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An Idealized Model of Buoyant Gas Spreading and Silt Release beneath Rough Cave Ceilings

Sandy Hardian Susanto Herho^{1,2,3,*}, Dasapta Erwin Irawan¹,
Rusmawan Suwarman⁴, and Deny Juanda Puradimaja¹

¹Applied Geology Research Group, Bandung Institute of Technology, Bandung, West Java 40132, Indonesia

²Department of Earth and Planetary Sciences, University of California, Riverside, CA 92521, USA

³Indonesian Subaquatic Sport Association (POSSI), North Jakarta, DKI Jakarta 14240, Indonesia

⁴Atmospheric Science Research Group, Bandung Institute of Technology, Bandung, West Java 40132, Indonesia

*Correspondence: sh001@ucr.edu

Abstract

Gas exhaled by open-circuit divers collects under the ceilings of submerged caves, migrates along the rock, and dislodges loose silt that then falls through the passage, a process that cave divers call percolation. This study develops an idealized model of the process. Buoyant gas under a self-affine rough ceiling is treated as a quasi-static sequence of pools that fill to their spill points, release gas that climbs to the next high point, and merge where their levels meet. Moving contact lines strip silt from the rock, and the released silt settles as a spectrum of grain sizes that reduces the distance at which a diver can see. The computation agrees exactly with an independent depression-filling algorithm, conserves gas volume to round-off, and reproduces the known exponents of invasion percolation in its capillary limit. A balance between the relief of the ceiling and its dip defines a pool scale that organizes where gas collects, and the area covered by gas from a single source grows more slowly than the volume supplied, as predicted by the self-affine geometry. The cutoff at the pool scale is soft, so large pools persist on steeper ceilings than the scale alone would suggest. In a reference passage, the lead diver of a team releases far more silt than the diver behind, and a team releases far less than the sum of its members acting alone, because the lead diver fills the pools and strips the paths that later divers reuse. This ordering holds across every detachment efficiency and ceiling geometry tested. Visibility toward the exit at eye level continues to decline for an hour or more after a team has passed, as fine silt descends from a nearly opaque layer beneath the ceiling. Absolute silt masses and sighting ranges depend on quantities that have not been measured in caves, and the results are best read as scaling relations and comparisons between scenarios.

Keywords: cave diving; fill-spill-merge; invasion percolation; sediment resuspension; self-affine roughness; underwater visibility

1 Introduction

Flooded caves are among the most demanding environments in recreational and scientific diving. Karst aquifers host extensive conduit networks in which passages may run for kilometers below the water table, with walls, floors, and ceilings shaped by dissolution and coated with fine sediment deposited by slow flow [10]. A diver in such a passage has no direct ascent to the surface, carries a finite supply of gas, and must retrace the route in. The fatality record of cave diving has been reviewed systematically [5], and much of the training that cave divers receive is devoted to preserving visibility and to following a continuous guideline to the exit. Visibility in a cave can change within seconds. A careless fin stroke near the floor raises a cloud of silt, and in still water such a cloud may persist for long periods, because the finest particles settle very slowly and little current exists to carry them away [7].

A second source of lost visibility originates on the ceiling rather than the floor. Gas exhaled through an open-circuit regulator rises as a stream of bubbles, spreads beneath the rock, collects in high points, and moves along the ceiling as further gas arrives. Where the gas-water interface sweeps across the rock, it loosens silt, clay, and small flakes that then fall into the passage. Cave divers refer to this as percolation, and the term is used in practice to explain why a passage that appeared clear behind a team can be turbid on the return. The phenomenon is widely recognized among divers but, as far as we are aware, it has not been treated quantitatively. The questions it raises are practical and physical at the same time: where the exhaled gas goes on a rough, inclined ceiling, how much silt it releases, how that depends on the number of divers and their breathing, and how long the released silt affects visibility along the route out.

The motion of gas under a rough ceiling belongs to a class of problems that has been studied extensively in porous media. When a nonwetting fluid displaces a wetting one slowly, capillary forces control the advance, and the invading fluid enters the accessible pore of lowest entry pressure at every step. Wilkinson and Willemsen [28] showed that this rule, invasion percolation, produces a critical percolation cluster. The phase diagram of immiscible displacement, organized by the capillary number and the viscosity ratio, places slow gas invasion in the regime of capillary fingering [18], and the advance of the front proceeds through abrupt pore-scale jumps

rather than smoothly [19]. Gravity modifies this picture. A density difference adds a hydrostatic gradient to the entry thresholds [27], which stabilizes or destabilizes the front depending on its direction. In the destabilizing case, a buoyant nonwetting fluid rises in a single finger whose width decreases with the gradient as a power law fixed by the exponents of percolation [3, 20], and the rising fluid can break into clusters that migrate and fragment [26].

A cave ceiling differs from a porous medium in that its relief is resolved at the scale of interest, centimeters to meters, rather than hidden in pore geometry. Buoyant gas under such a ceiling behaves like water on a landscape turned upside down. Water on a land surface fills depressions, spills over their lowest outlet, and joins depressions downslope, a fill-and-spill behavior first described in hydrological field studies [23] and since formalized as an exact hierarchy of filling, spilling, and merging for routing flow over digital elevation models [2]. The same hierarchy, applied to the inverted ceiling, describes how gas pools fill and overflow. The statistics of the relief then matter. Natural rock surfaces, including fracture and fault surfaces, are commonly self-affine over many decades of length, with a Hurst exponent that sets how relief grows with scale [6]. Combined with the mean dip of a passage, self-affine relief introduces a competition between roughness, which creates closed high points, and inclination, which removes them.

The release of silt is controlled by contact lines. The edge of a gas pool does not advance continuously over rock. Surface heterogeneity and contact-angle hysteresis pin the contact line until the driving pressure exceeds a threshold [8, 17], which adds a capillary contribution to the purely gravitational picture. When the line does move, it exerts a capillary force on attached particles that exceeds their adhesion for particles larger than about a micrometer [22], and the efficiency of detachment depends on the shape of the particles [1]. Each passage of a gas-water interface can therefore strip loose particles from rock.

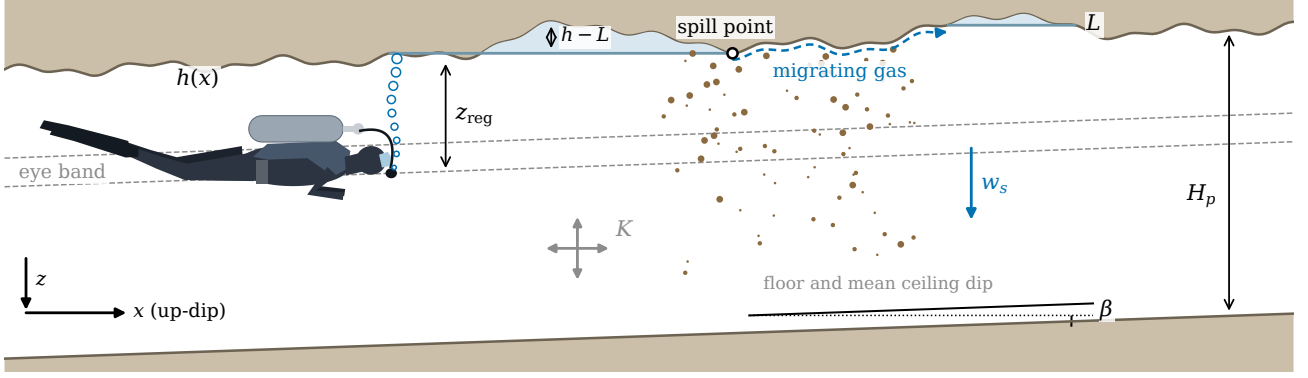
Once released, the particles settle. Their settling velocity grows with the square of their size in the Stokes regime and more slowly for larger grains [9], and in saline or organic-rich water fine particles may flocculate and settle faster than primary grains [29]. Their effect on a diver depends on the attenuation of light that they produce. Beam attenuation per unit mass is largest for the finest particles [4], and the distance at which a dark target remains visible is inversely proportional to the beam attenuation coefficient [30]. Visibility is therefore governed by the fine fraction, which settles most slowly. Lagrangian random-walk models are a standard way to compute the vertical distribution of such particles in a mixed water column [25].

This study links these elements in an idealized model of percolation. Its aims are to identify which geometric and operational factors control how much silt exhaled gas releases, where that silt originates relative to the divers, and how it affects visibility along the route out of a cave. The model does not attempt to predict the behavior of a particular passage. No observational data are used. Every parameter is a round value within a range we consider plausible, and the sensitivity of the results to the quantities that have not been measured is reported alongside them. The contributions are fourfold. We formulate the motion of exhaled gas under a rough ceiling as an exact fill-spill-merge computation with capillary pinning, and we verify it against an independent algorithm and known results of percolation theory. We derive a pool scale and an area-volume law from the self-affine geometry and test them against simulations. We quantify how silt release depends on the position of a diver in a team, the breathing rate, and holding station. Finally, we follow the released silt through the water column to the sighting range at eye level on the route out. The verification follows the practice adopted for related idealized solvers [13, 15, 16, 12].

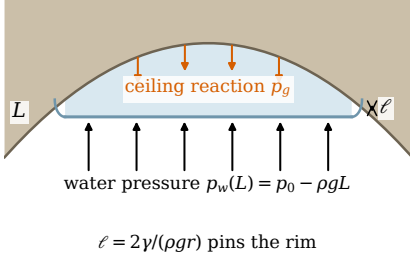
2 Methods

The configuration is shown in Fig. 1. A passage of length 12.8 m and width $W = 3.2$ m is bounded above by a rough ceiling and below by a floor at a distance $H_p = 2$ m beneath the mean ceiling. The coordinate x runs along the passage and is positive up-dip, y runs across it, and z is depth, measured downward, so that the elevation of a point is $-z$. Divers swim along the centerline with their second stage at $z_{\text{reg}} = 0.8$ m below the mean ceiling, and visibility is evaluated in an eye band between 0.6 and 0.8 m below it. All quantities are in SI units unless stated otherwise: lengths in m, times in s, pressures in Pa, forces in N, gas volumes in m^3 (quoted in liters where convenient), silt mass in g, and silt concentration in g m^{-3} , which equals mg L^{-1} . Throughout, the symbol $\equiv$ introduces a definition and $=$ denotes an equality that is derived or imposed as a model relation.

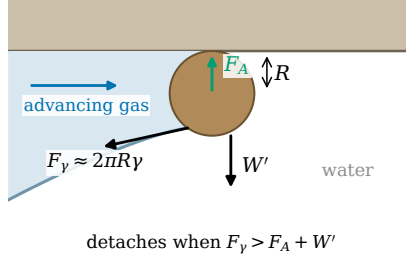
(a) Model section along the passage, to scale



(b) Gas pool



(c) Grain under a contact line



(d) Settling grain

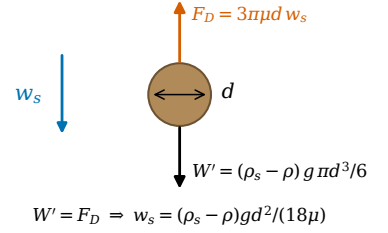


Figure 1: Model section and free-body diagrams. (a) Section along the passage, to scale. An idealized rough ceiling $h(x)$ with dip β holds gas pools with flat interfaces at levels L . The down-dip pool is full to its spill point, and gas crossing the saddle climbs along the ceiling to the next pool. An open-water diver exhales from the second stage at z_{reg} below the mean ceiling; silt stripped by moving contact lines settles at w_s and spreads with eddy diffusivity K . (b) Gas pool. The water pressure $p_w(L)$ on the flat interface balances the ceiling reaction, which carries the gas pressure p_g ; at the rim the contact line is pinned until the gas thickness exceeds ℓ . (c) Loose grain of radius R under an advancing contact line, with capillary force F_γ , adhesion F_A , and net weight W' . (d) Settling grain of diameter d , for which the net weight W' balances the Stokes drag F_D at the settling velocity w_s .

The ceiling elevation $h(x, y)$ (m), positive upward, is the sum of a random relief h_r , a uniform rise along the passage at the dip angle β , and a cross-passage arch of height A (m),

$$h(x, y) = h_r(x, y) + x \tan \beta - A \left(\frac{2y}{W} - 1 \right)^2. \quad (1)$$

The arch term vanishes on the centerline $y = W/2$ and equals $-A$ at the walls, so the ceiling is highest along the axis of the passage, as in a phreatic tube. The relief is written as a Fourier integral, $h_r(\mathbf{x}) = \int \hat{h}(\mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{x}} d^2k$, with independent Gaussian coefficients of spectral density $\langle |\hat{h}(\mathbf{k})|^2 \rangle = P(k)$ (m^4), where $k = |\mathbf{k}|$ (m^{-1}). The mean square height difference over a lag $\mathbf{r}$ is then

$$S^2(r) \equiv \langle [h_r(\mathbf{x} + \mathbf{r}) - h_r(\mathbf{x})]^2 \rangle = 2 \int P(k) [1 - \cos(\mathbf{k} \cdot \mathbf{r})] d^2k. \quad (2)$$

For a power-law spectrum $P(k) = P_0 k^{-(2+2H)}$ with $0 < H < 1$, the substitution $\mathbf{q} = r\mathbf{k}$ in Eq. (2) gives $S^2(r) = r^{2H} 2P_0 \int q^{-(2+2H)} [1 - \cos(q \cos \varphi)] d^2q$, in which the remaining integral is a pure number that converges at small q because $1 - \cos$ vanishes as q^2 and at large q because $H > 0$. The relief is therefore self-affine,

$$S(r) = \sigma_1 \left(\frac{r}{r_0} \right)^H, \quad r_0 \equiv 1 \text{ m}, \quad (3)$$

where σ_1 (m) is the root-mean-square height difference at a lag of 1 m and H is the Hurst exponent. Because $H < 1$, the slope of the relief over a lag r , of order $S(r)/r \propto r^{H-1}$, grows as r decreases: the ceiling is steep at small scales and gentle at large ones. We take $H = 0.8$, within the range reported for natural rock surfaces [6], and $\sigma_1 = 0.10$ m.

The water is at rest. The vertical momentum balance then reduces to $dp/dz = \rho g$, with $\rho = 1000 \text{ kg m}^{-3}$ and $g = 9.81 \text{ m s}^{-2}$, which integrates to the hydrostatic pressure

$$p_w(z) = p_0 + \rho g z, \quad (4)$$

in Pa, where p_0 is the pressure at the reference level $z = 0$. The same balance applies inside the gas, with the gas density ρ_g in place of ρ . At a ceiling depth of 20 m, the gas is at about 3 bar and $\rho_g \approx 3.6 \text{ kg m}^{-3}$, so across a

pool 0.2 m deep the gas pressure changes by $\rho_g g \Delta z \approx 7$ Pa while the water pressure changes by $\rho g \Delta z \approx 2000$ Pa. We therefore treat the gas pressure p_g as uniform within a connected pool. On scales large compared with the capillary length, the pressure is continuous across the gas-water interface, so the interface lies where $p_w(z) = p_g$. By Eq. (4), this condition holds on a single horizontal plane,

$$z_L = \frac{p_g - p_0}{\rho g}, \quad L \equiv -z_L, \quad (5)$$

where L (m) is the elevation of the interface, the level of the pool. Fig. 1b shows the forces on the gas of one pool. The water pushes up on the flat interface with the pressure $p_w(z_L) = p_g$, and the rock pushes down on the gas with the same pressure, so the gas is in equilibrium for any shape of the ceiling. The force that the gas exerts on the rock in excess of the hydrostatic water pressure that would act there without gas is obtained by integrating the pressure difference over the horizontal projection of the covered ceiling, $\int [p_g - p_w(-h)] dA = \int \rho g (h - L) dA$. With Eq. (7) below, this force is $\rho g V$, the buoyancy of the pool, which the ceiling carries.

At the rim of a pool, the contact line does not advance onto rock as soon as the interface reaches it. Micro-roughness, cracks, and the hysteresis of the contact angle pin the line [8, 17], and gas can only enter a crevice of radius r (m) when the pressure difference across the meniscus overcomes the Young-Laplace pressure. For a water-wet surface with two principal curvatures of order $1/r$, this pressure is $2\gamma/r$, with $\gamma = 0.072$ N m⁻¹ the surface tension. At a point of the ceiling at elevation h under a pool of level L , the pressure difference between gas and water is, by Eqs. (4) and (5), $p_g - p_w(-h) = \rho g (h - L)$, the weight per unit area of a water column of height $h - L$. The contact line can advance only if this difference exceeds $2\gamma/r$, which gives

$$h - L \geq \ell \equiv \frac{2\gamma}{\rho g r} \quad \Longleftrightarrow \quad L \leq h_{\text{eff}} \equiv h - \ell. \quad (6)$$

The pinning height ℓ has units of (N m⁻¹)/(N m⁻³ · m) = m. Equation (6) states that the gas sees an effective ceiling lowered by ℓ : a cell is covered when the level of the pool lies at or below h_{eff} . The radius r is drawn independently for each cell from a log-normal distribution with median $r_{50} = 4$ mm and logarithmic standard deviation 0.5, which gives a median pinning height $\ell_{50} = 3.7$ mm, comparable to the capillary length $\ell_c \equiv (\gamma/\rho g)^{1/2} = 2.7$ mm. The ratio of the relief across one lattice cell of size a to the pinning height, $\text{Bo} \equiv S(a)/\ell_{50} = 1.19$ for $a = 2$ cm, plays the role of a Bond number: gravity acting through the relief and capillarity acting through pinning are of comparable importance at the scale of one cell, and gravity dominates at larger scales because S grows with r .

The volume of gas held by a pool follows from the same picture. Below each covered point of the ceiling, the gas occupies the column between the effective ceiling and the level, so that

$$V = \int_{\mathcal{A}} (h_{\text{eff}} - L) dA, \quad (7)$$

where $\mathcal{A}$ is the covered area. The integrand is positive by Eq. (6), and V has units of m³. On a lattice of spacing a , the midpoint rule over the n covered cells i of a pool $\mathcal{P}$ gives

$$V = a^2 \sum_{i \in \mathcal{P}} (h_{\text{eff},i} - L) = a^2 (\Sigma_{\mathcal{P}} - nL), \quad \Sigma_{\mathcal{P}} \equiv \sum_{i \in \mathcal{P}} h_{\text{eff},i}. \quad (8)$$

Differentiating Eq. (8) with respect to L at fixed $\mathcal{P}$ gives $\partial V/\partial L = -na^2$, so that a small injection ΔV lowers the level by

$$\Delta L = -\frac{\Delta V}{na^2}. \quad (9)$$

Physically, added gas spreads over the whole covered area and pushes the interface down uniformly. As L descends, the first perimeter cell to satisfy Eq. (6) is the one with the highest effective ceiling, so the pool grows by admitting perimeter cells in decreasing order of h_{eff} . This is the invasion-percolation rule with threshold $-h_{\text{eff}}$ [28]. A perimeter cell j is a spill point when one of its neighbors m outside the pool lies higher, $h_{\text{eff},m} > h_{\text{eff},j}$. At $L = h_{\text{eff},j}$, gas reaching j is no longer confined: buoyancy drives it toward higher ceiling, which lies outside the pool. The pool is then full. Any further gas crosses the saddle at j and rises along the path of steepest ascent of h_{eff} , $c_{k+1} = \arg \max_{m \in \mathcal{N}(c_k)} h_{\text{eff},m}$, where $\mathcal{N}$ denotes the four lattice neighbors, until it reaches a crest, where it starts a new pool, or a cell of an existing pool, which it joins. When two pools meet at a common saddle with equal levels, they merge into one. Gas that reaches either open end of the passage leaves the modeled section. This is the fill-spill-merge construction of landscape hydrology [23, 2], applied to the inverted ceiling.

In this description, gas that crosses a saddle does not remain hydraulically connected to the pool it left. A connected gas body would, by Eq. (5), have one level and would flood every cell above the saddle elevation at once. Raising a gas layer by only 1 cm under 1 m² of ceiling requires 10 L, about seven breaths, so the volume delivered by a breath cannot maintain such a connection. Gas that crosses a saddle instead forms a thin stream that rises to the next crest, as in buoyancy-driven migration with fragmentation [26].

When every pool is full, the covered set no longer depends on the order in which gas arrived, because each pool is filled to the elevation of its spill point. That state is the filling of the depressions of $f \equiv -h_{\text{eff}}$ from the outlets at the two ends of the passage,

$$F_m = \max(f_m, F_c), \quad h_{\text{eff}} - L = F - f, \quad (10)$$

where F (m) is the filled surface and c is the neighbor through which cell m is first reached in order of increasing

F. Equation (10) provides an exact check on the event-driven computation of Eqs. (8) and (9).

Each diver exhales, once per breathing period T_b (s), a volume $\text{RMV } T_b/60$ at the ambient pressure of the regulator, where RMV is the respiratory minute volume in L min^{-1} and the factor 60 converts minutes to seconds. Exhaled bubbles reach thermal equilibrium with the water quickly, so their expansion between the regulator and the ceiling is isothermal, $pV = \text{const}$ (Boyle's law). With Eq. (4) evaluated at the regulator, at water depth $D + z_{\text{reg}}$, and at the ceiling, at water depth D , the volume per breath at the ceiling is

$$V_b = \frac{\text{RMV } T_b}{60} \frac{p_{\text{atm}} + \rho g (D + z_{\text{reg}})}{p_{\text{atm}} + \rho g D}, \quad (11)$$

in liters, with $p_{\text{atm}} = 101\,325$ Pa. For $\text{RMV} = 20 \text{ L min}^{-1}$, $T_b = 4$ s, $D = 20$ m, and $z_{\text{reg}} = 0.8$ m, the pressure ratio is 1.027 and $V_b = 1.37$ L. The bubbles reach the ceiling in a Gaussian footprint of root-mean-square radius 0.12 m centered above the regulator. A closed-circuit rebreather exhales no gas in level swimming and is represented by $\text{RMV} = 0$.

Silt is released by moving contact lines (Fig. 1c). A particle of radius R held at a gas-water interface experiences a capillary force of maximum magnitude $F_\gamma \approx 2\pi R\gamma$ (N), the surface tension acting along its line of contact. Its net weight in water is $W' = (\rho_s - \rho) g \frac{4}{3}\pi R^3$, with $\rho_s = 2650 \text{ kg m}^{-3}$ for quartz-density grains. The ratio

$$\frac{F_\gamma}{W'} = \frac{3\gamma}{2(\rho_s - \rho) g R^2} \quad (12)$$

is dimensionless and equals about 7×10^4 for $R = 10 \text{ }\mu\text{m}$, so weight plays no role in detachment, and the outcome is decided by the competition between F_γ and the adhesion F_A of the particle to the rock. Experiments show that a moving air-water interface removes a large fraction of attached particles larger than about a micrometer [22]. Because the adhesion of silt on a cave ceiling is not known, we represent the outcome by two efficiencies. When a contact line crosses a cell for the first time, it removes the fraction ϕ_c (dimensionless) of the loose silt in that cell. When a volume v (m^3) of migrating gas crosses a cell, each bubble sweeps the cell anew, and we take the remaining mass to decay as $m \rightarrow m e^{-v/v_d}$, with a stripping volume v_d (m^3). For a sequence of parcels v_k , the remaining mass is multiplied by

$$\prod_k e^{-v_k/v_d} = \exp\left(-\frac{1}{v_d} \sum_k v_k\right), \quad (13)$$

which depends only on the total volume, so the result does not depend on how a breath is divided into parcels. The loose inventory is uniform, $m_0 = 50 \text{ g m}^{-2}$, so each cell initially holds $m_0 a^2 = 0.02$ g, with $\phi_c = 0.5$ and $v_d = 2 \text{ mL}$.

The geometry of the ceiling sets a scale for the pools. Over a horizontal distance ℓ along the passage, the relief typically rises or falls by $S(\ell)$ of Eq. (3), while the dip raises the ceiling by $\ell \tan \beta$. A closed high point of width ℓ can hold gas only if the relief can reverse the rise of the dip across it. Equating the two, $\sigma_1(\ell/r_0)^H = \ell \tan \beta$, and solving for ℓ gives the pool scale

$$r^* = r_0 \left(\frac{\sigma_1}{r_0 \tan \beta} \right)^{1/(1-H)}, \quad (14)$$

which has units of m because $\sigma_1/(r_0 \tan \beta)$ is dimensionless. Since $H < 1$, the relief dominates below r^* and the dip above it. Because the exponent $1/(1-H)$ is 5 for $H = 0.8$, r^* changes by orders of magnitude over a few degrees of dip. The scale is not a sharp cutoff. For Gaussian relief, the height difference across a width ℓ is normally distributed with standard deviation $S(\ell)$, so the probability that it exceeds the rise of the dip is $\frac{1}{2} \text{erfc}[\ell \tan \beta / (\sqrt{2} S(\ell))]$. Using Eq. (14), the argument satisfies $[\ell \tan \beta / S(\ell)]^2 = (\ell/r^*)^{2(1-H)}$, and the leading behavior of the complementary error function gives

$$\text{Pr}(\ell) \sim \exp\left[-\frac{1}{2} \left(\frac{\ell}{r^*}\right)^{2(1-H)}\right]. \quad (15)$$

This is a stretched exponential with exponent $2(1-H)$, which is 0.4 for $H = 0.8$. It decays slowly, so pools several times larger than r^* remain likely, and r^* acts as a scaling variable rather than a limit.

Below r^* , the dip is negligible and the pools follow the self-affine relief alone. A pool of width ℓ covers an area $A \sim \ell^2$ and is filled to a depth set by the relief, $h - L \sim S(\ell) \propto \ell^H$. By Eq. (7), its volume is $V \sim A S(\ell) \propto \ell^{2+H}$. Eliminating ℓ gives

$$V \propto A^{(2+H)/2} \quad \text{and} \quad A \propto V^{2/(2+H)}. \quad (16)$$

The first relation applies across the pools of one ceiling. The second applies to gas fed from a single point, whose pools fill and merge into a cluster that remains self-similar as it grows. The exponent $2/(2+H)$ is 0.800 for $H = 0.5$ and 0.714 for $H = 0.8$. It is less than one, so each additional liter of gas covers less new ceiling than the liter before it.

When the relief is negligible compared with the pinning heights, as on a smooth, nearly flat ceiling, the level differences that drive fragmentation vanish, the gas remains connected, and its contact line advances into the perimeter cell with the lowest entry pressure. We normalize the entry pressures $2\gamma/r_i$ to dimensionless thresholds t_i uniform on $(0, 1)$. Water enclosed by gas on a ceiling remains connected to the water below, so no region can be trapped, and the process is invasion percolation without trapping [28]. Its cluster is a critical percolation

cluster, whose mass grows with its radius of gyration R_g as $N \propto R_g^{D_f}$ with $D_f = 91/48$, and the thresholds it accepts are bounded by the site percolation threshold $p_c = 0.592746$ of the square lattice [21]. A dip adds the hydrostatic term to the thresholds [27], $t_i = u_i - G y_i$, where y_i is the up-dip position in lattice units and the dimensionless gradient $G = \rho g a \tan \beta / \Delta p_c$ is the change in hydrostatic pressure per lattice step divided by the spread Δp_c of entry pressures. The finger width follows from a self-consistency argument. Near p_c , percolation clusters are correlated over a length $\xi \propto |p - p_c|^{-\nu}$, with $\nu = 4/3$ in two dimensions. Across a finger of width w , the gradient shifts the thresholds by Gw , which sets the local distance from p_c . Setting $w \sim \xi$ gives $w \propto (Gw)^{-\nu}$, or

$$w \propto G^{-\nu/(1+\nu)} = G^{-4/7}, \quad (17)$$

in lattice units, the scaling established for gravity-destabilized invasion [3, 20].

A released grain of diameter d settles under its net weight $W' = (\rho_s - \rho) g \pi d^3 / 6$ and the Stokes drag $F_D = 3\pi\mu d w_s$, with $\mu = 1.0 \times 10^{-3}$ Pa s (Fig. 1d). Its velocity relaxes to the terminal value within a time of order $\rho_s d^2 / (18\mu)$, less than 3 ms for the largest grains, after which the two forces balance,

$$(\rho_s - \rho) g \frac{\pi d^3}{6} = 3\pi\mu d w_s \quad \implies \quad w_s = \frac{(\rho_s - \rho) g d^2}{18\mu}, \quad (18)$$

in m s^{-1} . Stokes drag assumes a particle Reynolds number $\text{Re}_p = w_s d \rho / \mu$ well below one. This holds for all classes up to $64 \mu\text{m}$, for which $\text{Re}_p \leq 0.24$, but not for the coarsest class of $128 \mu\text{m}$, with $\text{Re}_p \approx 1.9$, whose settling velocity is therefore overestimated [9]. That class carries 0.4 percent of the mass and contributes little to attenuation.

The mass of each size class is conserved. In a width-averaged section of the passage, its concentration $C(x, z, t)$ (g m^{-3}) changes only through the divergence of its flux, $\partial_t C + \nabla \cdot \mathbf{J} = 0$. The flux combines advection by the along-passage current U (m s^{-1}), settling at w_s , and turbulent mixing represented by Fick's law with an eddy diffusivity K ($\text{m}^2 \text{s}^{-1}$), $\mathbf{J} = (UC - K\partial_x C, w_s C - K\partial_z C)$. With constant coefficients, this gives

$$\frac{\partial C}{\partial t} + U \frac{\partial C}{\partial x} + w_s \frac{\partial C}{\partial z} = K \left(\frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial z^2} \right). \quad (19)$$

No silt crosses the ceiling, so the vertical flux vanishes there, $J_z = w_s C - K\partial_z C = 0$ at $z = 0$. Silt that reaches the floor deposits and is not resuspended, $C = 0$ at $z = H_p$. The reference case has $U = 0$ and $K = 10^{-5} \text{ m}^2 \text{s}^{-1}$, which represents weak residual mixing in still cave water.

The time that a grain spends in the water column sets how long it can affect visibility. Let $T(z)$ (s) be the mean time for a grain at depth z to reach the floor. Over a short interval δt the grain moves by $w_s \delta t$ plus a random displacement of variance $2K\delta t$, so $T(z) = \delta t + \langle T(z + \delta z) \rangle$. Expanding to first order in δt gives the backward equation

$$K \frac{d^2 T}{dz^2} + w_s \frac{dT}{dz} = -1, \quad \left. \frac{dT}{dz} \right|_{z=0} = 0, \quad T(H_p) = 0, \quad (20)$$

where the first boundary condition expresses reflection at the ceiling and the second absorption at the floor. With $u \equiv dT/dz$, Eq. (20) becomes $u' + (w_s/K)u = -1/K$, whose solution with $u(0) = 0$ is $u = -(1 - e^{-w_s z/K})/w_s$. Integrating from 0 to H_p and using $T(H_p) = 0$ gives, for a grain released at the ceiling,

$$T(0) = \frac{H_p}{w_s} - \frac{K}{w_s^2} (1 - e^{-\text{Pe}}), \quad \text{Pe} \equiv \frac{w_s H_p}{K}. \quad (21)$$

The Péclet number Pe compares settling with mixing. For $\text{Pe} \gg 1$, $T(0) \rightarrow H_p/w_s$, the time to fall through the passage. For $\text{Pe} \ll 1$, expanding the exponential to second order gives $T(0) \rightarrow H_p^2/(2K)$, the time to diffuse across it. The released mass is divided among 22 size classes from 1 to $128 \mu\text{m}$, spaced by a factor $2^{1/3}$, according to a log-normal distribution with median $8 \mu\text{m}$ and logarithmic standard deviation 1. Their settling velocities range from 9×10^{-4} to 14.7 mm s^{-1} , and their residence times from about 2 min to about two days.

Visibility is set by the attenuation of a light beam. A grain of diameter d removes light over a cross-section $Q_c \pi d^2 / 4$, where the extinction efficiency Q_c approaches 2 for grains much larger than the wavelength. A mass concentration C_k of class k corresponds to $C_k / (\rho_s \pi d_k^3 / 6)$ grains per m^3 , with ρ_s expressed in g m^{-3} . The beam attenuation coefficient is the product of number concentration and cross-section,

$$c = c_w + \sum_k \frac{C_k}{\rho_s \pi d_k^3 / 6} \frac{Q_c \pi d_k^2}{4} = c_w + \sum_k b_k C_k, \quad b_k = \frac{3}{\rho_s d_k}, \quad (22)$$

in m^{-1} , where c_w is the attenuation of clear water and b_k ($\text{m}^2 \text{g}^{-1}$) is the mass-specific attenuation. Because b_k varies as $1/d_k$, a given mass of fine silt attenuates light far more than the same mass of coarse silt, consistent with the dependence of mass-specific attenuation on particle size observed in natural waters [4]. In uniform water, a black target is visible to a distance $4.8/c$ [30], which corresponds to an optical depth cy of 4.8. Along a path of nonuniform attenuation, the optical depth is the integral of c , so we take the sighting range y (m) toward the exit, from a point x_0 looking down-dip, as the distance at which

$$\int_0^y c(x_0 - s) ds = 4.8, \quad (23)$$

and we set $c_w = 0.16 \text{ m}^{-1}$, so that the clear-water sighting range is 30 m. The reference parameters are listed in

Table 1.

Table 1: Reference parameters. All values are illustrative.

Quantity	Symbol	Value	Unit
Lattice spacing	a	0.02	m
Passage length, width, height	–	12.8, 3.2, 2.0	m
Hurst exponent, relief at 1 m	H, σ_1	0.8, 0.10	–, m
Dip, arch	β, A	2, 0.15	deg, m
Median pinning radius, log-SD	r_{50}	4, 0.5	mm, –
Respiratory minute volume	RMV	20	L min ^{−1}
Breathing period, exhalation	T_b	4.0, 1.6	s
Swimming speed, spacing	–	0.2, 20	m s ^{−1} , s
Ceiling depth, regulator depth	D, z_{reg}	20, 0.8	m
Loose silt inventory	m_0	50	g m ^{−2}
Detachment constants	ϕ_c, v_d	0.5, 2	–, mL
Median grain size, log-SD	d_{50}	8, 1.0	μm, –
Eddy diffusivity	K	10 ^{−5}	m ² s ^{−1}
Clear-water sighting range	$4.8/c_w$	30	m

The ceiling is discretized on a lattice of spacing $a = 2$ cm, with 640×160 cells for the reference passage. The relief is generated by spectral synthesis of Eq. (2) on a lattice four times larger in each direction and cropped, which removes the periodicity of the discrete Fourier transform that would otherwise flatten $S(r)$ at lags approaching the domain size. It is rescaled so that the mean of the two lattice directions satisfies Eq. (3) at $r = r_0$. Each cell has four neighbors. The boundaries across the passage are closed, and each cell at an open end has an additional exit node, placed one cell beyond it with the elevation given by the extension of the dip.

The fill-spill-merge computation, `FillSpill`, is event driven. Each pool stores n , $\Sigma \mathcal{P}$, and L , so that Eq. (8) is evaluated in constant time, and a priority queue of its perimeter ordered by h_{eff} . An injection ΔV is applied through Eq. (9). If the level would pass below the highest perimeter cell, the volume needed to reach that cell, $(L - h_{\text{eff},j})na^2$, is consumed and the cell is either admitted or, if it is a spill point, the remainder is routed along the precomputed steepest-ascent path to the next pool. Merging appends the smaller pool to the larger. Because every step conserves the sum of the pool volumes and the volume leaving the section, the total volume is conserved to round-off. Each breath of Eq. (11) is divided into 40 parcels released at random times during an exhalation of 1.6 s and at random points of the footprint. The saturated state of Eq. (10) is computed independently by priority-flood filling, `priority_flood`, which shares no code with the event-driven computation. In the connected limit, invasion percolation is computed with a binary heap on lattices of up to 1201^2 sites.

Equation (19) is the Fokker-Planck equation of the stochastic differential equations $dX = U dt + \sqrt{2K} dB_x$ and $dZ = w_s dt + \sqrt{2K} dB_z$, where B_x and B_z are independent Brownian motions. We therefore solve it with Lagrangian particles, as is standard for particles in a mixed water column [25]. With constant coefficients, the increments over a step Δt are exactly Gaussian, and

$$x^{n+1} = x^n + U\Delta t + \sqrt{2K\Delta t}\xi_x, \quad z^{n+1} = \left| z^n + w_s\Delta t + \sqrt{2K\Delta t}\xi_z \right|, \quad (24)$$

where ξ_x and ξ_z are independent standard normal variables. The absolute value reflects particles at the ceiling, which enforces the no-flux condition, and particles that reach the floor are removed, which enforces deposition. Detection of the floor at the ends of steps introduces a bias of order $(\Delta t)^{1/2}$ in residence times, which we measure against Eq. (21). The released mass of each size class is represented by about 1.2×10^4 particles of equal mass, with stochastic rounding so that the expected mass is exact. The concentration is the particle mass in each bin divided by the bin volume, $W \Delta x \Delta z$, with $\Delta x = 0.1$ m and $\Delta z = 0.05$ m, and the step is $\Delta t = 2$ s. Equation (23) is evaluated by cumulative summation of Eq. (22) over the bins between each point and the down-dip end of the section, with clear water beyond it.

We verify the computation with five tests. The event-driven computation is compared with Eq. (10) on twelve ceilings with dips from 0 to 8°, and its volume balance is recorded over 20 000 random injections. For a paraboloidal dome, $h = -\varrho^2/(2R)$ with ϱ the horizontal distance from its apex and R (m) its radius of curvature, Eq. (7) gives a covered disk of area $A = 2\pi R|L|$ and a volume $V = \pi RL^2$, so that $A = 2\sqrt{\pi RV}$ exactly. This is compared with the computation at four lattice spacings. The connected limit is compared with D_f and p_c , and the gradient limit with Eq. (17). The Lagrangian scheme is compared with Eq. (21) over three decades of Pe and three step sizes. Ensembles use twelve ceiling realizations unless stated otherwise, and confidence intervals are obtained by bootstrap resampling over realizations. The code is written in Python with NumPy [11], SciPy [24], and Matplotlib [14].

3 Results

The computation passes every verification test (Fig. 2, Table 2). The saturated state of the event-driven computation agrees with the independent priority-flood filling of Eq. (10) with a maximum absolute difference of zero,

over 147 456 cells of which 27 086 hold gas, on twelve ceilings with dips from 0 to 8° . The agreement is exact, not approximate, because both algorithms fill every depression of $-h_{\text{eff}}$ to the elevation of its outlet, and the elevation of an outlet is a value taken from the same lattice. Over 20 000 random injections, the relative volume residual remains below 6×10^{-15} , close to the precision of double-precision arithmetic summed over many pools, which confirms that the bookkeeping of Eqs. (8) and (9) neither creates nor loses gas at spills or merges. The covered area of the paraboloidal dome converges to its exact value with an observed order between 1.3 and 2.3, reaching a relative error of 2.3×10^{-4} at $a = 1$ cm for the largest volume. The order exceeds one because the contact line is a closed curve whose pixelation errors partly cancel around the circumference. The Lagrangian residence time exceeds Eq. (21) by a positive bias. The bias is largest at small Péclet number, where grains approach the floor many times by diffusion before crossing it, and it halves when the step is reduced by a factor of four, as expected for a bias of order $(\Delta t)^{1/2}$ that arises from detecting floor crossings only at the ends of steps. For the smallest step it is below 4 percent. In the reference computations, the step is a small fraction of the residence time of the fine classes that control visibility, and only the arrival of the coarsest class at the floor may be late by a few percent.

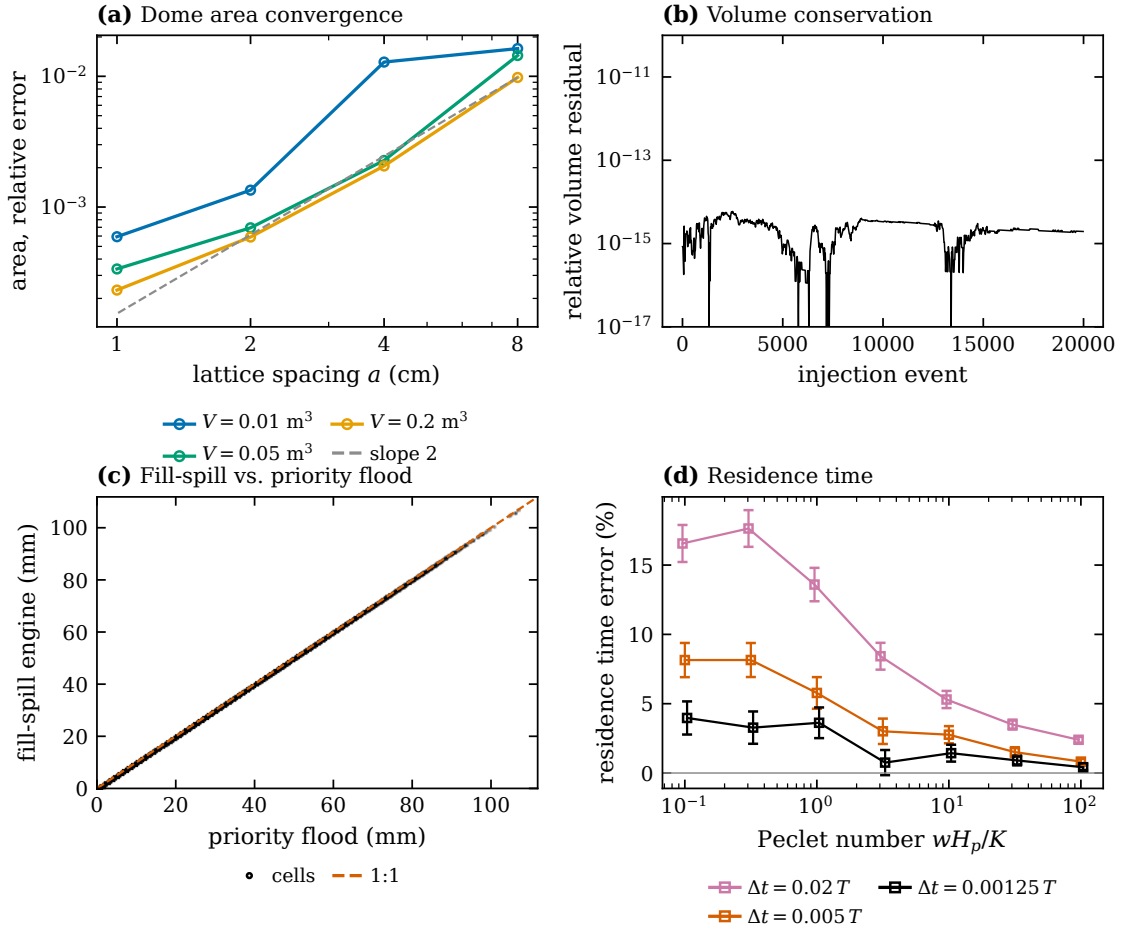


Figure 2: Verification. (a) Relative error of the covered area of a paraboloidal dome against lattice spacing for three gas volumes, with a slope-2 reference. (b) Relative volume residual after each of 20 000 random injections on a dipping rough ceiling. (c) Saturated gas thickness from the fill-spill engine against the priority-flood filling, all cells of twelve ceilings with dips from 0 to 8° . (d) Relative error of the Lagrangian mean residence time against the exact result of Eq. (21), for three time steps given as fractions of the exact residence time T ; bars are two standard errors.

Table 2: Measured residuals and exponents. Intervals are 95 percent bootstrap intervals.

Test	Expected	Measured
Engine vs priority flood, max abs (m)	0	0
Volume residual, 20 000 events	0	5.9×10^{-15}
Dome area, $a = 8$ to 1 cm, largest V	$\rightarrow 0$	9.8×10^{-3} to 2.3×10^{-4}
Cluster dimension D_f	1.8958	1.878 [1.847, 1.907]
Acceptance half-density	$p_c = 0.5927$	0.588
Finger exponent	-0.5714	-0.578 [-0.595, -0.562]
Relief exponent, $H = 0.5$ / 0.8	0.5 / 0.8	0.511 / 0.767
Point source $A \propto V^x$, $H = 0.5$	0.800	0.784 [0.745, 0.820]
Point source $A \propto V^x$, $H = 0.8$	0.714	0.727 [0.691, 0.765]
Pools $V \propto A^x$, $H = 0.5$	1.250	1.304 [1.292, 1.316]
Pools $V \propto A^x$, $H = 0.8$	1.400	1.435 [1.416, 1.455]
Residence time, smallest step	0	+0.4 to +4.0 percent

The synthetic ceilings follow Eq. (3) (Fig. 3b). The fitted exponent is 0.511 for an input of 0.5 and 0.767 for an input of 0.8. The second is reduced because the spectrum is truncated at the lattice scale, which removes the steepest small-scale relief and lowers the apparent exponent most when H is close to one. When every pool of an untilted ceiling is full, gas covers isolated pools of all sizes up to that of the domain (Fig. 3a). The largest pools lie over the long-wavelength highs of the relief, and the smallest occupy local highs only a few cells across. Across these pools, the volume grows with the area as predicted by Eq. (16) (Fig. 3c). The fitted exponents, 1.304 for $H = 0.5$ and 1.435 for $H = 0.8$ from 1939 and 1042 pools of at least ten cells, exceed the closed-form values of 1.25 and 1.40 by 3 to 4 percent, and their intervals exclude them. We attribute the difference to the reduced relief exponent at the lattice scale, which affects the smallest pools most and makes them shallower than the self-affine scaling implies. The slope of the relation nevertheless separates the two Hurst exponents clearly, so the relation between the volume and area of pools is a sensitive measure of the roughness of the ceiling.

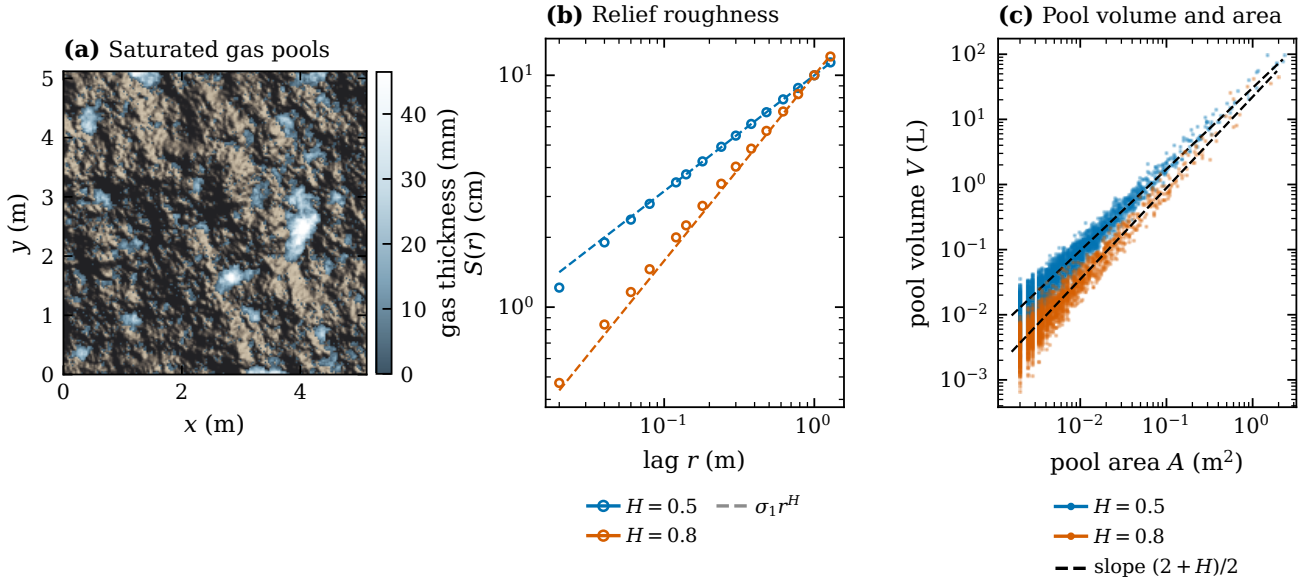


Figure 3: The ceiling and the pools it can hold. (a) Effective ceiling relief (hillshade) and gas thickness of the saturated state, in which every inverted depression is full to its spill point, for $H = 0.8$, $\sigma_1 = 0.10$ m, and no dip. (b) Structure function $S(r)$ for $H = 0.5$ and 0.8, eight realizations each, with $\sigma_1 r^H$ (dashed). (c) Volume against area of every pool with at least five cells in the saturated states, with slope $(2 + H)/2$ (dashed).

In the connected capillary limit, the invaded set reproduces the exact properties of critical percolation (Fig. 4). The number of invaded sites grows with the radius of gyration with an exponent of 1.878, whose interval [1.847, 1.907] contains 91/48. The density of accepted thresholds is uniform below p_c and falls to half its plateau at 0.588, slightly below p_c because each run stops when the cluster first reaches the edge of the lattice, before the invasion has accepted the full share of thresholds just below p_c . The color sequence of Fig. 4a shows how the cluster grows. Long periods of slow, local advance alternate with bursts in which the front runs through a chain of easy sites, the lattice counterpart of the pore-scale jumps observed in slow drainage [19]. On a dipping smooth ceiling, the gas rises in a single finger whose width decreases with the threshold gradient (Fig. 5). The measured exponent, -0.578 with an interval of [-0.595, -0.562], contains $-4/7$, as found in experiments and simulations of gravity-destabilized invasion [3, 20]. The fingers in Fig. 5a show the internal structure of percolation clusters

up to the finger width, with loops and dead ends, and a nearly straight path above it. On a smooth ceiling with even a weak dip, exhaled gas therefore follows a narrow track up-dip, and the track becomes narrower as the dip or the density contrast increases relative to the spread of pinning pressures.

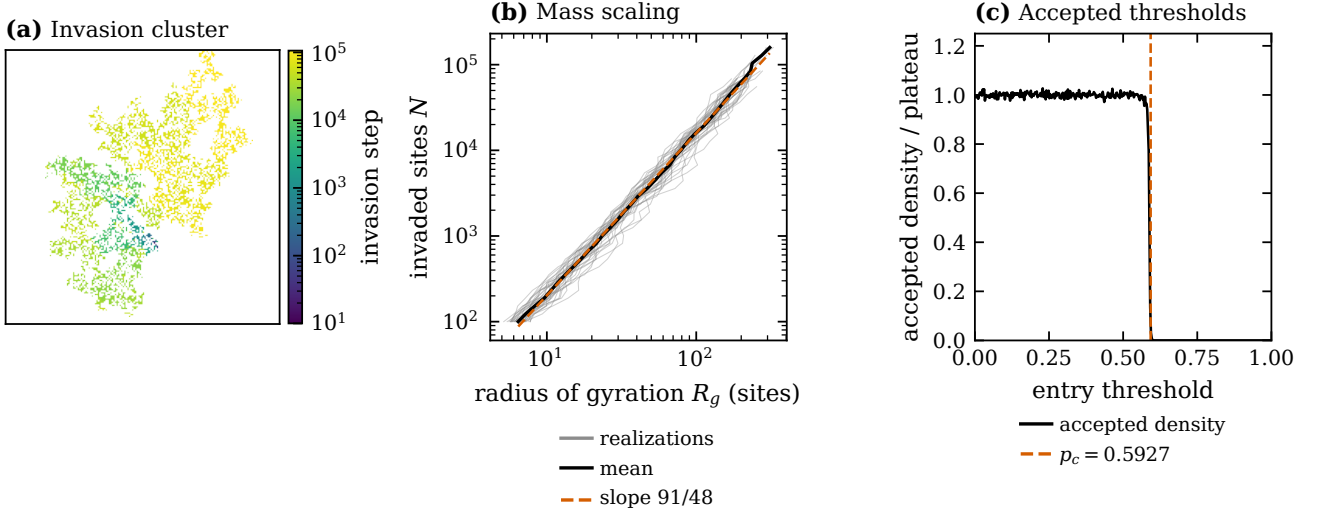


Figure 4: The connected capillary limit. (a) One invasion-percolation cluster on a 1201^2 lattice, colored by invasion step. (b) Invaded sites against radius of gyration during growth, 32 realizations and their mean, with slope 91/48 (dashed). (c) Density of accepted thresholds normalized by its plateau, with p_c (dashed).

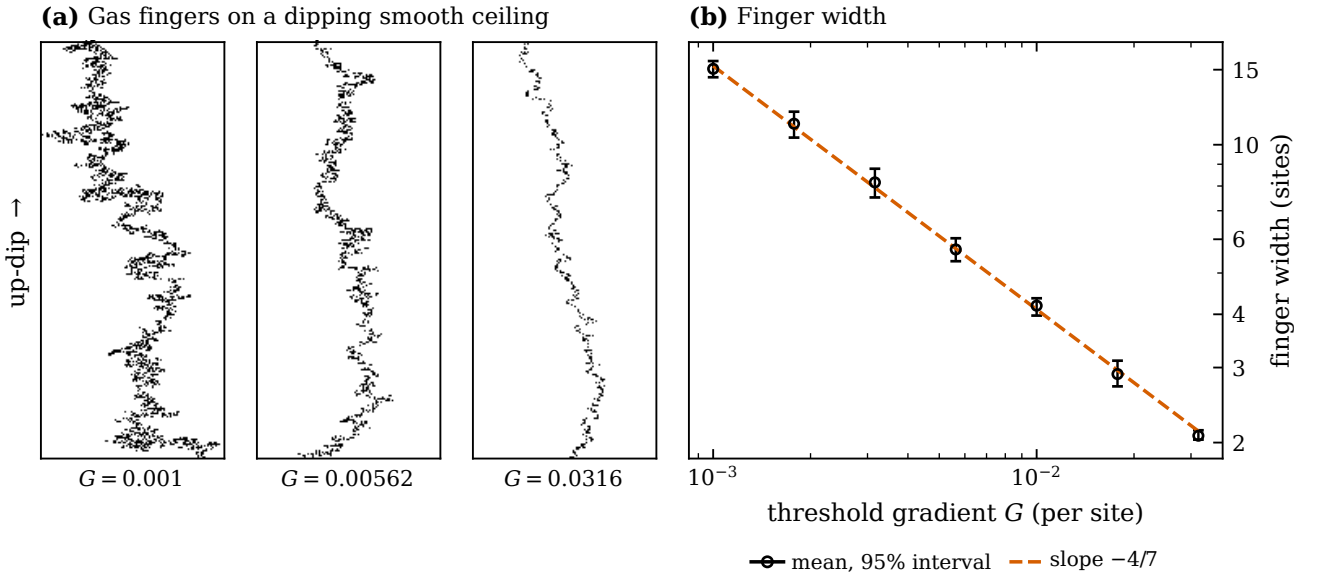


Figure 5: Gas on a dipping smooth ceiling. (a) Invasion fingers for three threshold gradients G , on strips 800 sites wide. (b) Root-mean-square lateral width of the finger against G , means of eight realizations with 95 percent bootstrap intervals, with slope $-4/7$ (dashed).

On rough dipping ceilings, the pool scale r^* organizes the pool sizes but does not cap them (Fig. 6). For $H = 0.5$, the fraction of pooled area held in pools larger than ℓ collapses onto a single curve of ℓ/r^* above about $1.5r^*$, for values of r^* from 0.1 to 0.8 m that correspond to dips from 17.5° to 6.4° . Below $1.5r^*$, the curves separate because pools only a few cells across are affected by the lattice and the pinning noise. For $H = 0.8$, the curves do not collapse within a domain of 10 m. The stretched exponential of Eq. (15) decays with an exponent of 0.4 in this case, so pools of ten times r^* remain likely and the largest pools are limited by the size of the domain rather than by r^* . The same large pool persists in Fig. 6a as the dip increases from 6.0° to 9.0° , shrinking at its margins but not disappearing, because it occupies a long-wavelength high whose relief exceeds the rise of the dip across it. The fraction of the ceiling that can hold gas decreases with the dip (Fig. 6d), from 32.6 to 18.0 percent between 6.4° and 17.5° for $H = 0.5$, and from 21.3 to 14.1 percent between 6.0° and 9.0° for $H = 0.8$. The rougher ceiling at small scale, $H = 0.5$, holds more gas at a given dip because its relief reverses more often over short distances. Because r^* varies with the fifth power of $\sigma_1/(r_0 \tan \beta)$ for $H = 0.8$, a change of a few degrees in

dip changes the pool scale by orders of magnitude, from 25 m at 3° to 0.18 m at 8° for $\sigma_1 = 0.10$ m.

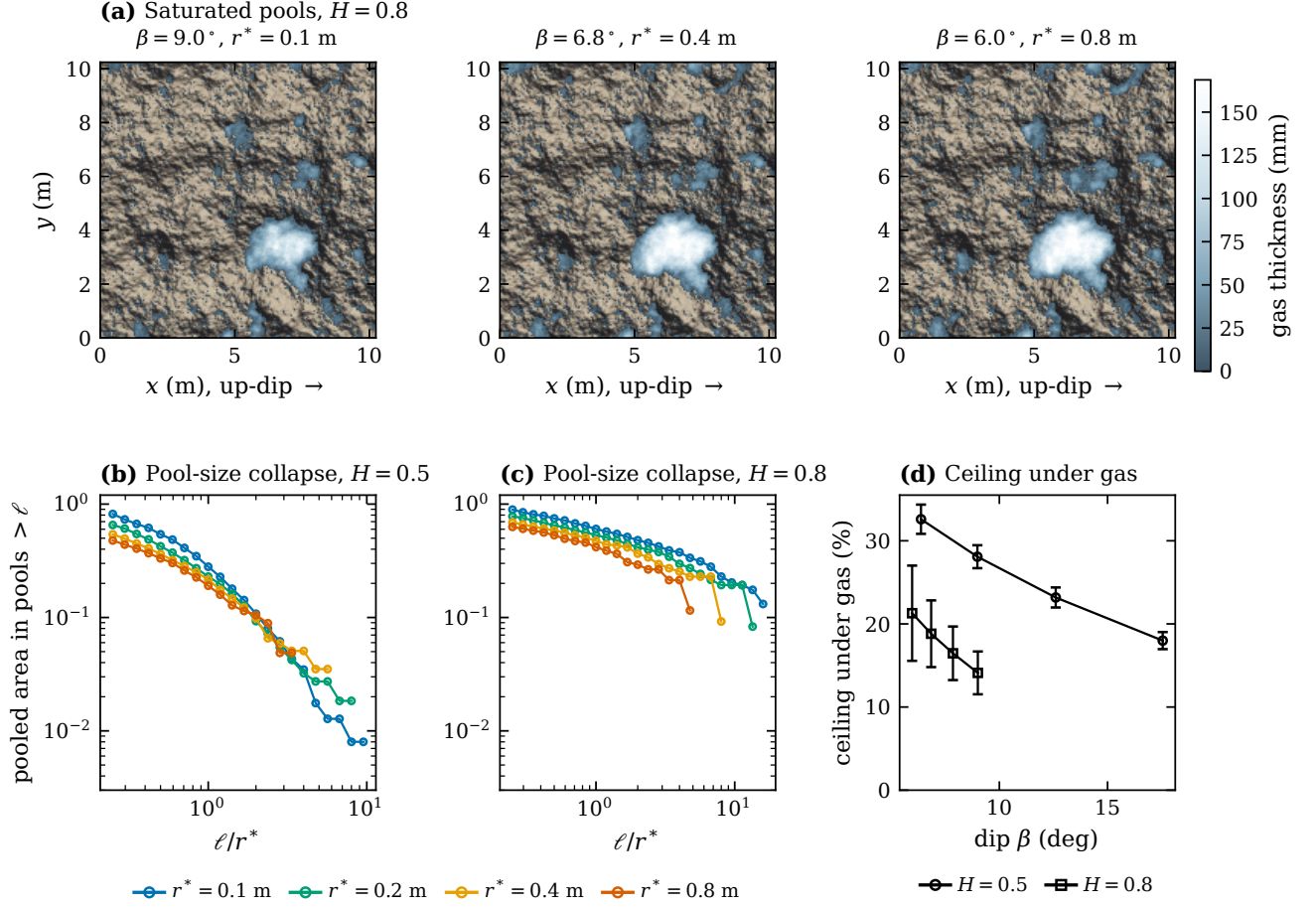


Figure 6: Pools on dipping rough ceilings, 512×512 cells (10.24 m square), $\sigma_1 = 0.10$ m, no arch or pinning, six realizations per dip. (a) Saturated states for $H = 0.8$ at three dips. (b, c) Fraction of the pooled area held in pools of linear size $\sqrt{A}$ larger than ℓ , against ℓ/r^* , for $H = 0.5$ and 0.8 . (d) Fraction of the ceiling that holds gas against dip; bars are one standard deviation.

Gas released at one point covers an area that grows as $V^{0.784}$ for $H = 0.5$ and $V^{0.727}$ for $H = 0.8$ (Fig. 7a). Both intervals contain the exponents of Eq. (16), so the second relation holds even though the first deviates slightly, which suggests that the growth of a single cluster is dominated by its larger pools, where the lattice cutoff matters least. A diver who holds station therefore covers progressively less new ceiling with each breath, and the silt released slows with time (Fig. 7b). Within the first minute the release is nearly the same at every dip, because the first breaths fill the pools directly above the diver. Thereafter the curves separate. After 10 min, the median release within the modeled section is 130 g on a level ceiling, 149 g at 3° , 123 g at 6° , and 63 g at 10° . A small dip increases the release because gas leaving the first pools travels along the ceiling and strips new paths. A larger dip reduces the release within the section because the gas escapes from it, 47 percent at 6° and 94 percent at 10° , and continues to strip silt farther up-dip. The silt does not fall where the diver hovers (Fig. 7c). Its mass-weighted position lies 0.8 m up-dip of the diver on a level ceiling, where the offset arises only from the relief and the arch, and 2.8 to 3.2 m up-dip at dips of 3 to 10° , the last value being limited by the end of the section. The spread among ceilings is large, with interquartile ranges of about one meter, because the position of the nearest closed high up-dip of the diver differs from one ceiling to the next.

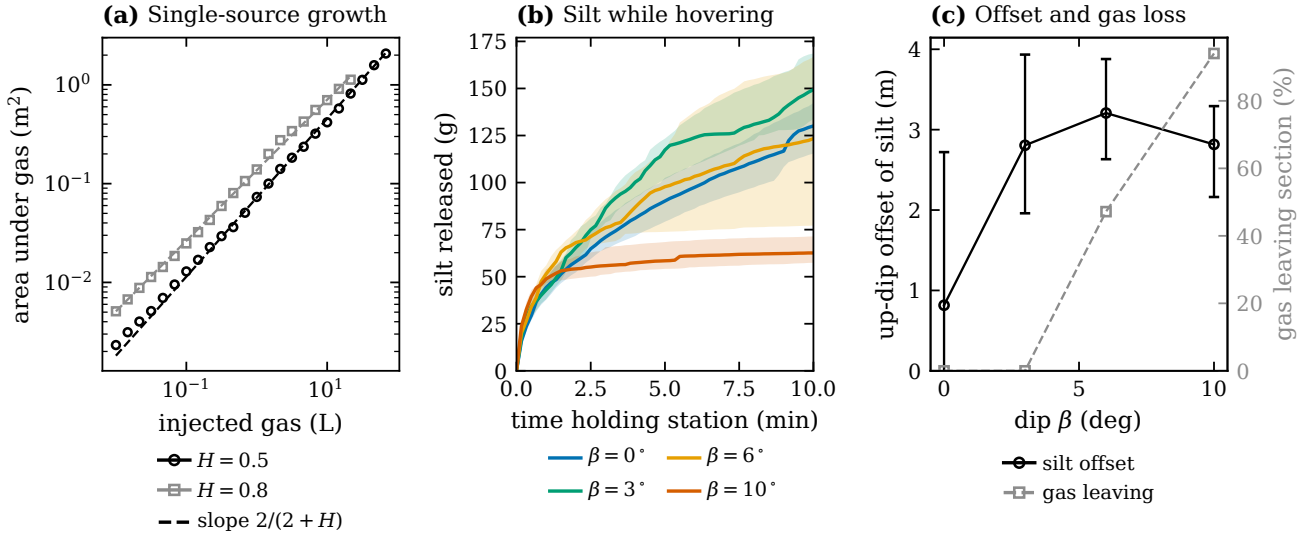


Figure 7: Gas released at one place. (a) Covered area against injected volume for a point source on ceilings with no dip, arch, or pinning, eight realizations per Hurst exponent, with slope $2/(2 + H)$ (dashed). (b) Silt released within the 12.8 m section by one diver (RMV 20 L min^{-1}) holding station for 10 min in the reference passage at four dips, median and interquartile range of twelve ceilings. (c) Mass-weighted up-dip distance of the released silt from the diver (left axis) and fraction of the exhaled gas leaving the section (right axis).

When a team swims through the reference passage, the silt released by each diver depends strongly on position (Fig. 8a). The four divers of a team release 121, 52, 34, and 27 g in order, so that the lead diver releases 2.3 times as much as the second, and each following diver releases less than the one before. Doubling the respiratory minute volume to 40 L min^{-1} increases the release of the lead diver by 19 percent, to 144 g, and of the second diver by 11 percent, and does not change the ordering. The total released by the team grows sublinearly with its size (Fig. 8b), from 122 g for one diver to 172, 210, and 240 g for two, three, and four divers, a ratio of 1.96 between four divers and one. The mechanism is visible in the gas states. The gas of the lead diver fills new pools and opens new migration paths, and the contact lines it moves cross clean rock for the first time. The gas of the following divers arrives at the same places, because they follow the same line, and largely enters pools that are already full and travels paths that have already been stripped. Their contribution comes mainly from the margins of the existing pools and from new paths opened when the added gas makes a pool spill elsewhere. Along the passage, a team in transit releases silt almost uniformly, at about 15 g per meter of passage, whereas a diver who holds station for 5 min concentrates a release of 99 g within about 2 m of the hovering position, with a peak of about 35 g per meter located up-dip of the diver (Fig. 8c). The hovering diver thus releases less silt in total than a team of three in transit, 99 g against 210 g, but deposits it in a section one-sixth as long.

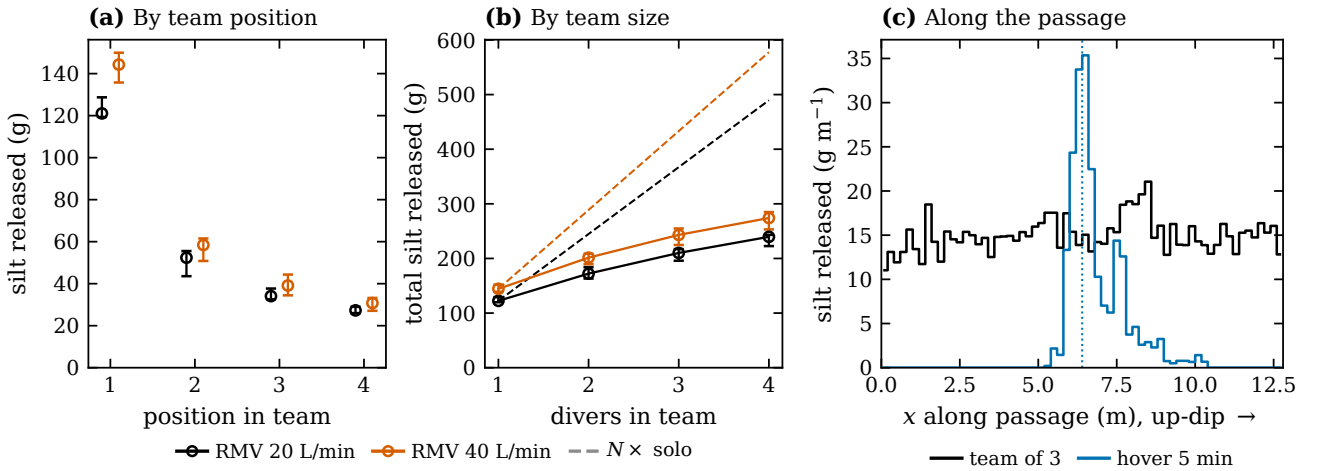


Figure 8: Silt released by a team. Reference passage, twelve ceilings, divers swimming up-dip at 0.2 m s^{-1} , 20 s apart. (a) Silt released by each diver of a team of four, for RMV 20 and 40 L min^{-1} , median and interquartile range. (b) Total silt released by teams of one to four divers, with N times the solo median (dashed). (c) Silt released per meter of passage by a team of three and by one diver holding station for 5 min at $x = 6.4 \text{ m}$, medians.

The ordering by position survives every combination of detachment constants and ceiling geometry that we tested (Table 3). Over contact-line fractions ϕ_c from 0.2 to 0.8 and stripping volumes v_d from 0.2 to 20 mL, the lead diver releases 2.3 to 2.5 times the silt of the second, and a team of four releases 1.86 to 1.92 times the silt of a solo diver. The absolute mass scales almost in proportion to ϕ_c , while v_d has little effect because a single breath of 1.37 L exceeds even the largest value by a factor of about seventy, so that any path crossed by a breath is stripped almost completely. Varying the relief amplitude from 0.05 to 0.20 m, the pinning radius from 2 to 8 mm, removing the arch, or increasing the dip to 6° changes the ratio of lead to second diver to between 2.2 and 2.8 and the ratio of team to solo to between 1.79 and 1.97. The ratio of lead to second diver is largest on the smoothest ceiling, $\sigma_1 = 0.05$ m, where pools are small and the lead diver’s gas reaches and strips a larger fraction of the accessible paths, and it is insensitive to the pinning radius, which changes the effective ceiling by millimeters against a relief of centimeters.

Table 3: Sensitivity of the team results, team of four, medians of six ceilings. Ratios are lead to second diver and team of four to solo diver.

Case	Lead diver (g)	Lead/second	Team/solo
Reference ($\phi_c = 0.5$, $v_d = 2$ mL)	120.7	2.36	1.90
$\phi_c = 0.2$, $v_d = 2$ mL	96.6	2.49	1.86
$\phi_c = 0.8$, $v_d = 2$ mL	148.1	2.35	1.91
$\phi_c = 0.5$, $v_d = 20$ mL	112.9	2.32	1.91
$\sigma_1 = 0.05$ m	127.2	2.80	1.79
$\sigma_1 = 0.20$ m	110.5	2.23	1.97
$r_{50} = 2$ mm	120.6	2.36	1.90
$r_{50} = 8$ mm	121.6	2.36	1.89
No arch	137.4	2.40	1.91
Dip 6°	120.9	2.48	1.88

Silt released by a team of three spreads through the passage over the following hour (Fig. 9). Coarse grains, above about $60 \mu\text{m}$, settle the full 2 m height in about 10 min or less, whereas the fine fraction remains in a layer under the ceiling and descends slowly. Thirty minutes after the team entered, the concentration exceeds 10 mg L^{-1} in the upper 0.3 m and is of order 1 mg L^{-1} at eye level (Fig. 9a). The layer is not uniform along the passage. It is thickest beneath the places where the gas pooled and spilled, and these maxima persist because horizontal mixing over 30 min spreads silt by only a few tens of centimeters at the reference diffusivity. The sighting range in the top 0.1 m below the ceiling stays below 1 m for about 45 min. At eye level, the median sighting range toward the exit, taken at the worst point along the passage, decreases from 27.7 m after 15 min to 23.6 m after 29 min, 15.7 m after 45 min, 10.5 m after 59 min, and 6.9 m after 89 min (Fig. 9b). The decline accelerates between about 30 and 50 min, when grains of about 20 to $30 \mu\text{m}$, whose Stokes fall time over the 1.3 m from ceiling to eye level is 25 to 60 min, which carry a large share of the mass and attenuate efficiently, reach eye level. For a team breathing 40 L min^{-1} , the corresponding values are 27.3, 22.6, 13.3, 9.2, and 6.0 m. After a diver holds station for 5 min, the decrease is slower, to 18.9 m after 59 min and 10.9 m after 89 min, because less silt is released and it is confined to a shorter part of the passage, which a sight line toward the exit crosses for a shorter distance. In every case, visibility at eye level continues to worsen for an hour or more after the divers have passed, which is the period during which a team would typically return along the same passage.

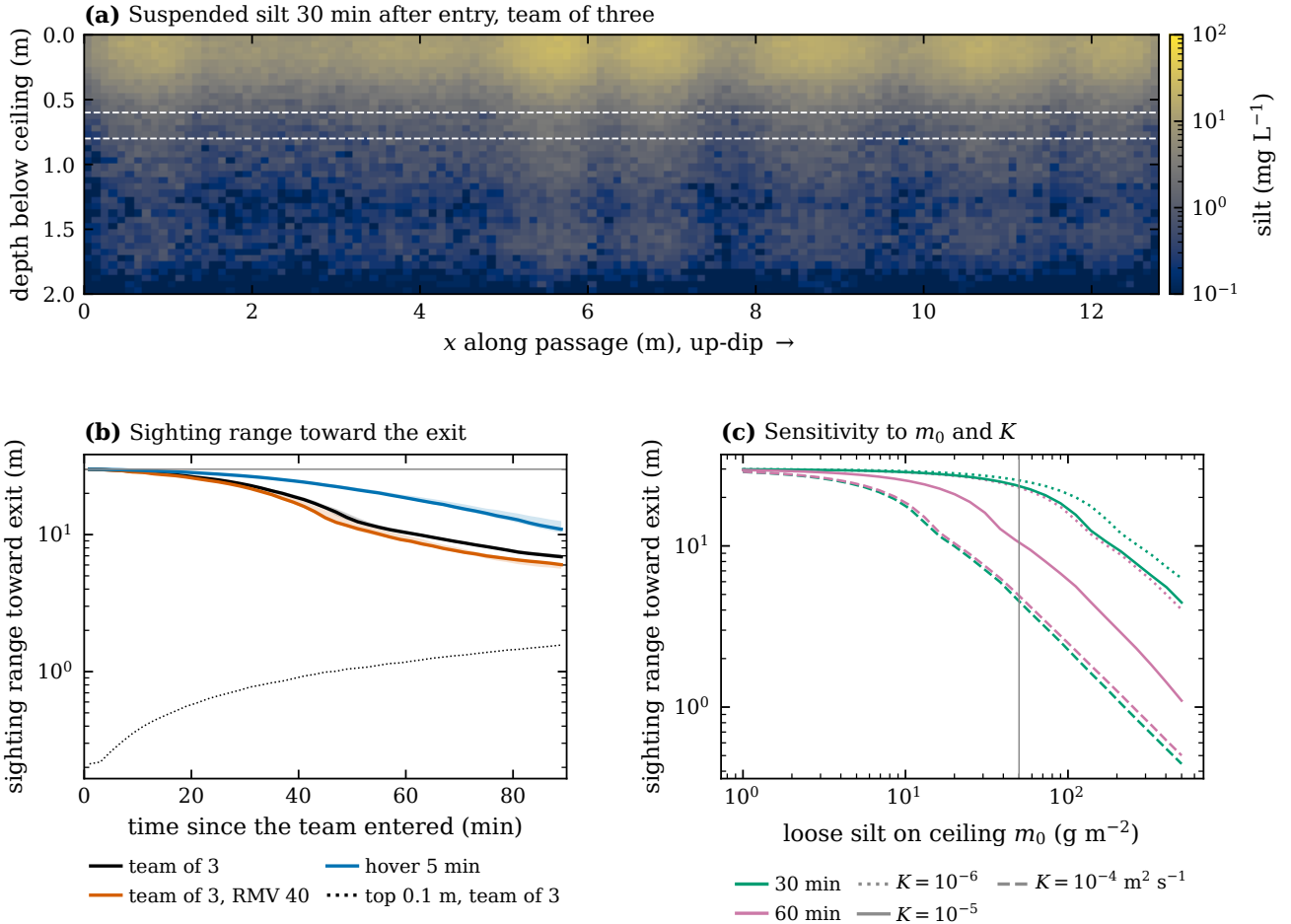


Figure 9: Visibility after the team has passed. Released silt settles as 22 Stokes classes with $K = 10^{-5}$ m² s⁻¹ in the width-averaged reference passage. (a) Suspended silt 30 min after a team of three entered, one ceiling, logarithmic scale; dashed lines bound the eye band. (b) Smallest sighting range toward the exit along the passage in the eye band, median and interquartile range of six ceilings, for three scenarios, and in the top 0.1 m below the ceiling for the team of three (dotted). (c) Smallest eye-band sighting range 30 and 60 min after entry against the loose silt inventory m_0 , for three eddy diffusivities; the vertical line marks the reference inventory.

The sighting ranges depend strongly on two quantities that are not known for caves (Fig. 9c). Because the released mass is proportional to m_0 , the attenuation added by silt scales with it, and at the reference diffusivity the eye-band sighting range after 60 min decreases from 25.5 m for $m_0 = 10$ g m⁻² to 10.6 m at the reference inventory and 3.1 m at 200 g m⁻². The dependence on the eddy diffusivity is equally strong but less intuitive. At the reference inventory, the same quantity is 23.0 m for $K = 10^{-6}$ m² s⁻¹ and 4.9 m for $K = 10^{-4}$ m² s⁻¹. Stronger mixing carries the fine fraction away from the ceiling and down to eye level sooner, and because the finest grains settle more slowly than they mix, it then keeps them distributed through the water column instead of in a layer that the eye band lies beneath. In nearly still water the silt remains concentrated near the ceiling, which is opaque but lies above the eyes of a diver in a passage of this height.

4 Discussion

The results separate two kinds of control on percolation, geometric and operational. The geometric control is the balance between the relief of the ceiling and its dip, which Eq. (14) condenses into one length. On rough ceilings with little dip, gas pools near where it arrives, and the silt it releases falls close to the diver. On steeper ceilings, gas leaves the first pools along migration paths and strips silt meters away from its source, in places that a diver may not associate with the bubbles. Because r^* depends on the dip through the power $1/(1-H)$, which is large when the relief is persistent, the transition between these behaviors can occur over a few degrees. The cutoff is soft, however, and long-wavelength highs continue to hold large pools on ceilings steeper than r^* alone would suggest. In this respect the problem resembles the gravity-modified invasion of porous media, in which a hydrostatic gradient competes with a random distribution of thresholds [27]. The difference is that the thresholds here are themselves correlated over all scales by the self-affine relief, so that the competition is decided scale by scale and yields a power law in the dip rather than a single crossover.

The ceiling also places the problem between two limits of immiscible displacement. Where the relief of a single cell exceeds the pinning height, gas behaves as a gravity-dominated fluid that fills and spills, and its pools fragment

and reconnect in the manner described for buoyancy-driven migration [26]. Where the ceiling is smooth, pinning dominates, gas remains connected, and the advance is invasion percolation in the capillary-fingering regime of the displacement phase diagram [18], with a finger width set by the dip [3, 20]. The ratio of relief at the lattice scale to the pinning height, close to one in the reference case, marks the boundary. Real ceilings may span both regimes, with smooth flowstone or bedding planes adjacent to scalloped or fractured rock. The model treats the two limits separately and does not represent the crossover between them, which would require resolving the meniscus at the scale of the pinning height.

The operational control is the history of the ceiling. The amount of silt released depends far more on whether a ceiling has already been swept than on how much gas reaches it. This is a consequence of the hierarchy of the fill-spill-merge process. The pools and paths available to gas are fixed by the geometry, and the first gas to arrive occupies them. Later gas follows the same routes, because the steepest-ascent paths and spill points do not change, and it crosses rock that has already been stripped. Breathing harder adds gas mainly to pools and paths that already exist, which explains why doubling the respiratory minute volume increases the release of the lead diver by only 19 percent and why the release from a diver who holds station slows with time in the manner set by Eq. (16). The same reasoning implies that the release from a passage is limited by its loose inventory and its accessible area, not by the volume of gas, once the accessible area has been swept.

The timing of the visibility loss follows from the physics of settling and attenuation. The silt that controls visibility is the fine fraction, both because fine grains attenuate light most per unit mass, as Eq. (22) shows and as observed in natural waters [4, 7], and because they remain in suspension longest, as Eq. (21) shows. A team that passes beneath a silty ceiling therefore leaves behind a layer that is opaque near the rock and that descends toward eye level over the following tens of minutes. Visibility on the return may then be worse than on entry even if the passage appeared clear immediately behind the team. The strong dependence on the eddy diffusivity shows that the residual mixing of a passage, including the wake of the divers themselves, matters as much as the amount of silt.

These results are consistent with common practice in cave diving, although they do not test it. Practice discourages lingering under silty ceilings, and in the model a stationary diver concentrates the release in a short section of the passage up-dip of the hovering position. Closed-circuit rebreathers are valued in silty systems, and in the model a diver who exhales no gas releases no silt from the ceiling. The model adds three features that are less obvious from experience. First, the position of a diver in a team affects the silt that diver releases more than the rate at which the diver breathes. Second, on a dipping passage the silt released by a hovering diver falls meters up-dip, so its effect may be met by a team continuing in that direction rather than by the diver who caused it. Third, visibility at eye level continues to decline long after the team has passed. We emphasize that these are properties of the model under the assumptions stated here, and that their relevance to real caves depends on measurements that have not yet been made.

The model has several limitations. The gas is quasi-static. Bubble dynamics, inertia at spill points, the dissolution of trapped gas, and receding contact lines are not represented, and the volume of the pinned meniscus is neglected. Contact-angle hysteresis is represented only through a pinning height for advancing lines [17]. Pools are disconnected fill-spill pools, the connected capillary limit is computed separately, and the crossover between the two is not modeled. Silt detached under a pool is released to the water at once, although some may remain on the interface or fall onto the rock beneath the pool. The detachment efficiencies are not derived from particle-scale forces but represent the outcome of competing capillary and adhesive forces whose magnitude depends on particle shape and surface chemistry [22, 1]. The water column is averaged across the width of the passage, which reduces local peaks of concentration and therefore overestimates the sighting range near the centerline. The eddy diffusivity is constant, whereas mixing in a passage after a team has passed is likely to decay with time, a situation for which random-walk models require additional care [25]. Settling follows the Stokes law, which overestimates the velocity of the coarsest class [9], and flocculation, which can increase the settling velocity of fine particles by orders of magnitude in saline or organic-rich water [29], is not included. Floc formation would shorten the persistence of the fine layer in anchialine or marine caves and may make the reference results conservative there. Finally, the loose silt inventory, the detachment constants, the pinning radius, the statistics of the relief, the grain sizes, and the eddy diffusivity are illustrative, and no measurement of them on a cave ceiling is known to us. The absolute silt masses scale with m_0 , and the sighting ranges change by an order of magnitude between $K = 10^{-6}$ and $10^{-4} \text{ m}^2 \text{ s}^{-1}$.

Several of the results are, in contrast, independent of these choices. The agreement of the event-driven computation with priority flooding, the conservation of volume, the percolation limits, the pool scale, and the area-volume exponents are properties of the geometry. The ordering of divers by position held for every combination of detachment constants and ceiling geometry tested, because it follows from the reuse of a fixed hierarchy of pools and paths rather than from the values of the constants. These findings suggest measurements that would test the model. Laboratory experiments with gas injected beneath inverted rough plates of known self-affine relief could test the pool scale and the area-volume law directly. Repeated photographs of a silty cave ceiling before and after the passage of divers could measure stripping efficiencies. Transmissometer profiles taken at fixed points in a passage after a team has passed could test the predicted timing of visibility loss and constrain the eddy diffusivity.

5 Conclusions

We developed an idealized model of percolation, the release of silt by exhaled gas beneath the ceilings of submerged caves. The model combines an exact fill-spill-merge description of buoyant gas on a self-affine rough ceiling with capillary pinning of the contact line, detachment of silt by moving contact lines, Lagrangian settling of a spectrum of grain sizes, and the beam-attenuation criterion for the sighting range of a diver. The computation agrees exactly with an independent depression-filling algorithm, conserves gas volume to round-off, and recovers the known exponents of invasion percolation and of gravity-destabilized invasion in its capillary limit. A balance between relief and dip defines a pool scale that organizes where gas collects, and the area that gas from a single source covers grows more slowly than the volume supplied, in agreement with the self-affine geometry. The cutoff at the pool scale is soft, so large pools persist on ceilings steeper than the scale alone suggests. The silt released by a team depends mainly on the history of the ceiling. The lead diver releases far more than the diver behind, later divers add progressively less, and breathing harder adds comparatively little, because the gas of later divers reuses pools and paths that the lead diver has already filled and stripped. A diver who holds station concentrates the release in a short section up-dip of the hovering position. Visibility at eye level toward the exit continues to decline for an hour or more after a team has passed, as fine silt descends from a nearly opaque layer beneath the ceiling. These conclusions hold across the detachment efficiencies and ceiling geometries tested, whereas absolute silt masses and sighting ranges depend on the loose silt inventory and on mixing in still cave water. Measurements of the loose silt on cave ceilings, of the efficiency with which moving contact lines remove it, and of mixing after the passage of divers would allow these comparisons to be converted into predictions for particular passages.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this article.

Declaration of generative AI use

During the preparation of this work, the authors used Claude Sonnet 5 (Anthropic) solely to refine the grammar and vocabulary of the manuscript. The authors reviewed and edited the resulting text and take full responsibility for the conceptualization, model development, mathematical derivations, computational results, and their interpretation.

Code and data availability

This study uses no external data. The analysis code and all of its outputs, including CSV tables of the data behind each figure panel, the figures in vector and raster form, plain-text computation reports, animations, and the scripts that regenerate them, are openly available at https://github.com/sandyherho/cave_percolation under the MIT License.

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References

- [1] Aramrak, S.; Flury, M.; Harsh, J. B.; Zollars, R. L.; Davis, H. P. Does Colloid Shape Affect Detachment of Colloids by a Moving Air–Water Interface? *Langmuir* **2013**, *29*(19), 5770–5780. <https://doi.org/10.1021/la400252q>
- [2] Barnes, R.; Callaghan, K. L.; Wickert, A. D. Computing water flow through complex landscapes, Part 3: Fill-Spill-Merge: flow routing in depression hierarchies. *Earth Surf. Dynam.* **2021**, *9*(1), 105. <https://doi.org/10.5194/esurf-9-105-2021>
- [3] Birovljev, A.; Furuberg, L.; Feder, J.; Jøssang, T.; Mly, K. J.; Aharony, A. Gravity invasion percolation in two dimensions: Experiment and simulation. *Phys. Rev. Lett.* **1991**, *67*(5), 584–587. <https://doi.org/10.1103/PhysRevLett.67.584>
- [4] Boss, E.; Taylor, L.; Gilbert, S.; Gundersen, K.; Hawley, N.; Janzen, C.; Johengen, T.; Purcell, H.; Robertson, C.; Schar, D. W. H.; Smith, G. J.; Tamburri, M. N. Comparison of inherent optical properties as a surrogate for particulate matter concentration in coastal waters. *Limnol. Oceanogr. Methods* **2009**, *7*, 803–810. <https://doi.org/10.4319/lom.2009.7.803>
- [5] Buzzacott, P. L.; Zeigler, E.; Denoble, P.; Vann, R. American Cave Diving Fatalities 1969–2007. *Int. J. Aquat. Res. Educ.* **2009**, *3*(2), 162–177. <https://doi.org/10.25035/ijare.03.02.07>
- [6] Candela, T.; Renard, F.; Klinger, Y.; Mair, K.; Schmittbuhl, J.; Brodsky, E. E. Roughness of fault surfaces over nine decades of length scales. *J. Geophys. Res. Solid Earth* **2012**, *117*, B08409. <https://doi.org/10.1029/2011JB009041>
- [7] Davies-Colley, R. J.; Smith, D. G. TURBIDITY SUSPENDED SEDIMENT, AND WATER CLAR-

- ITY: A REVIEW. *J. Am. Water Resour. Assoc.* **2001**, *37*(5), 1085–1101. <https://doi.org/10.1111/j.1752-1688.2001.tb03624.x>
- [8] de Gennes, P. G. Wetting: statics and dynamics. *Rev. Mod. Phys.* **1985**, *57*(3), 827–863. <https://doi.org/10.1103/RevModPhys.57.827>
- [9] Ferguson, R. I.; Church, M. A Simple Universal Equation for Grain Settling Velocity. *J. Sediment. Res.* **2004**, *74*(6), 933–937. <https://doi.org/10.1306/051204740933>
- [10] Ford, D.; Williams, P. *Karst Hydrogeology and Geomorphology*; Wiley, **2007**. <https://doi.org/10.1002/9781118684986>
- [11] Harris, C. R.; Millman, K. J.; van der Walt, S. J.; Gommers, R.; Virtanen, P.; Cournapeau, D.; Wieser, E.; Taylor, J.; Berg, S.; Smith, N. J.; Kern, R.; Picus, M.; Hoyer, S.; van Kerkwijk, M. H.; Brett, M.; Haldane, A.; del Río, J. F.; Wiebe, M.; Peterson, P.; Gérard-Marchant, P.; Sheppard, K.; Reddy, T.; Weckesser, W.; Abbasi, H.; Gohlke, C.; Oliphant, T. E. Array programming with NumPy. *Nature* **2020**, *585*, 357–362. <https://doi.org/10.1038/s41586-020-2649-2>
- [12] Herho, S. H. S.; Anwar, I. P.; Khadami, F.; Ndruru, T. R. E. B. N.; Suwarman, R.; Irawan, D. E. WAVE-ATTENUATION-1D: AN IDEALIZED ONE-DIMENSIONAL FRAMEWORK FOR WAVE ATTENUATION THROUGH COASTAL VEGETATION USING NUMBA-ACCELERATED SHALLOW WATER EQUATIONS. *J. Theor. Appl. Mech.* **2026**, *56*, 89–102. <https://doi.org/10.55787/jtams.2026.1.AI00236>
- [13] Herho, S. H. S.; Trilaksono, N. J.; Fajary, F. R.; Napitupulu, G.; Anwar, I. P.; Khadami, F.; Irawan, D. E. kh2d-solver: A Python library for idealized two-dimensional incompressible Kelvin-Helmholtz instability. *Appl. Comput. Mech.* **2025**, *19*(2), 125–156. <https://doi.org/10.24132/acm.2025.1040>
- [14] Hunter, J. D. Matplotlib: A 2D graphics environment. *Comput. Sci. Eng.* **2007**, *9*(3), 90–95. <https://doi.org/10.1109/MCSE.2007.55>
- [15] Irawan, D. E.; Herho, S. H. S.; Anwar, I. P.; Khadami, F.; Pamumpuni, A.; Kartiko, R. D.; Riawan, E.; Suwarman, R.; Puradimaja, D. J. amerta: A Python library for idealized 1D Saint-Venant dam-break simulation. *Front. Water* **2026**, *8*, 1900409. <https://doi.org/10.3389/frwa.2026.1900409>
- [16] Irawan, D. E.; Herho, S. H. S.; Pamumpuni, A.; Kartiko, R. D.; Khadami, F.; Anwar, I. P.; Sujatmiko, K. A.; Handayani, A. P.; Fajary, F. R.; Suwarman, R. An Open-Source Pseudo-Spectral Solver for Idealized Korteweg-de Vries Soliton Simulations. *Water* **2026**, *18*(7), 779. <https://doi.org/10.3390/w18070779>
- [17] Joanny, J. F.; de Gennes, P. G. A model for contact angle hysteresis. *J. Chem. Phys.* **1984**, *81*(1), 552–562. <https://doi.org/10.1063/1.447337>
- [18] Lenormand, R.; Touboul, E.; Zarcone, C. Numerical models and experiments on immiscible displacements in porous media. *J. Fluid Mech.* **1988**, *189*, 165–187. <https://doi.org/10.1017/S0022112088000953>
- [19] Måløy, K. J.; Furuberg, L.; Feder, J.; Jøssang, T. Dynamics of slow drainage in porous media. *Phys. Rev. Lett.* **1992**, *68*(14), 2161–2164. <https://doi.org/10.1103/PhysRevLett.68.2161>
- [20] Meakin, P.; Feder, J.; Frette, V.; Jøssang, T. Invasion percolation in a destabilizing gradient. *Phys. Rev. A* **1992**, *46*, 3357. <https://doi.org/10.1103/PhysRevA.46.3357>
- [21] Newman, M. E. J.; Ziff, R. M. Efficient Monte Carlo Algorithm and High-Precision Results for Percolation. *Phys. Rev. Lett.* **2000**, *85*(19), 4104–4107. <https://doi.org/10.1103/PhysRevLett.85.4104>
- [22] Sharma, P.; Flury, M.; Zhou, J. Detachment of colloids from a solid surface by a moving air-water interface. *J. Colloid Interface Sci.* **2008**, *326*(1), 143–150. <https://doi.org/10.1016/j.jcis.2008.07.030>
- [23] Spence, C.; Woo, M.-k. Hydrology of subarctic Canadian shield: soil-filled valleys. *J. Hydrol.* **2003**, *279*(1–4), 151–166. [https://doi.org/10.1016/S0022-1694\(03\)00175-6](https://doi.org/10.1016/S0022-1694(03)00175-6)
- [24] Virtanen, P.; Gommers, R.; Oliphant, T. E.; Haberland, M.; Reddy, T.; Cournapeau, D.; Burovski, E.; Peterson, P.; Weckesser, W.; Bright, J.; van der Walt, S. J.; Brett, M.; Wilson, J.; Millman, K. J.; Mayorov, N.; Nelson, A. R. J.; Jones, E.; Kern, R.; Larson, E.; Carey, C. J.; Polat, İ.; Feng, Y.; Moore, E. W.; VanderPlas, J.; Laxalde, D.; Perktold, J.; Cimrman, R.; Henriksen, I.; Quintero, E. A.; Harris, C. R.; Archibald, A. M.; Ribeiro, A. H.; Pedregosa, F.; van Mulbregt, P.; SciPy 1.0 Contributors. SciPy 1.0: fundamental algorithms for scientific computing in Python. *Nat. Methods* **2020**, *17*, 261–272. <https://doi.org/10.1038/s41592-019-0686-2>
- [25] Visser, A. W. Using random walk models to simulate the vertical distribution of particles in a turbulent water column. *Mar. Ecol. Prog. Ser.* **1997**, *158*, 275–281. <https://doi.org/10.3354/meps158275>
- [26] Wagner, G.; Meakin, P.; Feder, J.; Jøssang, T. Buoyancy-driven invasion percolation with migration and fragmentation. *Physica A* **1997**, *245*(3–4), 217–230. [https://doi.org/10.1016/S0378-4371\(97\)00324-5](https://doi.org/10.1016/S0378-4371(97)00324-5)
- [27] Wilkinson, D. Percolation model of immiscible displacement in the presence of buoyancy forces. *Phys. Rev. A* **1984**, *30*(1), 520–531. <https://doi.org/10.1103/PhysRevA.30.520>
- [28] Wilkinson, D.; Willemsen, J. F. Invasion percolation: a new form of percolation theory. *J. Phys. A: Math. Gen.* **1983**, *16*, 3365. <https://doi.org/10.1088/0305-4470/16/14/028>
- [29] Winterwerp, J. C. On the flocculation and settling velocity of estuarine mud. *Cont. Shelf Res.* **2002**, *22*(9), 1339–1360. [https://doi.org/10.1016/S0278-4343\(02\)00010-9](https://doi.org/10.1016/S0278-4343(02)00010-9)
- [30] Zaneveld, J. R. V.; Pegau, W. S. Robust underwater visibility parameter. *Opt. Express* **2003**, *11*(23), 2997–3009. <https://doi.org/10.1364/OE.11.002997>