2}{4\omega^2};\ -\frac{\beta}{2\omega},\ 0,\ -\frac{\beta y}{c_g}\right) \tag{10}$$

corresponding to a constant scalar potential $\phi = -\alpha\beta^2/4\omega^2$, which exerts no force, and a three-dimensional vector potential $\mathbf{A}_W = -(\beta/2\omega)\hat{\mathbf{x}} - \left(\beta y/c_g\right)\hat{\mathbf{z}}$, where $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$ are unit vectors pointing east, north, and upward. This vector potential produces a force field analogous to a magnetic field,

$$\mathbf{B}_W = \nabla \times \mathbf{A}_W = -\frac{\beta}{c_g}\hat{\mathbf{x}} \tag{11}$$

which is uniform and points to the west. The Hamiltonian is then

$$H = \alpha(-i\nabla - \mathbf{A}_W)^2 + \phi \tag{12}$$

with the charge set to unity. Substituting the wave function (9) into the Schrödinger equation $H\Psi = i\,\partial\Psi/\,\partial t$ and dividing throughout by α returns precisely Eq. (1), the wave equation sought. No term has been inserted by hand: the $\beta k/\omega$ term is generated by the cross term between $-i\,\partial/\,\partial x$ and A_{Wx}, and the residual $\beta^2/4\omega^2$ left over from that square is removed by the constant scalar potential, which is physically inert.

(ii) A second choice that yields the same equation of motion is to set

$$k_z = 0\,, \qquad A_{Wz} = -\frac{f_0+\beta y}{c_g} \tag{13}$$

leaving A_{Wx} and ϕ as before. The resulting $\mathbf{B}_W$-field is again given by Eq. (11). This version requires no vertical dependence of the wave function, which is more in line with the original problem of purely horizontal basic-state motion. The potential is assumed here; the companion paper[28] derives it. As shown below, models (i) and (ii) are not two different models at all, but the same physical system written in two gauges.

Gauge structure of the representation

The four-vector potential of Eq. (10) is not unique. In units with $c = \hbar = e = 1$, the transformation

$$\mathbf{A}_W \to \mathbf{A}_W + \boldsymbol{\nabla}\chi\,, \qquad \phi \to \phi - \frac{\partial \chi}{\partial t}\,, \qquad \Psi \to \Psi\exp(i\chi) \tag{14}$$

for any single-valued function $\chi(\mathbf{r},t)$ leaves both the force field $\mathbf{B}_W = \boldsymbol{\nabla}\times\mathbf{A}_W$ and the Schrödinger equation invariant. The physically meaningful quantity is therefore not the canonical momentum operator $\mathbf{p} = -i\boldsymbol{\nabla}$, which is gauge dependent, but the gauge-invariant kinematic momentum operator $\boldsymbol{\pi} = -i\boldsymbol{\nabla} - \mathbf{A}_W$. This distinction, a formality in many textbook problems, is the key to the difficulty encountered above.

The x-component of the vector potential, $A_{Wx} = -\beta/2\omega$, is a constant and therefore contributes nothing to $\mathbf{B}_W$: it is pure gauge. It can be removed entirely by choosing $\chi_1 = (\beta/2\omega)x$ in Eq. (14), which gives $A_{Wx} \to 0$ and, simultaneously,

$$\Psi = v(y)e^{ikx} \to v(y)\exp\left[i\left(k + \frac{\beta}{2\omega}\right)x\right] \tag{15}$$

Equation (15) is exactly the phase shift that had to be imposed by hand in Eq. (7). In the gauge $A_{Wx} = 0$ the term π_x^2 of the Hamiltonian becomes a pure multiplication by $(k + \beta/2\omega)^2$, and π_z^2 becomes $\left(\beta/c_g\right)^2(y-y_0)^2$; the Hamiltonian is then precisely the harmonic oscillator

Hamiltonian with the potential of Eq. (8) and the wave function of Eq. (7). In other words, **the harmonic oscillator representation is the four-vector potential representation written in a particular gauge.** Its two apparently arbitrary ingredients are not arbitrary at all: the phase shift is the gauge function χ_1, and the constant $-\alpha\beta^2/4\omega^2$ subtracted from the potential is the scalar potential of Eq. (10).

The physical content is the following. The zonal wavenumber k of the shallow water solution is the *canonical* momentum of the particle, while the gauge-invariant *kinematic* momentum, the one that enters the energy, is

$$\pi_x = k + \frac{\beta}{2\omega} \tag{16}$$

The term $\beta k/\omega$, which the harmonic oscillator model could not produce, is neither spurious nor a gauge artifact: it is the cross term that arises when the energy is expressed in terms of the canonical rather than the kinematic momentum, the difference between the two, $\beta/2\omega$, being the analogue of eA_x/c in the electromagnetic problem. This was anticipated in the earlier work[26], where the first-order term in k was attributed to the gauge used in deriving the shallow water equations, although the gauge function itself was not identified.

Models (i) and (ii) differ only in how the kinematic momentum along z is apportioned between the wavenumber and the potential: in model (i), $\pi_z = k_z - A_{Wz} = f_0/c_g + \beta y/c_g$; in model (ii), $\pi_z = 0 + (f_0 + \beta y)/c_g$. These are identical, and the two descriptions are connected by the gauge function $\chi_2 = -f_0 z/c_g$, which converts $A_{Wz} = -\beta y/c_g$ into $A_{Wz} = -(f_0 + \beta y)/c_g$ while simultaneously removing the factor $\exp(ik_z z)$ from the wave function (9). The two are therefore the same physical system, and the question of which is better is one of convenience rather than of physics: model (ii) is preferable because it requires no vertical wavenumber and hence no fictitious third dimension, the auxiliary z-dependence of model (i) being entirely a gauge artifact. The vertical wavenumber cannot be chosen freely: taking $k_z \neq 0$ in model (ii) replaces f_0 by $f_0 + k_z c_g$ in Eq. (1), which amounts to relocating the reference latitude, so that reproducing Eq. (1) fixes $\pi_z = (f_0 + \beta y)/c_g$ uniquely, whatever gauge is used to express it.

Because the Schrödinger equation is invariant under Eq. (14), the physics depends on $\mathbf{A}_W$ only through $\mathbf{B}_W$: two four-vector potentials producing the same $\mathbf{B}_W$ describe the same particle, and hence the same shallow water waves, however different they may look. The one exception in the electromagnetic case is the Aharonov–Bohm effect, in which the wave function acquires an observable phase from a region where $\mathbf{B}$ vanishes but $\mathbf{A}$ does not. Since $\mathbf{B}_W$ here is uniform and unbounded, no such effect arises, but the possibility is worth keeping in mind for geometries in which β varies or the domain is not simply connected.

The dispersion relation is a Landau spectrum

The particle constructed above is a charged particle moving in a uniform magnetic-like field. The corresponding electromagnetic problem — a free electron in a uniform magnetic field — was solved by Landau[30] in order to explain the diamagnetism of metals; for more recent summaries see refs.[31,32]. The energy levels are

$$E = \frac{\hbar^2 k_\parallel^2}{2m} + \left(n + \frac{1}{2}\right)\hbar\omega_c \tag{17}$$

where $k_\parallel$ is the kinematic wavenumber along $\mathbf{B}$, $n = 0,1,2, \ldots$ is the Landau level index, and $\omega_c = |e|B/mc$ is the cyclotron frequency (not to be confused with the Larmor precession frequency, which is half of it). The dispersion relation between E and $k_\parallel$ is a family of parabolas, one for each Landau level.

The earlier work[26] noted that Eq. (17) and the shallow water dispersion relation have the same form, drew an analogy between the two, and observed that the units of the two systems are not the same. The correspondence is in fact stronger than an analogy. It is exact, and Eq. (3) can be *derived* from Eq. (17) rather than compared with it.

Take the Hamiltonian of Eq. (12). The field $\mathbf{B}_W$ is uniform and directed along x, so the motion separates into free motion along $\mathbf{B}_W$, with the kinematic momentum π_x of Eq. (16), and Landau-quantized motion in the perpendicular (y, z) plane. In the units adopted here α plays the role of $\hbar^2/2m$, so that $m = 1/2\alpha$, while $e = c = \hbar = 1$ and $B_W = \beta/c_g$; the cyclotron frequency is therefore $\hbar\omega_c = |e|B/m = 2\alpha\beta/c_g$. The total energy, including the constant scalar potential of Eq. (10), is

$$E = \alpha\left(k + \frac{\beta}{2\omega}\right)^2 + (2n+1)\frac{\alpha\beta}{c_g} - \frac{\alpha\beta^2}{4\omega^2} \tag{18}$$

Setting $E = \alpha\omega^2/c_g^2$, as required by the wave function (9), and dividing by α, the square expands to give $k^2 + \beta k/\omega + \beta^2/4\omega^2$, the last term of which is cancelled identically by the scalar potential. What remains is exactly Eq. (3). The dispersion relation of tropical shallow water waves is thus the Landau spectrum of the Hamiltonian of Eq. (12); the Hermite index n of the Matsuno solution, Eq. (2), is the Landau level index of Eq. (17); and the Hermite functions of Eq. (2) are the Landau wave functions in the gauge $A_{Wx} = 0$. This makes the appearance of the factor $(2n + 1)$ in Eq. (3) — otherwise simply a property of the Hermite equation — a physical rather than a mathematical fact: it is the quantization of the particle's gyration about the analogue magnetic field. Figure 1a shows the dispersion relation read in this way, the branches of each meridional mode labelled by the Landau level index in place of the Hermite index.

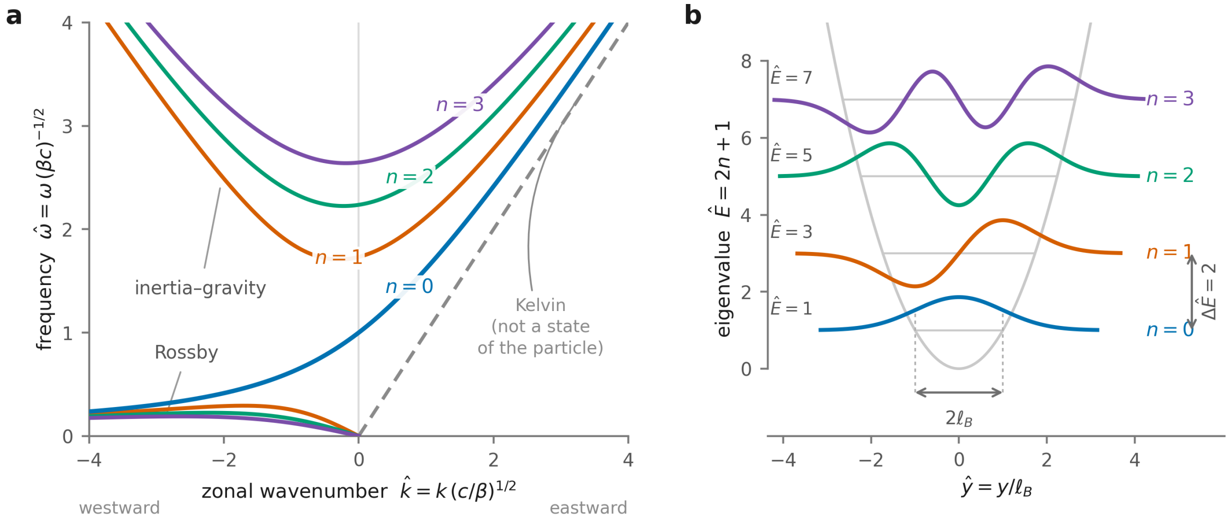

Fig. 1 | The Matsuno dispersion relation as a Landau spectrum. ***a***, Dispersion relation of Eq. (3) in non-dimensional form: frequency $\hat{\omega} = \omega(\beta c)^{-1/2}$ against zonal wavenumber $\hat{k} = k(c/\beta)^{1/2}$, for the meridional modes $n = 0$–3, coloured by Landau level index. Only positive frequencies are shown, so that each level appears as an inertia–gravity branch spanning both signs of $\hat{k}$ together with, for $n \geq 1$, a separate westward Rossby branch. The curves of a given colour are one Landau level of the Hamiltonian of Eq. (12), and the level index is the Hermite index of Eq. (2). The lowest level, $n = 0$, is the mixed Rossby–gravity (Yanai) wave, and is the only level on which the Rossby and gravity regimes join in a single continuous curve, Eq. (23), because the cubic Eq. (20) factorizes there and at no other level. The Kelvin wave (grey dashed) is shown for reference only: it is annihilated by the reduction to Eq. (1) and is not a state of the particle. ***b***, The corresponding meridional structures: the Landau wave functions $v_n \propto H_n(\hat{y})\exp(-\hat{y}^2/2)$ of Eq. (2), each drawn on its eigenvalue $\hat{E} = 2n + 1$ in the parabolic well of Eq. (1), with $\hat{y} = y/\ell_B$. Grey horizontal segments mark the classical turning points. The level spacing is uniform, as it is for a charged particle in a uniform field, and the width of the $n = 0$ Gaussian is the analogue magnetic length $\ell_B = (c/\beta)^{1/2}$ of Eq. (19), the equatorial Rossby radius of deformation.

The analogue magnetic field, magnetic length and guiding centre

The only force acting on the particle is the analogue magnetic field of Eq. (11): uniform, westward, and produced solely by the meridional gradient of the planetary vorticity. The "charge" of the particle is a coupling constant that measures how strongly the wave field responds to that gradient, and has been set to unity. The scalar potential is constant and exerts no force at all; it is present only to keep the energy zero point consistent with the definition of the canonical momentum.

The physical statement contained in this is that the β-effect acts on a shallow water wave packet in the same way that a uniform magnetic field acts on a charged particle. The meridional trapping of equatorial waves — the equatorial waveguide — is thereby identified as cyclotron gyration, and its quantization as Landau quantization. For the Earth's atmosphere with $\beta \approx 2.3 \times$

$10^{-11}\ m^{-1}s^{-1}$ and a first baroclinic mode speed $c_g \approx 30\ m\,s^{-1}$, the analogue field strength is $B_W = \beta/c_g \approx 7.6 \times 10^{-13}\ m^{-2}$.

For an electron in a magnetic field, the natural length scale is the magnetic length $\ell_B = (\hbar c/|e|B)^{1/2}$, the characteristic radius of the lowest Landau orbit. For the analogue field of Eq. (11), in the present units, this is simply $B_W^{-1/2}$, so that

$$\ell_B = \left(\frac{c_g}{\beta}\right)^{1/2} \equiv L_{eq} \tag{19}$$

which is the equatorial Rossby radius of deformation. This is not a coincidence but an identity: the Gaussian factor in Eq. (2) is $\exp[-(y-y_0)^2/2\ell_B^2]$, which is the lowest Landau wave function. The meridional width of an equatorially trapped wave and the cyclotron radius of the analogue particle are the same quantity. For the parameters quoted above, $\ell_B \approx 1150$ km; for a shallower equivalent depth, say $c_g \approx 2.8\,m\,s^{-1}$, it is about 350 km. Correspondingly, the Landau level spacing measured in the units of Eq. (3) is $\beta/c_g = \ell_B^{-2}$, so that the spectrum of meridional modes is fixed by the deformation radius alone. Figure 1b shows the corresponding ladder of meridional structures.

In the electromagnetic Landau problem each level is enormously degenerate: for a given n, the wave function may be centered on any guiding-center coordinate. Here the guiding center is the latitude at which the kinematic momentum $\pi_z = (f_0 + \beta y)/c_g$ vanishes, that is, $y = y_0 = -f_0/\beta$, the latitude at which the Coriolis parameter itself vanishes. The freedom to displace the guiding center is exactly the freedom to choose k_z, and reproducing Eq. (1) forces k_z to a single value. The two-dimensionality of the system therefore removes the Landau degeneracy and pins the guiding center to the zero-f latitude — the particle-representation statement of the familiar fact that equatorial waves are trapped about the equator and not about some arbitrary latitude.

Finally, since Ψ is proportional to $v(y)$, the probability density of the particle is $\Psi^*\Psi \propto |v(y)|^2$, which is proportional to the meridional kinetic energy density of the wave, $\frac{1}{2}\rho H v^2$. The particle is therefore most likely to be found where the meridional motion of the fluid is most vigorous. This gives the quantum mechanical formalism a direct fluid dynamical reading: the "position" of the particle is the latitude at which the wave carries its meridional energy, and the spreading of $|\Psi|^2$ over the deformation radius is the meridional extent of the waveguide.

A Hamiltonian that depends on its own eigenvalue

An unusual feature of the present construction is that the potential depends on the frequency: $A_{Wx} = -\beta/2\omega$ and $\phi = -\alpha\beta^2/4\omega^2$ both contain ω, while the energy is $E = \alpha\omega^2/c_g^2$. The Hamiltonian is therefore a function of its own eigenvalue, $H = H(E)$, and the eigenvalue problem is implicit rather than linear. This is the sharpest disanalogy with the electromagnetic Landau problem, in which H is independent of E.

Far from being a defect, this is the feature that produces the most characteristic property of the tropical wave spectrum. Multiplying Eq. (3) by ω gives

$$\frac{\omega^3}{c_g^2} - \omega\left[k^2 + \frac{(2n+1)\beta}{c_g}\right] - \beta k = 0 \qquad (20)$$

a cubic with three roots for each n: two of high frequency, corresponding to eastward- and westward-propagating inertia–gravity waves, and one of low frequency, corresponding to the westward-propagating Rossby wave. Had A_{Wx} been independent of ω — that is, had the vector potential not depended on the particle's own energy — the term $\beta k/\omega$ would be absent, Eq. (3)

would be quadratic in ω, and only the two gravity wave branches would survive. **The Rossby wave exists, in this representation, precisely because the particle moves in a field that depends on its own energy.** The usual care must of course be taken with the $n = 0$ case, for which one of the three roots is spurious[6].

It would be natural to read this as indicating that the four-vector potential is a self-consistent field, determined together with the state of the particle rather than imposed from outside. That reading is not correct: the frequency enters through the elimination of u and Φ by which Eq. (1) was obtained, and the companion paper[28] shows that the parent Lagrangian contains no reference to ω at all. The self-referential character of H is the price of reducing three coupled fields to one.

The Yanai wave as the lowest Landau level

The particle picture is at its most informative for the gravest meridional mode, $n = 0$, whose solution is the mixed Rossby–gravity or Yanai wave[5,6]. Because the field $\mathbf{B}_W$ of Eq. (11) is uniform, the perpendicular part of the Hamiltonian (12) can be factorized in the standard way. Defining

$$a = \frac{\pi_y - i\pi_z}{\sqrt{2\beta/c_g}}\,, \qquad a^\dagger = \frac{\pi_y + i\pi_z}{\sqrt{2\beta/c_g}}\,, \qquad [a, a^\dagger] = 1 \tag{21}$$

where $\pi_y = -i\,\partial/\partial y$ and $\pi_z = \beta(y - y_0)/c_g$ are the gauge-invariant kinematic momenta perpendicular to $\mathbf{B}_W$, the perpendicular kinetic energy becomes $\pi_y^2 + \pi_z^2 = \left(2\beta/c_g\right)\left(a^\dagger a + \frac{1}{2}\right)$, whose eigenvalues are $(2n+1)\beta/c_g$, exactly the term appearing on the right of Eq. (3). The Yanai wave is the state annihilated by a, and it therefore satisfies a **first-order** equation rather than the second-order structure equation (1):

$$a\,v = 0 \;\Rightarrow\; \frac{dv}{dy} + \frac{\beta(y-y_0)}{c_g}v = 0 \;\Rightarrow\; v \propto exp\left[-\frac{\beta(y-y_0)^2}{2c_g}\right] \tag{22}$$

which reproduces Eq. (2) at $n = 0$. Three consequences follow.

First, the Yanai wave is the cyclotron ground state. Its meridional structure is a nodeless Gaussian of width exactly the analogue magnetic length of Eq. (19), and its contribution $\beta/c_g = \ell_B^{-2}$ to the dispersion relation is pure zero-point energy of the gyration, with no excitation of the cyclotron motion at all. Every other equatorial mode follows from it by repeated application of $a^\dagger$, which is why n counts nodes. That index cannot be altered by a gauge transformation: gauge transformations leave $\mathbf{B}_W$, and hence the whole Landau spectrum, invariant, and since $\Psi^*\Psi$ is gauge invariant, so is the nodal structure of v. Moving between meridional modes requires the ladder operator, not the gauge function.

Second, the symmetry of the mode follows immediately. With $H_0 = 1$, the meridional velocity v is symmetric about y_0, while u and Φ, obtained from v through a derivative and multiplication by $f_0 + \beta y$, are antisymmetric. The Yanai wave therefore has maximum cross-equatorial flow precisely where the Coriolis parameter vanishes, with a node in geopotential and zonal wind there — the opposite symmetry to the Kelvin and $n = 1$ Rossby modes. In the particle language, the guiding center of the lowest Landau orbit sits on the zero-f latitude, and the gyration is what carries fluid across it: the structural reason why the Yanai wave is the most effective equatorial mode for cross-equatorial momentum transport.

Third, $n = 0$ is the only level at which the Rossby and gravity branches do not separate. Setting $n = 0$ in Eq. (20) and clearing denominators, the cubic factorizes as $\left(\omega + kc_g\right)\left(\omega^2 - kc_g\omega - \beta c_g\right) = 0$. The root $\omega = -kc_g$ must be discarded: the reduction of the shallow water equations to Eq. (1) requires division by $c_g^2k^2 - \omega^2$, which vanishes there[6]. What survives is a quadratic, not a cubic, and taking $\omega > 0$,

$$\omega = \frac{1}{2}\left[kc_g + \sqrt{k^2c_g^2 + 4\beta c_g}\right] \quad (23)$$

a single continuous curve with $\omega \to kc_g + \beta/k$ as $k \to +\infty$ and $\omega \to -\beta/k$ as $k \to -\infty$. The mode is thus inertia–gravity-like at the eastward end and Rossby-like at the westward end, which is the content of the name "mixed Rossby–gravity wave." For every $n \geq 1$ the cubic does not factorize, the gravity branches retain a frequency floor, and a separate low-frequency Rossby branch exists; only the lowest Landau level bridges the two regimes. In the particle picture the reason is that the missing third root is the one associated with $v \equiv 0$, and the lowest Landau level is the last state before that limit.

These structural properties account for the prominence of the Yanai wave in observations. It was the first of Matsuno's modes identified in the atmosphere, in equatorial lower-stratospheric radiosonde data[5,11], and its antisymmetric geopotential signature makes it separable as a distinct peak in wavenumber–frequency spectra of satellite-observed convection[20,21]. Because it transports easterly momentum upward, it was assigned a central role, alongside the Kelvin wave, in the classical wave-driven theory of the quasi-biennial oscillation[33–35], although current assessments attribute the larger share of the momentum flux to a broad spectrum of smaller-scale gravity waves[36]. In the troposphere, its convective counterpart over the Pacific is associated with tropical-depression-type disturbances and thus with one recognized pathway to tropical cyclogenesis[37].

Discussion

The linearized shallow water equations on a general β-plane have been reformulated as a quantum mechanical particle problem. The meridional structure equation is exactly the Schrödinger equation for a particle of unit charge minimally coupled to a four-vector potential whose analogue magnetic field is uniform and westward, and no *ad hoc* term is required. A representation based on a simple harmonic oscillator reproduces the structure equation only if a phase shift and a constant potential are inserted by hand, and both are explained by the four-vector representation: the phase shift is the gauge function $\chi_1 = (\beta/2\omega)x$ that removes the constant part of A_{Wx}, and the constant potential is the scalar potential. The zonal wavenumber is the canonical momentum, the gauge-invariant kinematic momentum is $\pi_x = k + \beta/2\omega$, and the $\beta k/\omega$ term of the dispersion relation is the cross term generated when the energy is expressed through the canonical rather than the kinematic momentum. The two versions of the vector potential are the same physical system in two gauges, related by $\chi_2 = -f_0 z/c_g$. The Matsuno dispersion relation is the Landau spectrum of the constructed Hamiltonian, the Hermite index is the Landau level index, the equatorial radius of deformation is the analogue magnetic length, and the meridional trapping of equatorial waves is cyclotron gyration. Because the vector potential depends on the frequency the eigenvalue problem is implicit, and that dependence is the origin of the Rossby branch. The Yanai wave is the lowest Landau level, annihilated by the lowering operator and therefore satisfying a first-order equation, and its mixed Rossby–gravity character follows from the factorization of the $n = 0$ cubic, which occurs at no other meridional mode. The constant α cancels identically from the dispersion relation and is therefore not determined by the wave problem; it fixes only the map between wave frequency and particle energy.

Three qualifications should be stated plainly. The correspondence is a mathematical identity between two eigenvalue problems, not a claim that a material particle exists. What has been established is that the shallow water wave field of a given zonal wavenumber and meridional mode number can be labeled by, and computed from, a single particle state, the labels behaving exactly as the quantum numbers of a charged particle in a uniform magnetic field; the economy claimed is an economy of description. The units of the particle problem are likewise those of the wave problem, not those of quantum mechanics: the particle's "momentum" is a wavenumber, its "energy" a frequency, and its "mass" is fixed by α, whose value is a matter of normalization. No dimensional bridge to elementary particle physics is intended or implied.

The third qualification is more interesting, because it can be stated precisely. The representation is built on the structure equation for v and therefore does not describe modes for which $v \equiv 0$. The Kelvin wave is the important case. The reduction of the shallow water equations to Eq. (1) requires division by $c_g^2 k^2 - \omega^2$, which vanishes identically on the Kelvin branch $\omega = kc_g$; the mode is annihilated by the elimination of u and Φ, before any of the present construction is applied. It is the second casualty of that division, the first being the spurious root $\omega = -kc_g$ discarded at $n = 0$. In the particle language the Kelvin wave is formally the state $n = -1$: substituting $n = -1$ in Eq. (3) gives $\omega = kc_g$ identically. But the Landau ladder terminates, since a annihilates the ground state, and there is no level below $n = 0$. The Yanai wave is the gravest mode of Eq. (1) precisely because the state below it is the one with $v \equiv 0$, which is the Kelvin wave itself and not a state of the particle at all. Suggestively, the lowest Landau structure is nevertheless present in the Kelvin mode: on an equatorial β-plane its zonal velocity is $u \propto \exp\left(-\beta y^2/2c_g\right)$, the same Gaussian, of the same width ℓ_B, as the $n = 0$ wave function of Eq. (2). The structure lives in u and Φ rather than in v. A representation capturing both modes would therefore have to be built on a multi-component wave function carrying (u, v, Φ) together, rather than on a scalar amplitude obtained by reducing to one of them. This is presumably why the topological treatment of Delplace et al.[24], which never performs such a reduction, captures the Kelvin wave on the same footing as the Yanai wave, and it indicates that the limitation belongs to the present construction rather than to the condensed-matter analogy in general.

Three directions seem worth pursuing. The first is a slowly varying medium, in which β, c_g, or the depth vary in space and the analogue magnetic field becomes non-uniform; the semiclassical machinery developed for electrons in inhomogeneous fields would then apply directly, and the drift of the guiding center would give a compact description of wave propagation through a varying basic state. The second is the incorporation of the Kelvin mode along the lines just indicated. The third, and perhaps the most interesting, is the relation between the Landau description given here and the topological description[24], which identifies the same two modes as exceptional by way of a Chern number rather than a level index. Landau levels and Chern numbers are intimately connected in the condensed matter setting, and it would be surprising if the connection did not survive the translation to the equatorial waveguide.

A word on the relation to the operator treatments of the same system. The linearized shallow water equations need not be reduced to Eq. (1) at all: kept as a first-order three-component Hermitian operator in latitude they have been studied in detail, and that is the form in which the topological analysis of Delplace et al.[24] is carried out. Faure[38] solves the equatorial problem directly in that representation, using ladder operators acting on Hermite functions to generate the Kelvin, Yanai and higher branches; Delplace[39] embeds it in a family of spin-*S* Hamiltonians whose spectral flows are governed by Berry–Chern monopoles; Rossi and Tarantola[40] read the Coriolis parameter explicitly as a Dirac mass and the equator as the interface between two topologically distinct regions; Tong[41] obtains the shallow water equations from a gauge theory; and Tauber, Delplace and Venaille[42] establish a bulk–interface correspondence for the equatorial waveguide. A related dictionary, in which the Coriolis frequency plays the part of a plasma frequency or of a rest mass and the speed of long gravity waves that of the speed of light, has been given by Heifetz and colleagues[43,44]. The relation between those descriptions and the present one is that the meridional structure equation, Eq. (1), is the product of two first-order operators, so that the Landau spectrum obtained here is a ladder in the square of the frequency rather than in the frequency itself; this is developed elsewhere (P.K.W., manuscript in preparation).

Methods

Linearized shallow water equations on a general β-plane

Shallow water waves on a β-plane have been considered by various investigators[6,7,9]. We consider a general β-plane that is not necessarily centered on the equator, so that the local Coriolis parameter is approximated by

$$f \simeq f_0 + \beta y \tag{24}$$

where f_0 is the Coriolis parameter at the reference latitude, y is the meridional coordinate, positive to the north, and $\beta = df/dy$ is assumed constant. Under this assumption, the linearized equations governing the perturbation zonal velocity u, meridional velocity v, and geopotential Φ are

$$\frac{\partial u}{\partial t} - (f_0 + \beta y)v + \frac{\partial \Phi}{\partial x} = 0 \tag{25}$$

$$\frac{\partial v}{\partial t} + (f_0 + \beta y)u + \frac{\partial \Phi}{\partial y} = 0 \tag{26}$$

$$\frac{\partial \Phi}{\partial t} + c_g^2\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0 \tag{27}$$

where c_g is the speed of long gravity waves, defined by $c_g^2 = gH$, with g the gravitational acceleration and H the mean depth of the fluid. The Coriolis terms couple u and v to each other and do no net work on the flow, as they must. Assuming periodic solutions of the form

$$(u, v, \Phi) = \big(u'(y), v'(y), \Phi'(y)\big)\exp(ikx - i\omega t) \tag{28}$$

and eliminating u and Φ, which requires division by $c_g^2k^2 - \omega^2$, reduces Eqs. (25)–(27) to Eq. (1) for v', with the prime dropped.

Minimal coupling in classical and quantum mechanics

Only the elements needed above are recorded here; for the physical background see refs.[22,45]. Gaussian units are used, c denotes the speed of light, and the speed of long gravity waves is written c_g throughout, so that no confusion between the two can arise. For a particle of mass m and charge e moving in an electromagnetic field, the classical Hamiltonian is

$$H = \frac{1}{2m}\left(\mathbf{p} - \frac{e\mathbf{A}}{c}\right)^2 + e\phi \tag{29}$$

where ϕ is the scalar potential and $\mathbf{A}$ the vector potential of the magnetic field, $\mathbf{B} = \nabla \times \mathbf{A}$. The Hamiltonian is still the total energy, since the magnetic field does no work. The mechanical (kinematic) momentum, however, is now $\mathbf{p} - e\mathbf{A}/c$ rather than $\mathbf{p}$, while $\mathbf{p}$ itself is the canonical momentum, and it is the canonical momentum that is carried over to the quantum mechanical formulation. Replacing $\mathbf{p} \to -i\hbar\nabla$ and $E \to i\hbar\,\partial/\partial t$ in $H\Psi = E\Psi$, and taking $\Psi(\mathbf{r}, t) = \psi(\mathbf{r})\exp(-iEt/\hbar)$, gives the time-independent Schrödinger equation with minimal coupling,

$$\frac{1}{2m}\left(-i\hbar\nabla - \frac{e\mathbf{A}}{c}\right)^2 \psi + e\phi\psi = E\psi \tag{30}$$

with E the total energy of the particle. Note that the coupling must be written in terms of the momentum operator, as in Eq. (30), and not as $(\nabla - e\mathbf{A}/c)^2$, whose two terms differ dimensionally by a factor $\hbar$; writing it as $(\mathbf{p} - e\mathbf{A}/c)$ throughout is the convention adopted above.

The role of the constant α

The earlier work[26] observed that the formulation goes through with $\alpha = 1$, remarked that the physical meaning of α was at that time unclear, and suggested that it might prove to be an important characteristic number of the geophysical shallow water problem. The derivation given above settles the question. α cancels identically between the two sides of Eq. (18) and does not appear in Eq. (3), so the dispersion relation places no constraint on it whatever, and it is not a characteristic number of the wave problem in the sense anticipated. What it does fix is the relation between wave frequency and particle energy, $E = \alpha\omega^2/c_g^2$, and hence the rate at which the particle gyrates in real time; carrying the dimensions of $\hbar/m$, it can be given a numerical value only after a system of units has been chosen. One choice deserves mention. Written in the form

$$\frac{1}{2}\frac{\omega^2}{c_g^2} = \frac{1}{2}\left(k^2 + \frac{\beta k}{\omega}\right) + \left(n + \frac{1}{2}\right)\frac{\beta}{c_g} \tag{31}$$

Eq. (3) matches Eq. (17) term by term: $\frac{1}{2}\omega^2/c_g^2$ corresponds to E, $\frac{1}{2}(k^2 + \beta k/\omega)$ to $\hbar^2 k_\parallel^2/2m$, and β/c_g to $\hbar\omega_c$. This reading amounts to $\alpha = 1/2$, that is, $m = 1$ in units with $\hbar = 1$ — a convenient normalization, making the particle's energy exactly half its squared kinematic momentum as for an ordinary free particle of unit mass, but a choice of units and not a requirement of the physics.

Data availability

No datasets were generated or analysed during the current study. All results reported here are analytic and are fully reproducible from the equations given in the text.

Code availability

No custom code was used in this study.

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Acknowledgements

This research is supported by National Science and Technology Council of Taiwan under grants NSTC 115-2119-M-001-007 and NSTC 115-2111-M-001-013.

Author contributions

P.K.W. conceived the study, carried out the analysis, and wrote the manuscript.

Competing interests

The author declares no competing interests.