The four-vector potential of equatorial shallow water waves from Hamilton's principle

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Abstract

Linear shallow water waves on an equatorial β-plane can be represented as the states of a charged particle minimally coupled to a four-vector potential whose analogue magnetic field is uniform and proportional to β/c_g. There the potential is written down and verified rather than derived. We show that it need not be assumed. In the Eulerian variational principle for rotating flow the Coriolis coupling is already a minimal coupling, the frame velocity playing the part of a vector potential, and the quadratic Lagrangian for linear shallow water motion is that of a two-dimensional charged particle in a uniform field acting in the plane of parcel displacement. Eliminating the zonal displacement amplitude, which enters algebraically, returns the four-vector potential identically. The $\beta k/\omega$ term then originates in a single integration by parts, the vertical wavenumber is forced to vanish, and the potential's frequency dependence is an artifact of the reduction.

Introduction

Hamilton's principle occupies a central place in classical dynamics, and its extension to hydrodynamics has been studied by many investigators [1–8]. The Hamiltonian and variational structure of geophysical fluid dynamics in particular has been reviewed[9–11] and is treated at length elsewhere[12]. An Eulerian variational principle yielding the equations of atmospheric motion in rotating Earth coordinates was given previously[13].

In a companion paper[14] (hereafter Paper I; see also ref.[15]) it is shown that the meridional structure equation for linear shallow water waves on a general β-plane is exactly the Schrödinger equation for a particle of unit charge minimally coupled to a four-vector potential $A_{W\mu}$, whose analogue magnetic field

$$\mathbf{B}_W = \boldsymbol{\nabla} \times \mathbf{A}_W = -\frac{\beta}{c_g}\hat{\mathbf{x}} \tag{1}$$

is uniform and directed westward, c_g being the speed of long gravity waves. The dispersion relation of Matsuno[16] is then the Landau[17] spectrum of that Hamiltonian, the equatorial radius of deformation is the analogue magnetic length, and the meridional trapping of equatorial waves is cyclotron gyration.

That construction has one unsatisfactory feature. The potential is written down and then verified: it is shown to reproduce the structure equation, but it is not obtained from anything. A reader is entitled to ask where it comes from, and whether some other potential might have served as well. The purpose of the present note is to answer that question by deriving $A_{W\mu}$ from Hamilton's principle.

The route is short, and it rests on an observation that is already implicit in the variational principle for a rotating fluid: the coupling between the flow and the rotation of the frame *is* a minimal coupling, with the frame velocity playing the part of a vector potential and mass density the part of charge. The Coriolis force is then the corresponding Lorentz force. This correspondence between rotation and magnetism is of course long established in mechanics, and it has recently been put to use in the topological analysis of equatorial waves[18]. What does not appear to have been noted is that carrying the correspondence through the linearization and the normal-mode reduction of the shallow water system produces the four-vector potential of the particle representation exactly, and in doing so explains the one feature of that representation that has always looked contrived.

The route is set out below: the rotating-frame Lagrangian and the minimal-coupling structure of the Coriolis term; the quadratic Lagrangian for linear shallow water motion on a β-plane, identified as an elastic continuum of two-dimensional charged particles; the normal-mode reduction; and the identification of the result with the four-vector potential.

Results

The Coriolis coupling is a minimal coupling

For the adiabatic, inviscid flow of a compressible fluid in a rotating frame, the Lagrangian density is[13]

$$l = \frac{1}{2}\rho \mathbf{V}^2 + \rho(\mathbf{V} \cdot \mathbf{V}_e) + \frac{1}{2}\rho \mathbf{V}_e^2 - \rho(E + \phi) \tag{2}$$

where $\mathbf{V}$ is the velocity relative to the frame, ρ the density, E the internal energy, ϕ the external potential, and

$$\mathbf{V}_e = \mathbf{\Omega} \times \mathbf{r} \tag{3}$$

is the velocity of the rotating frame at the point $\mathbf{r}$, $\boldsymbol{\Omega}$ being the angular velocity of the rotation. The sum of the first three terms of Eq. (2) is $\frac{1}{2}\rho(\mathbf{V}+\mathbf{V}_e)^2$, the absolute kinetic energy, as it must be.

Compare Eq. (2) with the Lagrangian of a particle of mass m and charge e in an electromagnetic field,

$$L = \frac{1}{2}m\mathbf{v}^2 + \frac{e}{c}\mathbf{v}\cdot\mathbf{A} - e\varphi \tag{4}$$

where c is the speed of light, $\mathbf{A}$ the vector potential and φ the scalar potential. The correspondence of the second terms is exact: $\mathbf{V}_e$ stands in the place of $\mathbf{A}$, and mass density in the place of charge. The associated field is

$$\boldsymbol{\nabla}\times\mathbf{V}_e = 2\boldsymbol{\Omega} \tag{5}$$

the vorticity of solid-body rotation being twice the angular velocity.

That Eq. (5) really does play the part of a magnetic field may be checked directly. The canonical momentum conjugate to position, per unit mass, is

$$\mathbf{p} = \mathbf{V}+\mathbf{V}_e \tag{6}$$

so that the canonical momentum is the absolute velocity while the kinematic momentum is the relative velocity — the distinction between canonical and kinematic momentum familiar from the electromagnetic problem. The Euler–Lagrange equations of Eq. (2) then give

$$\frac{d\mathbf{V}}{dt} = -2\boldsymbol{\Omega}\times\mathbf{V} - \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{r}) + \cdots \tag{7}$$

the remaining terms coming from E and ϕ. The first term of Eq. (7) is the Coriolis force, and it is the Lorentz force $\mathbf{V}\times(\boldsymbol{\nabla}\times\mathbf{V}_e)$ belonging to the field (5). Its factor of two arises in exactly the way it does in electromagnetism: one contribution of $\boldsymbol{\Omega}\times\mathbf{V}$ from the time derivative of the canonical momentum (6), and one from the gradient of the coupling term. The second term of Eq. (7) is the centrifugal force, generated by the third term of Eq. (2). That term has no counterpart in Eq. (4); it is what makes the sum of the first three terms of Eq. (2) the absolute kinetic energy, and the correspondence with the electromagnetic Lagrangian is confined to the coupling term.

Equation (5) is a field in physical space with the dimensions of inverse time. It is not $\mathbf{B}_W$, which acts in physical space but has the dimensions of inverse length squared. The relation between the two is established in the Discussion.

The quadratic Lagrangian for linear shallow water motion

Consider now linear motion of a shallow homogeneous layer of mean depth H about a state of rest, on a β-plane on which the Coriolis parameter is

$$f = f_0 + \beta y \tag{8}$$

with y the meridional coordinate and β constant. Let $\boldsymbol{\xi} = (\xi, \eta)$ be the horizontal displacement of a fluid column from its rest position, so that the perturbation velocities and geopotential are

$$u = \frac{\partial \xi}{\partial t}\ , \qquad v = \frac{\partial \eta}{\partial t}\ , \qquad \Phi = -c_g^2\, \boldsymbol{\nabla} \cdot \boldsymbol{\xi} \tag{9}$$

with $c_g^2 = gH$.

The relevant part of Eq. (2) is the coupling term. Write it per unit mass as $\mathbf{V} \cdot \mathbf{R}$, where for horizontal motion $\mathbf{R}$ is any vector field satisfying $\boldsymbol{\nabla} \times \mathbf{R} = f\hat{\mathbf{z}}$, the local vertical component of $2\boldsymbol{\Omega}$. Evaluating $\mathbf{R}$ at the displaced position and expanding,

$$\dot{\boldsymbol{\xi}} \cdot \mathbf{R}(\mathbf{x}_0 + \boldsymbol{\xi}) = \dot{\boldsymbol{\xi}} \cdot \mathbf{R}(\mathbf{x}_0) + \dot{\xi}_i \xi_j \frac{\partial R_i}{\partial x_j} + O(\xi^3) \tag{10}$$

The first term is a total time derivative and does not contribute to the action. The second splits into a part symmetric in (i, j), which is also a total time derivative, and an antisymmetric part, for which

$$\frac{\partial R_i}{\partial x_j} - \frac{\partial R_j}{\partial x_i} = -\varepsilon_{ij} f \tag{11}$$

Retaining only the antisymmetric part and adding the kinetic and available potential energies, the shallow water Lagrangian density per unit mass, correct to second order in the displacement, is

$$L_2 = \tfrac{1}{2}\left(\dot{\xi}^2 + \dot{\eta}^2\right) + \tfrac{1}{2} f(y)\left(\xi\dot{\eta} - \eta\dot{\xi}\right) - \tfrac{1}{2} c_g^2 (\boldsymbol{\nabla}\cdot\boldsymbol{\xi})^2 \tag{12}$$

Here f is evaluated at the rest latitude of the fluid column and is therefore an explicit function of the independent variable y, not of the displacement field η; this distinction matters, and getting it wrong introduces a spurious quadratic term. The Euler–Lagrange equations of Eq. (12) are

$$\frac{\partial u}{\partial t} - fv + \frac{\partial \Phi}{\partial x} = 0\,, \qquad \frac{\partial v}{\partial t} + fu + \frac{\partial \Phi}{\partial y} = 0 \tag{13}$$

which, together with the definition of Φ in Eq. (9), are the linearized shallow water equations on a β-plane.

The middle term of Eq. (12) is already a minimal coupling. Writing it as $\dot{\boldsymbol{\xi}} \cdot \mathbf{A}_d$,

$$\mathbf{A}_d = \tfrac{1}{2} f\, \hat{\mathbf{z}} \times \boldsymbol{\xi}\,, \qquad \boldsymbol{\nabla}_\xi \times \mathbf{A}_d = f\, \hat{\mathbf{z}} \tag{14}$$

which is the symmetric gauge for a uniform field of strength f — but a field in the *plane of displacements*, not in physical space. Equation (12) therefore describes an elastic continuum of two-dimensional charged particles, one for each fluid column, each moving in a uniform magnetic field whose strength $f(y)$ varies from column to column, and coupled to its neighbours only through the divergence term. The β-effect is the variation of the field strength across the medium.

The physical content of Eq. (14) is easily checked. Dropping the elastic term, Eq. (12) gives $\ddot{\xi} = f\dot{\eta}$ and $\ddot{\eta} = -f\dot{\xi}$, that is, circular motion at angular frequency f. The cyclotron motion of the displacement-space particle is the inertial oscillation of the fluid column, and the cyclotron frequency is the inertial frequency.

Reduction to the meridional structure

Equation (12) is not yet in the form of the particle representation, whose Hamiltonian acts in physical space and whose field is uniform. The step that converts one into the other is the normal-mode reduction.

Consider a mode periodic in longitude and time, and write

$$\xi = Re\left[iX(y)e^{i(kx-\omega t)}\right], \qquad \eta = Re\left[Y(y)e^{i(kx-\omega t)}\right] \tag{15}$$

with X and Y real. Retaining complex amplitudes instead splits the quadratic form into two identical and uncoupled copies of what follows, so no generality is lost by fixing the relative phase in this way; the factor of i merely records the quadrature between the zonal and meridional motions familiar from the Matsuno solutions.

Averaging Eq. (12) over a wavelength and a period, and discarding an overall factor of four, the Lagrangian for the mode is

$$\mathcal{L} = \omega^2(X^2 + Y^2) - 2f\omega XY - c_g^2\left(\frac{dY}{dy} - kX\right)^2 \tag{16}$$

The amplitude X occurs algebraically in Eq. (16): no derivative of X appears, because $\partial\xi/\partial x$ contributes only ikX. It may therefore be eliminated exactly, and not as an approximation. Stationarity with respect to X gives

$$X = \frac{f\omega Y - c_g^2 k\, dY/dy}{\omega^2 - c_g^2 k^2} \tag{17}$$

which is the variational counterpart of the elimination of u and Φ by which the meridional structure equation is conventionally obtained. Substituting Eq. (17) back into Eq. (16) yields a reduced Lagrangian of the form

$$\mathcal{L}_{red} = a\left(\frac{dY}{dy}\right)^2 + b\,Y\frac{dY}{dy} + d\,Y^2 \tag{18}$$

with

$$a = \frac{c_g^2\omega^2}{c_g^2k^2-\omega^2}\,, \qquad b = -\frac{2c_g^2k\omega f}{c_g^2k^2-\omega^2} \tag{19}$$

and d a function of f, k, ω and c_g whose form is not needed explicitly.

The cross term of Eq. (18) is the crux. Integrating it by parts,

$$\int b\,Y\frac{dY}{dy}\,dy = \left[\frac{1}{2}bY^2\right] - \frac{1}{2}\int \frac{db}{dy}Y^2\,dy \tag{20}$$

the boundary contribution vanishing because $Y \to 0$ as $y \to \pm\infty$ for trapped modes. Since b is proportional to f by Eq. (19), db/dy is proportional to $df/dy = \beta$. The reduced Lagrangian becomes

$$\mathcal{L}_{red} = \lambda\left[-\left(\frac{dY}{dy}\right)^2 + \left(\frac{\omega^2}{c_g^2} - k^2 - \frac{\beta k}{\omega} - \frac{f^2}{c_g^2}\right)Y^2\right]\,, \qquad \lambda = \frac{c_g^2\omega^2}{c_g^2k^2-\omega^2} \tag{21}$$

where λ is a single constant, independent of y. Stationarity of Eq. (21) returns

$$\frac{d^2Y}{dy^2} + \left[\frac{\omega^2}{c_g^2} - k^2 - \frac{\beta k}{\omega} - \frac{(f_0+\beta y)^2}{c_g^2}\right]Y = 0 \tag{22}$$

which is the meridional structure equation of Matsuno[16], as it must be.

Identification with the four-vector potential

Equation (21) is not merely equivalent to the particle representation; it is that representation. Set

$$\Psi = Y(y)\exp(ikx) \tag{23}$$

and take

$$\mathbf{A}_W = -\frac{\beta}{2\omega}\hat{\mathbf{x}} - \frac{f_0+\beta y}{c_g}\hat{\mathbf{z}}\,, \qquad \phi_W = -\frac{\alpha\beta^2}{4\omega^2} \tag{24}$$

which is the four-vector potential of Paper I, α being the constant that plays there the role of $\hbar^2/2m$. Then Eq. (21) is identically

$$\mathcal{L}_{red} = -\lambda\left[|(-i\boldsymbol{\nabla} - \mathbf{A}_W)\Psi|^2 + \frac{\phi_W}{\alpha}|\Psi|^2 - \frac{\omega^2}{c_g^2}|\Psi|^2\right] \tag{25}$$

with no residual whatever. The identity follows from expanding the square of the kinematic momentum,

$$\left(k + \frac{\beta}{2\omega}\right)^2 = k^2 + \frac{\beta k}{\omega} + \frac{\beta^2}{4\omega^2} \tag{26}$$

the last term of which is cancelled by the constant scalar potential. Stationarity of Eq. (25) is the time-independent Schrödinger equation

$$\alpha(-i\boldsymbol{\nabla} - \mathbf{A}_W)^2\Psi + \phi_W\Psi = \alpha\frac{\omega^2}{c_g^2}\Psi \tag{27}$$

for any constant α, which is the equation of motion of Paper I. The four-vector potential is therefore a consequence of Hamilton's principle applied to a rotating shallow water fluid, and not an ansatz.

The derivation also fixes the vertical wavenumber. Paper I offers two versions of the potential, one with $k_z = f_0/c_g$ and $A_{Wz} = -\beta y/c_g$, the other with $k_z = 0$ and $A_{Wz} = -(f_0 + \beta y)/c_g$; the two are related by a gauge transformation. Repeating the comparison of Eq. (25) with k_z left free leaves a residual proportional to

$$k_z\left(2\beta y + c_g k_z + 2f_0\right) \tag{28}$$

which vanishes only for $k_z = 0$. The variational route therefore delivers the second version directly, and the vertical dependence of the first is exposed as superfluous without appeal to a gauge argument.

Discussion

The term $\beta k/\omega$ in Eq. (22) is the one that makes the particle representation interesting and the one that has always been hardest to motivate. A representation based on a simple harmonic oscillator cannot produce it, and reproducing it requires shifting the zonal wavenumber by $\beta/2\omega$ by hand. Earlier work[15] attributed the shift to "the special gauge taken in deriving the shallow water waves" without identifying the gauge, and Paper I identifies it as the gauge function $\chi = (\beta/2\omega)x$ that removes the constant part of A_{Wx}, so that k is the canonical and $k + \beta/2\omega$ the kinematic zonal momentum.

The derivation above supplies the missing mechanism. The term arises from the single integration by parts (20), in which the cross term $b\,Y\,dY/dy$ of the reduced Lagrangian is converted into a term proportional to $db/dy \propto \beta$. Nothing else in the calculation produces a β; the f^2/c_g^2 term of Eq. (22) comes from the algebra of the elimination, and β appears there only through f itself.

This is why the harmonic oscillator representation must fail. It writes down a potential proportional to f^2 and nothing else, and so has no cross term to integrate by parts. It is also why the gauge is "special" in the sense intended there[15]: the choice is not made, it is inherited from the boundary term discarded in Eq. (20).

The four-vector potential (24) depends on ω, through $A_{Wx} = -\beta/2\omega$ and through ϕ_W, while ω is also the eigenvalue. The Hamiltonian is thus a function of its own eigenvalue and Eq. (27) is an implicit rather than a linear eigenvalue problem. This is the sharpest disanalogy between the particle representation and the electromagnetic Landau problem, and it invites the reading that the potential is some kind of self-consistent or mean field, determined together with the state of the particle.

The derivation shows that this reading is wrong. The parent Lagrangian, Eq. (12), makes no reference to ω whatever. The frequency enters at exactly one point, the elimination (17) of the zonal displacement amplitude, and it enters there because that elimination is performed on a single normal mode. The implicit character of Eq. (27) is therefore the price of reducing three coupled fields to one, and not a property of the fluid system.

Paper I notes that the $\beta k/\omega$ term is what makes the dispersion relation cubic in ω and hence what produces the Rossby branch; without it the relation would be quadratic and admit only gravity waves. In the light of the present derivation that statement acquires a sharper form. The Rossby branch is a property of the coupled (u, v, Φ) system, and it appears in the meridional structure equation only because the elimination of u and Φ leaves its trace there.

Two magnetic-like fields have appeared, and the derivation makes their relation explicit. At the level of the fluid column the field is $f\hat{\mathbf{z}}$ of Eq. (14), acting in the plane of displacements, with the dimensions of inverse time; its cyclotron frequency is the inertial frequency. At the level of the wave the field is $\mathbf{B}_W = -(\beta/c_g)\hat{\mathbf{x}}$ of Eq. (1), acting in physical space, with the dimensions of inverse length squared; its magnetic length is the equatorial radius of deformation.

The step that converts the first into the second is the integration by parts (20), which replaces f by $df/dy = \beta$. The Landau quantization of the equatorial wave spectrum therefore originates in the *gradient* of the planetary vorticity and not in the planetary vorticity itself. This is as it should be:

a uniform f gives inertial oscillations but no meridional trapping, and it is the variation of f that confines the modes to the neighbourhood of the latitude at which f vanishes.

Three limitations should be stated.

First, the quadratic Lagrangian, Eq. (12), has been constructed in displacement form and verified against the linearized shallow water equations (13), rather than obtained by a formal linearization of the full three-dimensional Eulerian principle[13]. As shown in Methods, the constraints of that principle are not what stands in the way: the Lin constraint is satisfied identically in displacement variables, and conservation of mass is expressed by the relation between Φ and $\boldsymbol{\nabla} \cdot \boldsymbol{\xi}$ in Eq. (9). What is not demonstrated here is the columnar and hydrostatic reduction from three dimensions to a shallow layer, which is a separate approximation and would have to be carried out within the variational principle for the chain from Eq. (2) to Eq. (12) to be one of derivation throughout.

Second, the reduction is performed on a single normal mode. This is not a restriction on the shallow water problem, whose linear solutions are superpositions of such modes, but it is the reason the frequency appears in the reduced potential, and any attempt to write a potential valid for a general initial-value problem would have to work with the unreduced Lagrangian (12).

Third, the derivation inherits the restriction of the particle representation to modes with $v \not\equiv 0$. The Kelvin wave has no meridional velocity, is annihilated by the normal-mode reduction, and has no counterpart here. It is worth noting that the topological treatment of Delplace et al.[18] does capture the Kelvin wave on the same footing as the mixed Rossby–gravity wave, which suggests that the limitation belongs to this particular reduction rather than to the correspondence with condensed matter physics in general.

The four-vector potential that represents linear equatorial shallow water waves as a charged particle has been derived from Hamilton's principle. The chain is as follows.

1. In the variational principle for flow in a rotating frame, the coupling $\rho(\mathbf{V} \cdot \mathbf{V}_e)$ is a minimal coupling, with the frame velocity in the role of a vector potential and mass density in the role of charge. The associated field is $\boldsymbol{\nabla} \times \mathbf{V}_e = 2\boldsymbol{\Omega}$ and the Coriolis force is the corresponding Lorentz force.

2. Linearized about rest on a β-plane and written in terms of parcel displacement, the shallow water Lagrangian is Eq. (12): an elastic continuum of two-dimensional charged particles, each in a uniform field of strength f acting in the plane of displacements, whose cyclotron motion is the inertial oscillation.

3. For a normal mode the zonal displacement amplitude enters algebraically and can be eliminated exactly. The resulting reduced Lagrangian, Eq. (21), is identically the Lagrangian of a charged particle in the four-vector potential of the particle representation, Eq. (25), with no residual and with the vertical wavenumber forced to vanish.

4. The $\beta k/\omega$ term of the dispersion relation, which no harmonic-oscillator representation can produce and which must otherwise be inserted by hand, originates in a single integration by parts, in which f is replaced by β.

5. The dependence of the potential on the frequency, which makes the eigenvalue problem implicit, is an artifact of reducing three coupled fields to one. The parent Lagrangian contains no reference to the frequency at all.

The natural continuations are the columnar reduction of the full Eulerian principle noted above, and the treatment of a slowly varying medium, in which β or c_g varies and the analogue field ceases to be uniform. In the latter case the variational route is likely to be the more useful of the two, since it does not require the potential to be guessed anew for each configuration.

Methods

The Eulerian variational principle in a rotating frame

Equation (5) corrects a misprint in ref.[13], whose Eq. (3) gives $\boldsymbol{\nabla} \times \mathbf{V}_e = \boldsymbol{\Omega}$. The factor of two is required by the identity $\boldsymbol{\nabla} \times (\boldsymbol{\Omega} \times \mathbf{r}) = 2\boldsymbol{\Omega}$. It should be stressed that the misprint is isolated and affects no result of that paper, because the correct value is what the derivation there actually uses. Its Eq. (23), $\boldsymbol{\nabla}\left(\frac{1}{2}\mathbf{V}_e^2\right) = \mathbf{V}_e \times (\boldsymbol{\nabla} \times \mathbf{V}_e) + (\mathbf{V}_e \cdot \boldsymbol{\nabla})\mathbf{V}_e = -\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r})$, is an identity only when the factor of two is present; and the passage from its Eq. (21) to its Eq. (24) requires $-\mathbf{V} \times (\boldsymbol{\nabla} \times \mathbf{V}_e) = +2\boldsymbol{\Omega} \times \mathbf{V}$ in order to produce the Coriolis term at all. The equation of motion obtained there therefore stands as published.

The Lin constraint and the vortical part of the canonical momentum

The derivation of the quadratic Lagrangian was carried out in displacement variables, in which the constraints of the Eulerian principle play no visible part. It is worth saying what becomes of them, because for this problem the Lin constraint is not a technicality.

In the Eulerian principle the constraint is

$$\frac{d\mathbf{a}}{dt} = \frac{\partial \mathbf{a}}{\partial t} + (\mathbf{V} \cdot \boldsymbol{\nabla})\mathbf{a} = 0 \tag{29}$$

where $\mathbf{a}$ denotes the Lagrangian coordinates, or labels, of a fluid particle; Serrin[2] read it as the conservation of particle identity. It is imposed with a multiplier $\rho\gamma$, alongside the multiplier ξ that imposes conservation of mass. Varying the resulting functional with respect to $\mathbf{V}$, which occurs algebraically, gives

$$\mathbf{V} + \mathbf{V}_e = -\nabla\xi - \gamma\nabla a \tag{30}$$

only one component of the Lin constraint being required, as shown by Seliger and Whitham[5]. The left-hand side of Eq. (30) is the canonical momentum per unit mass, Eq. (6), the analogue of $m\mathbf{v} + e\mathbf{A}/c$. Equation (30) therefore resolves the canonical momentum into two pieces of quite different character. The mass-conservation multiplier ξ enters as a pure gradient and contributes nothing to the curl; it is a velocity potential, and in the language of the particle representation it is a gauge function of exactly the kind used in Paper I to remove the constant part of A_{Wx}. The Lin term is the only piece that can carry circulation:

$$\nabla \times (\mathbf{V} + \mathbf{V}_e) = -\nabla\gamma \times \nabla a = \boldsymbol{\zeta} + 2\boldsymbol{\Omega} \tag{31}$$

with $\boldsymbol{\zeta}$ the relative vorticity. The multipliers γ and a are thus Clebsch potentials for the absolute vorticity, and the role of the Lin constraint is precisely to supply the non-gauge part of the canonical momentum. Without it γ vanishes, Eq. (31) gives zero absolute vorticity, and one recovers the restriction of the Herivel principle to irrotational flow[1,3].

For the present problem that restriction is fatal rather than merely limiting. In a rotating frame the condition is $\boldsymbol{\zeta} + 2\boldsymbol{\Omega} = 0$, not $\boldsymbol{\zeta} = 0$, and a fluid at rest in the rotating frame has $\boldsymbol{\zeta} = 0$ and therefore absolute vorticity $2\boldsymbol{\Omega}$. The basic state about which the quadratic Lagrangian is constructed lies outside the Herivel principle altogether: without the Lin constraint there would be no state of rest to perturb, and hence no wave problem.

None of this is visible in Eq. (12), and the reason is instructive. In displacement variables the labels are the independent coordinates, so Eq. (29) is satisfied identically and the multipliers never arise. The vortical content that Eq. (30) assigns to $\gamma\nabla a$ is carried instead by the displacement field itself, and it is what the minimal coupling term of Eq. (12) encodes. The Lin constraint is therefore discharged by the choice of variables rather than neglected by it.

Data availability

No datasets were generated or analysed during the current study. All results reported here are analytic and are fully reproducible from the equations given in the text.

Code availability

No custom code was used in this study.

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Author contributions

P.K.W. conceived the study, carried out the analysis, and wrote the manuscript.

Competing interests

The author declares no competing interests.