

A Particle Representation of Tropical Shallow Water Waves

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ABSTRACT

The linearized shallow water equations on a general β -plane reduce to a single meridional structure equation for the meridional velocity. It is shown that this equation is exactly the Schrödinger equation for a “charged” particle minimally coupled to a four-vector potential whose analogue magnetic field, $\mathbf{B}_W = -(\beta/c_g)\hat{\mathbf{x}}$, is uniform and westward. No ad hoc term is required: every term of the structure equation is generated by the minimal coupling. A representation based on a simple harmonic oscillator, by contrast, reproduces the structure equation only if a phase shift and a constant potential are inserted by hand; both are identified here as gauge ingredients of the four-vector representation. The zonal wavenumber is the canonical momentum of the particle, the gauge-invariant kinematic momentum being $k + \beta/2\omega$. The analogue magnetic length is the equatorial Rossby radius of deformation, the Hermite index of the Matsuno solution is the Landau level index, and the Matsuno dispersion relation is recovered exactly as the Landau spectrum of the constructed Hamiltonian rather than merely resembling it. Because the vector potential depends on frequency, the eigenvalue problem is implicit, and this dependence is the origin of the Rossby branch. The mixed Rossby–gravity (Yanai) wave is the lowest Landau level: it satisfies a first-order equation, and its mixed character follows from a factorization of the dispersion relation that occurs for no other meridional mode. The Kelvin wave, whose meridional velocity vanishes, lies outside the representation.

29 **SIGNIFICANCE STATEMENT**

30 Large-scale waves trapped near the equator influence tropical weather and climate, but their
31 wave patterns are complicated and difficult to track. This study shows that each of these waves
32 can be described exactly as a single particle pushed by a magnetic-like force, which arises
33 because the effect of Earth's rotation changes with latitude. In this picture, the width of the
34 tropical wave channel equals the size of the particle's orbit, the familiar family of tropical wave
35 types becomes a ladder of particle energy levels, and one wave type, observed in the tropical
36 stratosphere and linked to tropical storm formation, is the lowest rung of that ladder. The particle
37 view offers a compact way to label and analyze these waves.

38

1. Introduction

Wave phenomena in the atmosphere and ocean play an important role in many geophysical processes, such as energy transport, climate variability, and cloud formation. Among the most important categories for synoptic- and large-scale phenomena are shallow water waves, whose wavelengths are much longer than the depth of the fluid; they have been studied in great detail and are summarized in many textbooks (Holton 1979; Pedlosky 1979; Gill 1982; Vallis 2017). Tropical shallow water waves have received special attention because the tropics are the energy source region of many atmospheric and oceanic processes. Important contributions to their theory and observation have been made by many investigators (e.g., Rosenthal 1965; Matsuno 1966; Yanai and Maruyama 1966; Blandford 1966; Lindzen 1967; Lindzen and Matsuno 1968; Yanai et al. 1968; Wallace and Kousky 1968; Mak 1969; Wallace 1971; Hayashi 1976; Wunsch and Gill 1976; Anderson and Gill 1979; Grose and Hoskins 1979; Itoh and Ghil 1988), and the observational picture has since been placed on a firm spectral footing (Wheeler and Kiladis 1999) and reviewed (Kiladis et al. 2009).

All of these studies have been concerned with the “wave” properties of the disturbances. For long-term climate studies, the role of the waves can be difficult to recognize: wave profiles are complicated, made more so by reflection, refraction, and diffraction, and in combination with complex boundaries the resulting signals are difficult to analyze, let alone to interpret. One way of reducing such complexity is to reformulate the waves as a system in which they can be regarded as particles, or at least be associated with particle properties. Unlike waves, which are spread out in space and time, particles are compact and usually easier to identify.

The present study is motivated by the wave–particle duality established in quantum physics early in the twentieth century (Schiff 1968; Saxon 1968): perhaps, with a suitable formulation, waves in the atmosphere and ocean can also be viewed as particles, and such a view may simplify their analysis. The bound electron of the hydrogen atom illustrates the potential gain. Its wave description requires a probability density $\Psi^*\Psi$ that is already a complicated function of space, and the time series of that signal at a point would be almost impossible to analyze; this complication disappears once the signal is recognized as a single particle, from which many properties of hydrogen can be deduced.

The idea that geophysical wave problems may be illuminated by the methods of condensed matter physics has since been pursued independently, and from a quite different direction.

Delplace et al. (2017) showed that the Kelvin and Yanai modes owe their existence to a topological invariant of the shallow water system—the first Chern number, of value two, of the bulk Poincaré wave bands—in close analogy with the protected edge states of a topological insulator (see also Venaille and Delplace 2021). Both approaches single out the Yanai wave as exceptional; the Kelvin wave, which that work treats on the same footing, lies outside the present representation (section 8a).

The question addressed here is whether geophysical shallow water waves can be formulated as particles and, if so, what rules of behavior those particles obey. The analysis indicates that they obey a Schrödinger equation with minimal coupling to a four-vector potential, just as charged quantum mechanical particles do. This paper develops and completes an argument first presented at the Eighth Conference on Atmospheric and Oceanic Waves and Stability (Wang 1991), from which the construction, and the observation that the resulting dispersion relation resembles the Landau spectrum of a free electron in a uniform magnetic field, are taken. Three matters left open in that work are settled here: (i) the gauge structure of the four-vector potential, which accounts for an ad hoc phase shift and shows two apparently different versions of the construction to be one physical system written in two gauges; (ii) the status of the correspondence with the Landau problem, shown below to be an identity rather than an analogy, together with the meaning of the constant α , which was explicitly left for future study; and (iii) the physical interpretation of the particles. The four-vector potential is written down and verified here rather than derived; it is obtained from Hamilton’s principle, applied to the rotating-frame Lagrangian of Wang (1984), in a companion paper (Wang 2026, hereafter Paper II).

The paper is organized as follows. Section 2 presents the governing equations and the meridional structure equation. Section 3 shows why a harmonic-oscillator representation is unsatisfactory, and section 4 constructs the four-vector potential representation. Its gauge structure is analyzed in section 5, and the identification of the dispersion relation with a Landau spectrum is made in section 6. Section 7 treats the mixed Rossby–gravity (Yanai) wave as the lowest Landau level. The results are discussed in section 8 and summarized in section 9. The role of the constant α is examined in appendix A.

2. Governing equations

a. Linearized shallow water equations on a general β -plane

Shallow water waves on a β -plane have been considered by various investigators (Matsuno 1966; Blandford 1966; Lindzen 1967). We consider a general β -plane that is not necessarily centered on the equator, so that the local Coriolis parameter is approximated by

$$f \simeq f_0 + \beta y, \quad (1)$$

where f_0 is the Coriolis parameter at the reference latitude, y is the meridional coordinate (positive northward), and $\beta = df/dy$ is assumed constant. The linearized equations governing the perturbation zonal velocity u , meridional velocity v , and geopotential Φ are

$$\frac{\partial u}{\partial t} - (f_0 + \beta y) v + \frac{\partial \Phi}{\partial x} = 0, \quad (2)$$

$$\frac{\partial v}{\partial t} + (f_0 + \beta y) u + \frac{\partial \Phi}{\partial y} = 0, \quad (3)$$

$$\frac{\partial \Phi}{\partial t} + c_g^2 \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = 0, \quad (4)$$

where c_g is the speed of long gravity waves, defined by $c_g^2 = gH$, with g the gravitational acceleration and H the mean depth of the fluid. The Coriolis terms couple u and v and do no net work on the flow, as they must. Solutions periodic in x and t are sought in the form

$$(u, v, \Phi) = [u'(y), v'(y), \Phi'(y)] \exp(ikx - i\omega t), \quad (5)$$

where k is the zonal wavenumber and ω the angular frequency.

b. The meridional structure equation

Eliminating u and Φ from (2)–(4), which requires division by $c_g^2 k^2 - \omega^2$, yields a single meridional structure equation for the meridional velocity (the prime is dropped hereafter):

$$\frac{d^2 v}{dy^2} + \left[\frac{\omega^2}{c_g^2} - k^2 - \frac{\beta k}{\omega} - \frac{(f_0 + \beta y)^2}{c_g^2} \right] v = 0. \quad (6)$$

Equation (6), with the boundary condition $v \rightarrow 0$ as $y \rightarrow \pm\infty$, poses an eigenvalue problem. Since $f_0 + \beta y = \beta(y - y_0)$ with $y_0 = -f_0/\beta$, the last term in the bracket is $-(\beta/c_g)^2 (y - y_0)^2$, so that the problem is that of a particle in a parabolic well centered on the latitude at which the Coriolis parameter vanishes. The solution is

$$v = H_n \left[\left(\frac{\beta}{c_g} \right)^{1/2} (y - y_0) \right] \exp \left[-\frac{\beta(y-y_0)^2}{2c_g} \right], \quad (7)$$

where H_n is the Hermite polynomial of order n ; the normalization constant is omitted because (6) is homogeneous (Matsuno 1966; Gill 1982). The dispersion relation is

$$\frac{\omega^2}{c_g^2} - k^2 - \frac{\beta k}{\omega} = \frac{(2n+1)\beta}{c_g}, \quad (8)$$

with $n = 0, 1, 2, \dots$. On an equatorial β -plane, $f_0 = 0$ and $y_0 = 0$, and the oscillation is symmetric about the equator; for a general β -plane the eigenfunctions are the same functions displaced to $y = y_0$, and the dispersion relation is unchanged. The trapped solutions are of practical interest chiefly in the tropics, because for a midlatitude reference latitude the turning latitude y_0 lies outside the range over which the β -plane approximation is defensible.

3. Failure of a harmonic-oscillator representation

Matsuno (1966) noted the similarity between (6) and the quantum mechanical problem of a particle bound in a harmonic potential, and one may be tempted to build a particle formulation directly on the Schrödinger equation for a harmonic oscillator. We first show that this is unsatisfactory. If a Schrödinger equation can be written down whose wave amplitude is the v of (6), it is justified to declare that the corresponding particle exists, even though its nature is not yet known; its behavior must then be interpreted consistently with quantum mechanics.

Consider the case in which the waves represent the amplitude of a particle moving in a simple harmonic oscillator potential proportional to $(\beta/c_g)^2 (y - y_0)^2$. The quantum mechanical operators for this particle are

$$E = i \frac{\partial}{\partial t}, \quad \mathbf{p} = -i\nabla, \quad H = \alpha(-i\nabla)^2 + \alpha \left(\frac{\beta}{c_g} \right)^2 (y - y_0)^2, \quad (9)$$

where α plays the role of $\hbar^2/2m$. Since elementary particles are not involved, there is no reason to expect $\hbar$ and m to retain their usual meanings, and it is convenient to absorb them into the single constant α , which carries the dimensions of $\hbar/m$ (section 4a and appendix A). Taking the plane-wave function

$$\Psi(\mathbf{r}, t) = v(y) \exp \left[ikx - i\alpha \left(\frac{\omega^2}{c_g^2} \right) t \right], \quad (10)$$

in which the energy term $E/\hbar$ has been written explicitly as $\alpha(\omega^2/c_g^2)$, the energy of a plane wave of angular frequency ω , and substituting it into the Schrödinger equation with the operators of (9), we obtain

$$\frac{d^2v}{dy^2} + \left[\frac{\omega^2}{c_g^2} - k^2 - \left(\frac{\beta}{c_g} \right)^2 (y - y_0)^2 \right] v = 0. \quad (11)$$

Comparison of (11) with (6) shows that the term $\beta k/\omega$ is missing. To repair this, the wave function of (10) may be changed to

$$\Psi(\mathbf{r}, t) = v(y) \exp \left[i \left(k + \frac{\beta}{2\omega} \right) x - i \alpha \frac{\omega^2}{c_g^2} t \right], \quad (12)$$

and, at the same time, the harmonic oscillator potential rewritten as

$$V = \alpha \left[\left(\frac{\beta}{c_g} \right)^2 (y - y_0)^2 - \frac{\beta^2}{4\omega^2} \right]. \quad (13)$$

With these two changes, (6) is recovered exactly. In writing (12) and (13), however, it has been asserted a priori that the zonal wavenumber must be shifted by $\beta/2\omega$ —equivalently, that the wave phase must be shifted by $\exp(i\beta x/2\omega)$ —and that the potential must be lowered by $\alpha\beta^2/4\omega^2$. A potential may always be shifted by an arbitrary constant without changing the physics, but there is no obvious justification for the phase shift; as it stands, it is no more than an ad hoc manipulation. It would be far more satisfactory to have these quantities specified by the operators, that is, by the external conditions in which the particle moves, rather than inserted by hand into the wave function.

4. A charged particle in a four-vector potential

a. Minimal coupling in classical and quantum mechanics

Only the elements needed below are recorded here; for the physical background, see Goldstein (1950) and Schiff (1968). Gaussian units are used, c denotes the speed of light, and the speed of long gravity waves is written c_g throughout so that no confusion between the two can arise. For a particle of mass m and charge e moving in an electromagnetic field, the classical Hamiltonian is

$$H = \frac{1}{2m} \left(\mathbf{p} - \frac{e\mathbf{A}}{c} \right)^2 + e\phi, \quad (14)$$

where ϕ is the scalar potential and $\mathbf{A}$ the vector potential of the magnetic field, $\mathbf{B} = \nabla \times \mathbf{A}$. The Hamiltonian is still the total energy, since the magnetic field does no work. The mechanical (kinematic) momentum, however, is now $\mathbf{p} - e\mathbf{A}/c$ rather than $\mathbf{p}$, while $\mathbf{p}$ itself is the canonical momentum, and it is the canonical momentum that is carried over to the quantum mechanical formulation. Replacing $\mathbf{p} \rightarrow -i\hbar\nabla$ and $E \rightarrow i\hbar \partial/\partial t$ in $H\Psi = E\Psi$, and taking $\Psi(\mathbf{r}, t) = \psi(\mathbf{r})\exp(-iEt/\hbar)$, gives the time-independent Schrödinger equation with minimal coupling,

$$\frac{1}{2m} \left(-i\hbar\nabla - \frac{e\mathbf{A}}{c} \right)^2 \psi + e\phi\psi = E\psi, \quad (15)$$

with E the total energy of the particle. The coupling must be written in terms of the momentum operator, as in (15), and not as $(\nabla - e\mathbf{A}/c)^2$, whose two terms differ dimensionally by a factor $\hbar$; writing it as $(\mathbf{p} - e\mathbf{A}/c)$ throughout is the convention adopted below.

b. The four-vector potential

We now derive (6) by a more satisfactory procedure (Wang 1991). The shallow water waves are assumed to be associated with the motion of a “charged” particle moving in a four-vector potential field, that is, in a field similar to the electromagnetic potential. The charge here is not an electric charge; it merely indicates that the particle responds to this field as an electrically charged particle responds to an external electromagnetic field, and may equally be called the coupling constant (Frauenfelder and Henley 1974). Units are used in which $c = \hbar = e = 1$, and the subscript W marks the analogue potentials throughout, so that they are never confused with the genuine electromagnetic $\mathbf{A}$ and $\mathbf{B}$.

Since a charged particle in such a field may in general perform three-dimensional motion, a wave function periodic in both the x - and z -directions is assumed:

$$\Psi(\mathbf{r}, t) = v(y)\exp\left(ikx + ik_z z - i\alpha \frac{\omega^2}{c_g^2} t\right). \quad (16)$$

There are two ways of choosing the potential so as to obtain the correct equation of motion.

(i) Let $k_z = f_0/c_g$ and let the four-vector potential $A_{W\mu}$ be

$$A_{W\mu} = \left(-\frac{\alpha\beta^2}{4\omega^2}; -\frac{\beta}{2\omega}, 0, -\frac{\beta y}{c_g} \right), \quad (17)$$

corresponding to a constant scalar potential $\phi_W = -\alpha\beta^2/4\omega^2$, which exerts no force, and a three-dimensional vector potential $\mathbf{A}_W = -(\beta/2\omega)\hat{\mathbf{x}} - (\beta y/c_g)\hat{\mathbf{z}}$, where $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ are unit vectors pointing east, north, and upward, respectively. This vector potential produces a force field analogous to a magnetic field,

$$\mathbf{B}_W = \nabla \times \mathbf{A}_W = -\frac{\beta}{c_g}\hat{\mathbf{x}}, \quad (18)$$

which is uniform and points westward. The Hamiltonian is then

$$H = \alpha(-i\nabla - \mathbf{A}_W)^2 + \phi_W, \quad (19)$$

with the charge set to unity. Substituting the wave function (16) into the Schrödinger equation $H\Psi = i\partial\Psi/\partial t$ and dividing throughout by α returns precisely (6). No term has been inserted by hand: the $\beta k/\omega$ term is generated by the cross term between $-i\partial/\partial x$ and A_{Wx} , and the residual $\beta^2/4\omega^2$ left over from that square is removed by the constant scalar potential, which is physically inert.

(ii) A second choice that yields the same equation of motion is

$$k_z = 0, \quad A_{Wz} = -\frac{f_0 + \beta y}{c_g}, \quad (20)$$

with A_{Wx} and ϕ_W as before. The resulting $\mathbf{B}_W$ field is again given by (18). This version requires no vertical dependence of the wave function, which is more in line with the original problem of purely horizontal basic-state motion. The potential is assumed here; Paper II derives it. As shown in section 5, models (i) and (ii) are not two different models at all, but the same physical system written in two gauges.

5. Gauge structure of the representation

The four-vector potential of (17) is not unique. In units with $c = \hbar = e = 1$, the transformation

$$\mathbf{A}_W \rightarrow \mathbf{A}_W + \nabla\chi, \quad \phi_W \rightarrow \phi_W - \frac{\partial\chi}{\partial t}, \quad \Psi \rightarrow \Psi \exp(i\chi) \quad (21)$$

for any single-valued function $\chi(\mathbf{r}, t)$ leaves both the force field $\mathbf{B}_W = \nabla \times \mathbf{A}_W$ and the Schrödinger equation invariant. The physically meaningful quantity is therefore not the canonical momentum operator $\mathbf{p} = -i\nabla$, which is gauge dependent, but the gauge-invariant kinematic momentum operator $\boldsymbol{\pi} = -i\nabla - \mathbf{A}_W$. This distinction, a formality in many textbook problems, is the key to the difficulty encountered in section 3.

The x -component of the vector potential, $A_{Wx} = -\beta/2\omega$, is a constant and therefore contributes nothing to $\mathbf{B}_W$: it is pure gauge. It can be removed entirely by choosing $\chi_1 = (\beta/2\omega)x$ in (21), which gives $A_{Wx} \rightarrow 0$ and, simultaneously,

$$\Psi = v(y)e^{ikx} \rightarrow v(y)\exp\left[i\left(k + \frac{\beta}{2\omega}\right)x\right]. \quad (22)$$

Equation (22) is exactly the phase shift that had to be imposed by hand in (12). In the gauge $A_{Wx} = 0$, the term π_x^2 of the Hamiltonian becomes a pure multiplication by $(k + \beta/2\omega)^2$, and π_z^2 becomes $(\beta/c_g)^2(y - y_0)^2$; the Hamiltonian is then precisely the harmonic oscillator Hamiltonian with the potential of (13) and the wave function of (12). In other words, the harmonic oscillator representation is the four-vector potential representation written in a particular gauge. Its two apparently arbitrary ingredients are not arbitrary at all: the phase shift is the gauge function χ_1 , and the constant $-\alpha\beta^2/4\omega^2$ subtracted from the potential is the scalar potential of (17).

The physical content is as follows. The zonal wavenumber k of the shallow water solution is the canonical momentum of the particle, whereas the gauge-invariant kinematic momentum, the one that enters the energy, is

$$\pi_x = k + \frac{\beta}{2\omega}. \quad (23)$$

The term $\beta k/\omega$, which the harmonic oscillator model could not produce, is neither spurious nor a gauge artifact: it is the cross term that arises when the energy is expressed in terms of the canonical rather than the kinematic momentum, the difference between the two, $\beta/2\omega$, being the analogue of eA_x/c in the electromagnetic problem. This was anticipated by Wang (1991), where the first-order term in k was attributed to the gauge used in deriving the shallow water equations, although the gauge function itself was not identified.

Models (i) and (ii) differ only in how the kinematic momentum along z is apportioned between the wavenumber and the potential: in model (i), $\pi_z = k_z - A_{Wz} = f_0/c_g + \beta y/c_g$; in model (ii), $\pi_z = 0 + (f_0 + \beta y)/c_g$. These are identical, and the two descriptions are connected by the gauge function $\chi_2 = -f_0 z/c_g$, which converts $A_{Wz} = -\beta y/c_g$ into $A_{Wz} = -(f_0 + \beta y)/c_g$ while simultaneously removing the factor $\exp(ik_z z)$ from the wave function (16). The two are therefore the same physical system, and the question of which is better is one of convenience rather than of physics. Model (ii) is preferable because it requires no vertical

wavenumber and hence no fictitious third dimension, the auxiliary z -dependence of model (i) being entirely a gauge artifact. The vertical wavenumber cannot be chosen freely: taking $k_z \neq 0$ in model (ii) replaces f_0 by $f_0 + k_z c_g$ in (6), which amounts to relocating the reference latitude, so that reproducing (6) fixes $\pi_z = (f_0 + \beta y)/c_g$ uniquely, whatever gauge is used to express it.

Because the Schrödinger equation is invariant under (21), the physics depends on $\mathbf{A}_W$ only through $\mathbf{B}_W$: two four-vector potentials producing the same $\mathbf{B}_W$ describe the same particle, and hence the same shallow water waves, however different they may appear. The one exception in the electromagnetic case is the Aharonov–Bohm effect, in which the wave function acquires an observable phase from a region where $\mathbf{B}$ vanishes but $\mathbf{A}$ does not. Since $\mathbf{B}_W$ here is uniform and unbounded, no such effect arises, but the possibility is worth keeping in mind for geometries in which β varies or the domain is not simply connected.

6. The dispersion relation as a Landau spectrum

a. Landau quantization

The particle constructed in section 4 is a charged particle moving in a uniform magnetic-like field. The corresponding electromagnetic problem—a free electron in a uniform magnetic field—was solved by Landau (1930) to explain the diamagnetism of metals; for later summaries, see Peierls (1955) and Seeger (1982). The energy levels are

$$E = \frac{\hbar^2 k_{\parallel}^2}{2m} + \left(n + \frac{1}{2}\right) \hbar \omega_c, \quad (24)$$

where $k_{\parallel}$ is the kinematic wavenumber along $\mathbf{B}$, $n = 0, 1, 2, \dots$ is the Landau level index, and $\omega_c = |e|B/mc$ is the cyclotron frequency (not to be confused with the Larmor precession frequency, which is half of it). The dispersion relation between E and $k_{\parallel}$ is a family of parabolas, one for each Landau level.

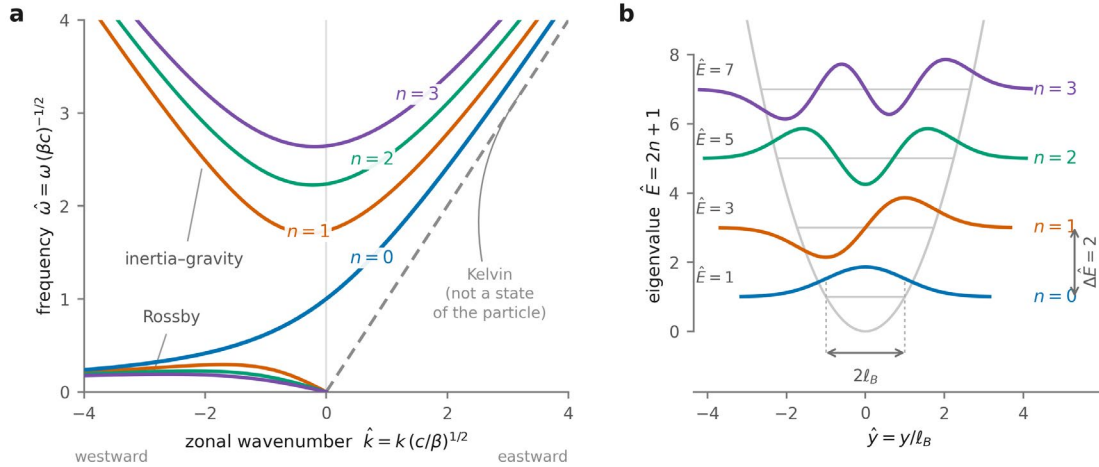
Wang (1991) noted that (24) and the shallow water dispersion relation have the same form, drew an analogy between the two, and observed that the units of the two systems are not the same. The correspondence is in fact stronger than an analogy: it is exact, and (8) can be derived from (24) rather than merely compared with it.

Take the Hamiltonian of (19). The field $\mathbf{B}_W$ is uniform and directed along x , so the motion separates into free motion along $\mathbf{B}_W$, with the kinematic momentum π_x of (23), and Landau-quantized motion in the perpendicular (y, z) plane. In the units adopted here, α plays the role of

283 $\hbar^2/2m$, so that $m = 1/2\alpha$, while $e = c = \hbar = 1$ and $B_W = \beta/c_g$; the cyclotron frequency is
 284 therefore $\hbar\omega_c = |e|B/m = 2\alpha\beta/c_g$. The total energy, including the constant scalar potential of
 285 (17), is

$$286 \quad E = \alpha \left(k + \frac{\beta}{2\omega} \right)^2 + (2n + 1) \frac{\alpha\beta}{c_g} - \frac{\alpha\beta^2}{4\omega^2}. \quad (25)$$

287 Setting $E = \alpha\omega^2/c_g^2$, as required by the wave function (16), and dividing by α , the square
 288 expands to give $k^2 + \beta k/\omega + \beta^2/4\omega^2$, the last term of which is cancelled identically by the
 289 scalar potential. What remains is exactly (8). The dispersion relation of tropical shallow water
 290 waves is thus the Landau spectrum of the Hamiltonian of (19); the Hermite index n of the
 291 Matsuno solution, (7), is the Landau level index of (24); and the Hermite functions of (7) are the
 292 Landau wave functions in the gauge $A_{Wx} = 0$. The appearance of the factor $(2n + 1)$ in (8)—
 293 otherwise simply a property of the Hermite equation—thereby becomes a physical rather than a
 294 mathematical fact: it is the quantization of the particle's gyration about the analogue magnetic
 295 field. Figure 1a shows the dispersion relation read in this way, with the branches of each
 296 meridional mode labeled by the Landau level index in place of the Hermite index.



297
 298 Fig. 1. The Matsuno dispersion relation as a Landau spectrum. (a) Dispersion relation of (8) in nondimensional
 299 form: frequency $\hat{\omega} = \omega(\beta c_g)^{-1/2}$ against zonal wavenumber $\hat{k} = k(c_g/\beta)^{1/2}$, for the meridional modes $n =$
 300 $0-3$, colored by Landau level index. Only positive frequencies are shown, so that each level appears as an
 301 inertia-gravity branch spanning both signs of $\hat{k}$ together with, for $n \geq 1$, a separate westward Rossby branch.
 302 The curves of a given color are one Landau level of the Hamiltonian of (19), and the level index is the Hermite
 303 index of (7). The lowest level, $n = 0$, is the mixed Rossby-gravity (Yanai) wave, and is the only level on
 304 which the Rossby and gravity regimes join in a single continuous curve, (30), because the cubic of (27)
 305 factorizes there and at no other level. The Kelvin wave (gray dashed) is shown for reference only: it is
 306 annihilated by the reduction to (6) and is not a state of the particle. (b) The corresponding meridional
 307 structures: the Landau wave functions $v_n \propto H_n(\hat{y}) \exp(-\hat{y}^2/2)$ of (7), each drawn on its eigenvalue $\hat{E} = 2n +$

1 in the parabolic well of (6), with $\hat{y} = y/\ell_B$. Gray horizontal segments mark the classical turning points. The level spacing is uniform, as it is for a charged particle in a uniform field, and the width of the $n = 0$ Gaussian is the analogue magnetic length $\ell_B = (c_g/\beta)^{1/2}$ of (26), the equatorial Rossby radius of deformation.

b. Analogue magnetic field, magnetic length, and guiding center

The only force acting on the particle is the analogue magnetic field of (18): uniform, westward, and produced solely by the meridional gradient of planetary vorticity. The “charge” of the particle is a coupling constant that measures how strongly the wave field responds to that gradient, and has been set to unity. The scalar potential is constant and exerts no force at all; it is present only to keep the energy zero point consistent with the definition of the canonical momentum.

The physical statement contained in this is that the β -effect acts on a shallow water wave packet in the same way that a uniform magnetic field acts on a charged particle. The meridional trapping of equatorial waves—the equatorial waveguide—is thereby identified as cyclotron gyration, and its quantization as Landau quantization. For the Earth’s atmosphere, with $\beta \approx 2.3 \times 10^{-11} \text{ m}^{-1} \text{ s}^{-1}$ and a representative gravity-wave speed $c_g \approx 30 \text{ m s}^{-1}$, the analogue field strength is $B_W = \beta/c_g \approx 7.6 \times 10^{-13} \text{ m}^{-2}$.

For an electron in a magnetic field, the natural length scale is the magnetic length $\ell_B = (\hbar c/|e|B)^{1/2}$, the characteristic radius of the lowest Landau orbit. For the analogue field of (18), in the present units, this is simply $B_W^{-1/2}$, so that

$$\ell_B = \left(\frac{c_g}{\beta}\right)^{1/2} \equiv L_{eq}, \quad (26)$$

which is the equatorial Rossby radius of deformation. The Gaussian factor in (7) is $\exp[-(y - y_0)^2/2\ell_B^2]$, which is the lowest Landau wave function. The meridional width of an equatorially trapped wave and the cyclotron radius of the analogue particle are the same quantity. For the parameters quoted above, $\ell_B \approx 1150 \text{ km}$; for a smaller equivalent depth, for example $c_g \approx 2.8 \text{ m s}^{-1}$, it is about 350 km. Correspondingly, the Landau level spacing measured in the units of (8) is $\beta/c_g = \ell_B^{-2}$, so that the spectrum of meridional modes is fixed by the deformation radius alone. Figure 1b shows the corresponding ladder of meridional structures.

In the electromagnetic Landau problem, each level is highly degenerate: for a given n , the wave function may be centered on any guiding-center coordinate. Here the guiding center is the latitude at which the kinematic momentum $\pi_z = (f_0 + \beta y)/c_g$ vanishes, that is, $y = y_0 =$

$-f_0/\beta$, the latitude at which the Coriolis parameter itself vanishes. The freedom to displace the guiding center is exactly the freedom to choose k_z , and reproducing (6) forces k_z to a single value. The two-dimensionality of the system therefore removes the Landau degeneracy and pins the guiding center to the zero- f latitude—the particle-representation statement of the familiar fact that equatorial waves are trapped about the equator and not about some arbitrary latitude.

Finally, since Ψ is proportional to $v(y)$, the probability density of the particle is $\Psi^*\Psi \propto |v(y)|^2$, which is proportional to the meridional kinetic energy density of the wave, $1/2 \rho H v^2$. The particle is therefore most likely to be found where the meridional motion of the fluid is most vigorous. This gives the quantum mechanical formalism a direct fluid dynamical interpretation: the “position” of the particle is the latitude at which the wave carries its meridional energy, and the spreading of $|\Psi|^2$ over the deformation radius is the meridional extent of the waveguide.

c. A Hamiltonian that depends on its own eigenvalue

An unusual feature of the present construction is that the potential depends on the frequency: $A_{Wx} = -\beta/2\omega$ and $\phi_W = -\alpha\beta^2/4\omega^2$ both contain ω , while the energy is $E = \alpha\omega^2/c_g^2$. The Hamiltonian is therefore a function of its own eigenvalue, $H = H(E)$, and the eigenvalue problem is implicit rather than linear. This is the sharpest disanalogy with the electromagnetic Landau problem, in which H is independent of E .

Far from being a defect, this is the feature that produces the most characteristic property of the tropical wave spectrum. Multiplying (8) by ω gives

$$\frac{\omega^3}{c_g^2} - \omega \left[k^2 + \frac{(2n+1)\beta}{c_g} \right] - \beta k = 0, \quad (27)$$

a cubic with three roots for each n : two of high frequency, corresponding to eastward- and westward-propagating inertia-gravity waves, and one of low frequency, corresponding to the westward-propagating Rossby wave. Had A_{Wx} been independent of ω —that is, had the vector potential not depended on the particle’s own energy—the term $\beta k/\omega$ would be absent, (8) would be quadratic in ω , and only the two gravity wave branches would survive. In this representation, the Rossby wave exists precisely because the particle moves in a field that depends on its own energy. The usual care must be taken with the case $n = 0$, for which one of the three roots is spurious (Matsuno 1966).

It would be natural to read this as indicating that the four-vector potential is a self-consistent field, determined together with the state of the particle rather than imposed from outside. That reading is not correct: the frequency enters through the elimination of u and Φ by which (6) was obtained, and Paper II shows that the parent Lagrangian contains no reference to ω at all. The self-referential character of H is the price of reducing three coupled fields to one.

7. The Yanai wave as the lowest Landau level

The particle picture is at its most informative for the gravest meridional mode, $n = 0$, whose solution is the mixed Rossby–gravity or Yanai wave (Matsuno 1966; Yanai and Maruyama 1966). Because the field $\mathbf{B}_W$ of (18) is uniform, the lowest level can be identified without solving (6) in full. Measuring latitude in units of the magnetic length, $\eta = (y - y_0)/\ell_B$, and writing $\hat{E} = \ell_B^2(\omega^2/c_g^2 - k^2 - \beta k/\omega)$, the structure equation (6) becomes

$$\frac{d^2 v}{d\eta^2} + (\hat{E} - \eta^2)v = 0. \quad (28)$$

Multiplying (28) by v , integrating over all latitudes, and integrating the derivative term by parts with $v \rightarrow 0$ as $|\eta| \rightarrow \infty$ gives $\hat{E} \int v^2 d\eta = \int [(dv/d\eta)^2 + \eta^2 v^2] d\eta$. Because $\int 2\eta v (dv/d\eta) d\eta = -\int v^2 d\eta$, the right-hand side can be written as a perfect square plus a remainder, so that

$$\hat{E} = 1 + \frac{\int_{-\infty}^{\infty} \left(\frac{dv}{d\eta} + \eta v\right)^2 d\eta}{\int_{-\infty}^{\infty} v^2 d\eta} \geq 1. \quad (29)$$

The two contributions to $\hat{E}$ are the meridional kinetic energy of the gyration, which comes from the curvature of v , and its potential energy in the parabolic well, which comes from the Coriolis term. Confining the wave closer to the equator lowers the second but raises the first, so their sum has a positive minimum. That minimum, $\hat{E} = 1$, is reached only when the squared term in (29) vanishes, that is, when v satisfies the first-order equation

$$\frac{dv}{dy} + \frac{\beta(y-y_0)}{c_g} v = 0 \Rightarrow v \propto \exp\left[-\frac{\beta(y-y_0)^2}{2c_g}\right],$$

which reproduces (7) at $n = 0$, and $\hat{E} = 1$ is exactly the right-hand side of (8) with $n = 0$. The integration by parts used here is the same manipulation of a cross term that, in Paper II, generates the $\beta k/\omega$ term of the structure equation. Three consequences follow.

First, the Yanai wave is the cyclotron ground state. Its meridional structure is a nodeless Gaussian whose width is exactly the analogue magnetic length of (26), and its contribution $\beta/c_g = \ell_B^{-2}$ to the dispersion relation is the smallest value allowed by (29): pure zero-point energy of the gyration, with no excitation of the cyclotron motion at all. The higher modes are the same Gaussian multiplied by the Hermite polynomial H_n , which has n zeros; each additional node raises $\hat{E}$ by 2, that is, raises the frequency term by $2\beta/c_g$, which is why n counts nodes and the levels are equally spaced. That index cannot be altered by a gauge transformation: gauge transformations leave $\mathbf{B}_W$, and hence the whole Landau spectrum, invariant, and since $\Psi^*\Psi$ is gauge invariant, so is the nodal structure of v . Moving from one meridional mode to another requires changing the meridional structure itself, by adding or removing nodes, which no gauge transformation can do.

Second, the symmetry of the mode follows immediately. With $H_0 = 1$, the meridional velocity v is symmetric about y_0 , whereas u and Φ , obtained from v through a derivative and multiplication by $f_0 + \beta y$, are antisymmetric. The Yanai wave therefore has maximum cross-equatorial flow precisely where the Coriolis parameter vanishes, with a node in geopotential and zonal wind there—the opposite symmetry to the Kelvin and $n = 1$ Rossby modes. In the particle language, the guiding center of the lowest Landau orbit sits on the zero- f latitude, and the gyration is what carries fluid across it: the structural reason why the Yanai wave is the most effective equatorial mode for cross-equatorial momentum transport.

Third, $n = 0$ is the only level at which the Rossby and gravity branches do not separate. Setting $n = 0$ in (27) and clearing denominators, the cubic factorizes as $(\omega + kc_g)(\omega^2 - kc_g\omega - \beta c_g) = 0$. The root $\omega = -kc_g$ must be discarded: the reduction of the shallow water equations to (6) requires division by $c_g^2 k^2 - \omega^2$, which vanishes there (Matsuno 1966). What survives is a quadratic, not a cubic, and, taking $\omega > 0$,

$$\omega = \frac{1}{2} \left[kc_g + (k^2 c_g^2 + 4\beta c_g)^{1/2} \right], \quad (30)$$

a single continuous curve with $\omega \rightarrow kc_g + \beta/k$ as $k \rightarrow +\infty$ and $\omega \rightarrow -\beta/k$ as $k \rightarrow -\infty$. The mode is thus inertia-gravity-like at the eastward end and Rossby-like at the westward end, which is the content of the name “mixed Rossby-gravity wave.” For every $n \geq 1$, the cubic does not factorize, the gravity branches retain a frequency floor, and a separate low-frequency Rossby

branch exists; only the lowest Landau level bridges the two regimes. In the particle picture, the reason is that the missing third root is the one associated with $v \equiv 0$, and the lowest Landau level is the last state before that limit.

These structural properties account for the prominence of the Yanai wave in observations. It was the first of Matsuno's modes to be identified in the atmosphere, in equatorial lower-stratospheric radiosonde data (Yanai and Maruyama 1966; Yanai et al. 1968), and its antisymmetric geopotential signature makes it separable as a distinct peak in wavenumber–frequency spectra of satellite-observed convection (Wheeler and Kiladis 1999; Kiladis et al. 2009). Because it transports easterly momentum upward, it was assigned a central role, alongside the Kelvin wave, in the classical wave-driven theory of the quasi-biennial oscillation (Lindzen and Holton 1968; Holton and Lindzen 1972; Baldwin et al. 2001), although current assessments attribute the larger share of the momentum flux to a broad spectrum of smaller-scale gravity waves (Anstey et al. 2022). In the troposphere, its convectively coupled counterpart over the Pacific is associated with tropical-depression-type disturbances and thus with one recognized pathway to tropical cyclogenesis (Takayabu and Nitta 1993).

8. Discussion

a. Limitations of the correspondence

Three qualifications should be stated plainly. First, the correspondence is a mathematical identity between two eigenvalue problems, not a claim that a material particle exists. What has been established is that the shallow water wave field of a given zonal wavenumber and meridional mode number can be labeled by, and computed from, a single particle state, the labels behaving exactly as the quantum numbers of a charged particle in a uniform magnetic field; the economy claimed is an economy of description.

Second, the units of the particle problem are those of the wave problem, not those of quantum mechanics: the particle's "momentum" is a wavenumber, its "energy" a frequency, and its "mass" is fixed by α , whose value is a matter of normalization (appendix A). No dimensional bridge to elementary particle physics is intended or implied.

The third qualification is more interesting, because it can be stated precisely. The representation is built on the structure equation for v and therefore does not describe modes for which $v \equiv 0$. The Kelvin wave is the important case. The reduction of the shallow water

equations to (6) requires division by $c_g^2 k^2 - \omega^2$, which vanishes identically on the Kelvin branch $\omega = kc_g$; the mode is annihilated by the elimination of u and Φ before any of the present construction is applied. It is the second casualty of that division, the first being the spurious root $\omega = -kc_g$ discarded at $n = 0$. In the particle language, the Kelvin wave is formally the state $n = -1$: substituting $n = -1$ in (8) gives $\omega = kc_g$ identically. The sequence of Landau levels, however, terminates: by (29), no meridional structure can lower the gyration energy below its ground-state value, so there is no level below $n = 0$. The Yanai wave is the gravest mode of (6) precisely because the state below it is the one with $v \equiv 0$, which is the Kelvin wave itself and not a state of the particle at all. Suggestively, the lowest Landau structure is nevertheless present in the Kelvin mode: on an equatorial β -plane its zonal velocity is $u \propto \exp(-\beta y^2/2c_g)$, the same Gaussian, of the same width ℓ_B , as the $n = 0$ wave function of (7). The structure resides in u and Φ rather than in v . A representation capturing both modes would therefore have to be built on a multicomponent wave function carrying (u, v, Φ) together, rather than on a scalar amplitude obtained by reducing to one of them. This is presumably why the topological treatment of Delplace et al. (2017), which never performs such a reduction, captures the Kelvin wave on the same footing as the Yanai wave, and it indicates that the limitation belongs to the present construction rather than to the condensed-matter analogy in general.

b. Relation to operator and topological treatments

The linearized shallow water equations need not be reduced to (6) at all: kept as a first-order, three-component Hermitian operator in latitude, they have been studied in detail, and that is the form in which the topological analysis of Delplace et al. (2017) is carried out. Faure (2023) solves the equatorial problem directly in that representation, using ladder operators acting on Hermite functions to generate the Kelvin, Yanai, and higher branches; Delplace (2022) embeds it in a family of spin- S Hamiltonians whose spectral flows are governed by Berry–Chern monopoles; Rossi and Tarantola (2024) interpret the Coriolis parameter explicitly as a Dirac mass and the equator as the interface between two topologically distinct regions; Tong (2023) obtains the shallow water equations from a gauge theory; and Tauber et al. (2019) establish a bulk–interface correspondence for the equatorial waveguide. A related dictionary, in which the Coriolis frequency plays the part of a plasma frequency or of a rest mass and the speed of long gravity waves that of the speed of light, has been given by Heifetz et al. (2021, 2022). The

relation between those descriptions and the present one is that the meridional structure equation, (6), is the product of two first-order operators, so that the Landau spectrum obtained here is a ladder in the square of the frequency rather than in the frequency itself; this will be developed elsewhere.

c. Future directions

Two directions seem worth pursuing. The first is a slowly varying medium, in which β , c_g , or the depth varies in space and the analogue magnetic field becomes nonuniform; the semiclassical machinery developed for electrons in inhomogeneous fields would then apply directly, and the drift of the guiding center would give a compact description of wave propagation through a varying basic state. The second is the incorporation of the Kelvin mode along the lines indicated in section 8a.

9. Summary

The linearized shallow water equations on a general β -plane have been reformulated as a quantum mechanical particle problem. The main results are as follows.

1. The meridional structure equation is exactly the Schrödinger equation for a particle of unit charge minimally coupled to a four-vector potential whose analogue magnetic field is uniform and westward, and no ad hoc term is required.
2. A representation based on a simple harmonic oscillator reproduces the structure equation only if a phase shift and a constant potential are inserted by hand. Both are explained by the four-vector representation: the phase shift is the gauge function $\chi_1 = (\beta/2\omega)x$ that removes the constant part of A_{Wx} , and the constant potential is the scalar potential.
3. The zonal wavenumber is the canonical momentum, the gauge-invariant kinematic momentum is $\pi_x = k + \beta/2\omega$, and the $\beta k/\omega$ term of the dispersion relation is the cross term generated when the energy is expressed through the canonical rather than the kinematic momentum. The two versions of the vector potential are the same physical system in two gauges, related by $\chi_2 = -f_0 z/c_g$.
4. The Matsuno dispersion relation is the Landau spectrum of the constructed Hamiltonian, the Hermite index is the Landau level index, the equatorial radius of deformation is the analogue magnetic length, and the meridional trapping of equatorial waves is cyclotron gyration.

5. Because the vector potential depends on the frequency, the eigenvalue problem is implicit, and that dependence is the origin of the Rossby branch.
6. The Yanai wave is the lowest Landau level, the state of minimum gyration energy, which satisfies a first-order equation; its mixed Rossby–gravity character follows from the factorization of the $n = 0$ cubic, which occurs for no other meridional mode.
7. The constant α cancels identically from the dispersion relation and is therefore not determined by the wave problem; it fixes only the map between wave frequency and particle energy.

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Data availability statement

No datasets were generated or analyzed during the current study. All results reported here are analytic and are fully reproducible from the equations given in the text; no custom code was used.

APPENDIX A

The role of the constant α

Wang (1991) observed that the formulation goes through with $\alpha = 1$, remarked that the physical meaning of α was at that time unclear, and suggested that it might prove to be an important characteristic number of the geophysical shallow water problem. The derivation given in section 6a settles the question. The constant α cancels identically between the two sides of (25) and does not appear in (8), so the dispersion relation places no constraint on it whatever, and it is not a characteristic number of the wave problem in the sense anticipated. What it does fix is the relation between wave frequency and particle energy, $E = \alpha\omega^2/c_g^2$, and hence the rate at which the particle gyrates in real time; since it carries the dimensions of $\hbar/m$, it can be given a numerical value only after a system of units has been chosen. One choice deserves mention. Written in the form

$$\frac{1}{2} \frac{\omega^2}{c_g^2} = \frac{1}{2} \left(k^2 + \frac{\beta k}{\omega} \right) + \left(n + \frac{1}{2} \right) \frac{\beta}{c_g}, \quad (\text{A1})$$

(8) matches (24) term by term: $1/2 \omega^2/c_g^2$ corresponds to E , $1/2 (k^2 + \beta k/\omega)$ to $\hbar^2 k_{\parallel}^2/2m$, and β/c_g to $\hbar\omega_c$. This reading amounts to $\alpha = 1/2$, that is, $m = 1$ in units with $\hbar = 1$ —a convenient normalization, making the particle’s energy exactly half its squared kinematic momentum as for an ordinary free particle of unit mass, but a choice of units and not a requirement of the physics.

References

- Anderson, D. L. T., and A. E. Gill, 1979: Beta dispersion of inertial waves. *J. Geophys. Res.*, 84, 1836–1842.
- Anstey, J. A., and Coauthors, 2022: Impacts, processes and projections of the quasi-biennial oscillation. *Nat. Rev. Earth Environ.*, 3, 588–603.
- Baldwin, M. P., and Coauthors, 2001: The quasi-biennial oscillation. *Rev. Geophys.*, 39, 179–229.
- Blandford, R. R., 1966: Mixed gravity-Rossby waves in the ocean. *Deep-Sea Res.*, 13, 941–961.
- Delplace, P., 2022: Berry–Chern monopoles and spectral flows. *SciPost Phys. Lect. Notes*, 39.
- Delplace, P., J. B. Marston, and A. Venaille, 2017: Topological origin of equatorial waves. *Science*, 358, 1075–1077.
- Faure, F., 2023: Manifestation of the topological index formula in quantum waves and geophysical waves. *Ann. Henri Lebesgue*, 6, 449–492.
- Frauenfelder, H., and E. M. Henley, 1974: *Subatomic Physics*. Prentice-Hall.
- Gill, A. E., 1982: *Atmosphere–Ocean Dynamics*. Academic Press.
- Goldstein, H., 1950: *Classical Mechanics*. Addison-Wesley.
- Grose, W. L., and B. J. Hoskins, 1979: On the influence of orography on large-scale atmospheric flow. *J. Atmos. Sci.*, 36, 223–234.
- Hayashi, Y., 1976: Non-singular resonance of equatorial waves under the radiation condition. *J. Atmos. Sci.*, 33, 183–201.
- Heifetz, E., L. R. M. Maas, J. Mak, and I. Pomerantz, 2021: Analogy between electro-magnetic waves in cold unmagnetized plasma and shallow water inertio-gravity waves in geophysical systems. *J. Phys. Commun.*, 5, 125006.

568 Heifetz, E., L. R. M. Maas, J. Mak, and I. Pomerantz, 2022: Inertio-gravity Poincaré waves and
 569 the quantum relativistic Klein–Gordon equation, near-inertial waves and the non-
 570 relativistic Schrödinger equation. *Phys. Fluids*, 34, 116608.
 571 Holton, J. R., 1979: *An Introduction to Dynamic Meteorology*. 2nd ed. Academic Press.
 572 Holton, J. R., and R. S. Lindzen, 1972: An updated theory for the quasi-biennial cycle of the
 573 tropical stratosphere. *J. Atmos. Sci.*, 29, 1076–1080.
 574 Itoh, H., and M. Ghil, 1988: The generation mechanism of mixed Rossby-gravity waves in the
 575 equatorial troposphere. *J. Atmos. Sci.*, 45, 585–604.
 576 Kiladis, G. N., M. C. Wheeler, P. T. Haertel, K. H. Straub, and P. E. Roundy, 2009:
 577 Convectively coupled equatorial waves. *Rev. Geophys.*, 47, RG2003.
 578 Landau, L., 1930: Diamagnetismus der Metalle. *Z. Phys.*, 64, 629–637.
 579 Lindzen, R. S., 1967: Planetary waves on beta-planes. *Mon. Wea. Rev.*, 95, 441–451.
 580 Lindzen, R. S., and J. R. Holton, 1968: A theory of the quasi-biennial oscillation. *J. Atmos. Sci.*,
 581 25, 1095–1107.
 582 Lindzen, R. S., and T. Matsuno, 1968: On the nature of large scale wave disturbances in the
 583 equatorial lower stratosphere. *J. Meteor. Soc. Japan*, 46, 215–221.
 584 Mak, M.-K., 1969: Laterally driven stochastic motions in the tropics. *J. Atmos. Sci.*, 26, 41–64.
 585 Matsuno, T., 1966: Quasi-geostrophic motions in the equatorial area. *J. Meteor. Soc. Japan*, 44,
 586 25–43.
 587 Pedlosky, J., 1979: *Geophysical Fluid Dynamics*. Springer-Verlag.
 588 Peierls, R. E., 1955: *Quantum Theory of Solids*. Clarendon Press.
 589 Rosenthal, S. L., 1965: Some preliminary considerations of tropospheric wave motions in
 590 equatorial latitudes. *Mon. Wea. Rev.*, 93, 605–612.
 591 Rossi, S., and A. Tarantola, 2024: Topology of 2D Dirac operators with variable mass and an
 592 application to shallow-water waves. *J. Phys. A: Math. Theor.*, 57, 065201.
 593 Saxon, D. S., 1968: *Elementary Quantum Mechanics*. Holden-Day.
 594 Schiff, L. I., 1968: *Quantum Mechanics*. 3rd ed. McGraw-Hill.
 595 Seeger, K., 1982: *Semiconductor Physics: An Introduction*. Springer-Verlag.
 596 Takayabu, Y. N., and T. Nitta, 1993: 3–5 day-period disturbances coupled with convection over
 597 the tropical Pacific Ocean. *J. Meteor. Soc. Japan*, 71, 221–246.

598 Tauber, C., P. Delplace, and A. Venaille, 2019: A bulk-interface correspondence for equatorial
599 waves. *J. Fluid Mech.*, 868, R2.

600 Tong, D., 2023: A gauge theory for shallow water. *SciPost Phys.*, 14, 102.

601 Vallis, G. K., 2017: *Atmospheric and Oceanic Fluid Dynamics: Fundamentals and Large-Scale*
602 *Circulation*. 2nd ed. Cambridge University Press.

603 Venaille, A., and P. Delplace, 2021: Wave topology brought to the coast. *Phys. Rev. Res.*, 3,
604 043002.

605 Wallace, J. M., 1971: Spectral studies of tropospheric wave disturbances in the tropical western
606 Pacific. *Rev. Geophys. Space Phys.*, 9, 557–612.

607 Wallace, J. M., and V. E. Kousky, 1968: Observational evidence of Kelvin waves in the tropical
608 stratosphere. *J. Atmos. Sci.*, 25, 900–907.

609 Wang, P. K., 1984: A brief review of the Eulerian variational principle for atmospheric motions
610 in rotating coordinates. *Atmos.–Ocean*, 22, 387–392.

611 Wang, P. K., 1991: A particle representation of tropical shallow water waves. Preprints, *Eighth*
612 *Conf. on Atmospheric and Oceanic Waves and Stability*, Amer. Meteor. Soc., 225–228.

613 Wang, P. K., 2026: The four-vector potential of equatorial shallow water waves from Hamilton’s
614 principle. EarthArXiv, <https://doi.org/10.31223/X53F7G>.

615 Wheeler, M., and G. N. Kiladis, 1999: Convectively coupled equatorial waves: Analysis of
616 clouds and temperature in the wavenumber–frequency domain. *J. Atmos. Sci.*, 56, 374–
617 399.

618 Wunsch, C., and A. E. Gill, 1976: Observations of equatorially trapped waves in Pacific sea level
619 variations. *Deep-Sea Res.*, 23, 371–390.

620 Yanai, M., and T. Maruyama, 1966: Stratospheric wave disturbances propagating over the
621 equatorial Pacific. *J. Meteor. Soc. Japan*, 44, 291–294.

622 Yanai, M., T. Maruyama, T. Nitta, and Y. Hayashi, 1968: Power spectra of large-scale
623 disturbances over the tropical Pacific. *J. Meteor. Soc. Japan*, 46, 308–323.