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An Idealized Model of Matrix Storage and the Tidal Response of Coastal Aquifers

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Abstract

Tides entering a coastal aquifer weaken and lag the sea as they travel inland, and classical theory requires the two rates to be equal. Field records often show the amplitude decaying faster than the phase lag grows, which is usually attributed to geometry or leakage. We examine a third mechanism, the exchange of water between a conduit network and the matrix blocks it surrounds, and ask when this exchange makes the aquifer behave as a fractional medium in time. The study is purely theoretical. For each tidal constituent, the measured decay and lag are exactly equivalent to a complex storage capacity at that frequency, so a single constituent cannot separate memory from leakage. Passive storage exchange and leakage can never make the lag rate exceed the decay rate, and diffusive exchange with blocks of any sizes cannot reduce their ratio below a fixed bound close to two fifths. A power law distribution of block sizes produces a fractional medium over a band of frequencies, with an order between one half and one set by the exponent of the distribution. In the nonlinear unconfined aquifer, the period mean of the squared head is conserved with or without memory, and the mean rise of the water table depends on the decay rate alone. For a karstic reference aquifer, one month of four tidal constituents reliably separates memory from leakage even when the matrix stores a fifth as much water as the conduits. All closed forms are verified numerically.

Keywords: Coastal aquifer; Dual porosity; Fractional diffusion; Matrix diffusion; Tidal method

1 Introduction

Coastal aquifers are an important source of water on many low-lying coasts and islands, and they are sensitive to both pumping and changes in sea level [13]. Their response to the sea controls the discharge of fresh groundwater to the coast [32] and the landward movement of seawater [48]. Ocean tides drive periodic head fluctuations in these aquifers, and the attenuation and phase lag of the fluctuations with distance from the shore are a standard, low-cost means of estimating aquifer diffusivity without pumping [44, 33]. The method requires only pairs of head records, one at the coast and one or more inland, and the harmonic analysis of each record into its tidal constituents [35]. It has therefore been applied widely to confined and unconfined aquifers, to islands, and to estuarine settings in which the forcing arrives from more than one boundary [25, 39, 45].

In the classical one-dimensional solution for a confined, homogeneous aquifer, the log-amplitude of each constituent decays inland at a rate a and its phase lag grows at a rate b , and the two rates are equal, so that either one returns the same diffusivity. Field records frequently depart from this equality. Diffusivities estimated from attenuation and from time lag often disagree, and the ratio of the two estimates has been analyzed as a slope factor in aquifers of variable thickness [9]. On a barrier island, tidal amplitudes were observed to decay faster than the phase lag increased, whereas storm surges of several days showed similar rates for both [20]. Diffusivities estimated from different constituents at the same wells can also differ [45], and oscillatory pumping tests have returned hydraulic properties that depend on the period of the forcing [37]. Leakage through confining layers [25], tidal forcing from more than one coast [39], wedge-shaped geometry [9], and the finite depth of unconfined aquifers [20] have all been proposed to explain such departures. Each of these explanations changes the boundary conditions or the geometry of the flow while keeping the storage of the aquifer instantaneous.

Fractured and karstic aquifers add a mechanism of a different kind, one that acts through storage rather than geometry. Karst aquifers supply water to many regions and are difficult to characterize, because their hydraulic behavior is dominated by a small volume of highly permeable conduits embedded in a much larger, less permeable matrix [17]. In these media most of the transmissivity resides in a network of fractures or conduits, while much of the storage resides in the matrix blocks between them. A change in conduit head is transmitted into the matrix by slow diffusion, and the water that the matrix takes up or releases lags behind the head that drives it. The response of the conduit head therefore depends on its own history. The first dual-porosity models represent this exchange by a first-order rate law [3, 47, 27], later models resolve diffusion within blocks of finite size [31], and homogenization theory has shown that the double-porosity structure is the correct limit of flow in a medium of

small blocks surrounded by a connected, permeable network [1]. Hydraulic tests in fractured rock are commonly interpreted with these models or with a generalized flow dimension [4], but the tidal method has generally been applied with a single storage coefficient.

Memory of this kind is closely tied to fractional calculus. In solute transport, a distribution of mass-transfer rates between mobile and immobile domains [14] or a power-law memory function [40] leads to fractional-order time derivatives, and fractional models have become widely used to describe anomalous transport in geological media [6, 7, 30]. Field applications of such models distinguish nonlocality in time, which represents trapping and release, from nonlocality in space, which represents fast paths [49], and fractional calculus has been applied to a growing range of problems in science and engineering [42]. Multirate mass transfer, continuous time random walks, and fractional derivatives have been shown to be closely related representations of the same physics [11]. Time-fractional versions of the groundwater flow equation have also been proposed [2], and fractional derivatives have been applied to pressure transients in fractured reservoirs [38]. The Caputo derivative, introduced to represent dissipation that is nearly independent of frequency [8], is the usual choice because it accepts conventional initial conditions [12, 29]. In most hydrological applications, however, the fractional order is a parameter fitted to data, its physical origin is left open, and its range is not constrained, a practice that has drawn criticism [34].

The tidal response of a dual-porosity aquifer brings these two lines of work together. A time-fractional diffusion equation of order γ gives a constant ratio $b/a = \tan(\gamma\pi/4)$, so a fractional aquifer would violate the classical equality in a specific, frequency-independent way. Matrix exchange, leakage, and fractional dynamics all lower the ratio, and a field analysis that fits one of them may achieve close agreement without having identified the mechanism. It is not known which storage structures produce fractional behavior in the tidal response, over what range of frequencies, whether the resulting ratio is bounded, or whether a tidal record of realistic length and noise can distinguish storage memory from leakage. Nor is it known whether memory leaves a trace in the nonlinear response of unconfined aquifers, where the mean water table rises above mean sea level by an amount that depends on the tidal amplitude and the aquifer properties [33, 36].

These questions are directly relevant to the coastal aquifers of the Indonesian Maritime Continent, many of which lie in uplifted reef limestones and other karstic rocks in which conduits and matrix blocks coexist. Tidal records from wells in such aquifers are relatively inexpensive to obtain and are often among the few hydraulic data available. If the departures of these records from the classical equality can be traced to the storage structure of the rock, the same records could constrain the matrix storage and the size of the blocks, quantities that control how much fresh water an aquifer holds and how quickly it responds to pumping or to sea level rise.

This study addresses these questions with an idealized model. We derive the tidal response of a confined aquifer whose conduits exchange water diffusively with matrix blocks, and we show that the attenuation and phase lag of each constituent are exactly equivalent to a complex storage capacity. We establish bounds on the ratio of phase lag to attenuation for passive storage exchange and for diffusive exchange with blocks of arbitrary sizes. We identify the conditions under which the aquifer behaves as a time-fractional medium and relate the fractional order to the distribution of block sizes. We derive the switch-on transient of a fractional medium in closed form, extend the analysis to the nonlinear unconfined aquifer, and quantify how much information a tidal record carries about the memory. No observational data are used. Every quantity reported follows from the model equations, and each closed form is checked against time-domain solvers that share no code with the frequency-domain theory.

2 Methods

2.1 Model description

The configuration is shown in Fig. 1. A confined aquifer of transmissivity T occupies $0 < x < L$ between two aquitards, with x measured inland from a vertical coastline. Its transmissivity resides in a conduit network of storativity S_m , and the network surrounds matrix blocks of storativity S_{im} per unit aquifer area, hydraulic diffusivity D_{im} , and half-thickness ℓ . The sea imposes the head $h(0, t) = A \cos \omega t$, and the landward boundary at $x = L$ carries no flux. Heads are measured from mean sea level. Throughout, the symbol $\equiv$ introduces a definition and $=$ denotes an equality that is derived or imposed as a model relation.

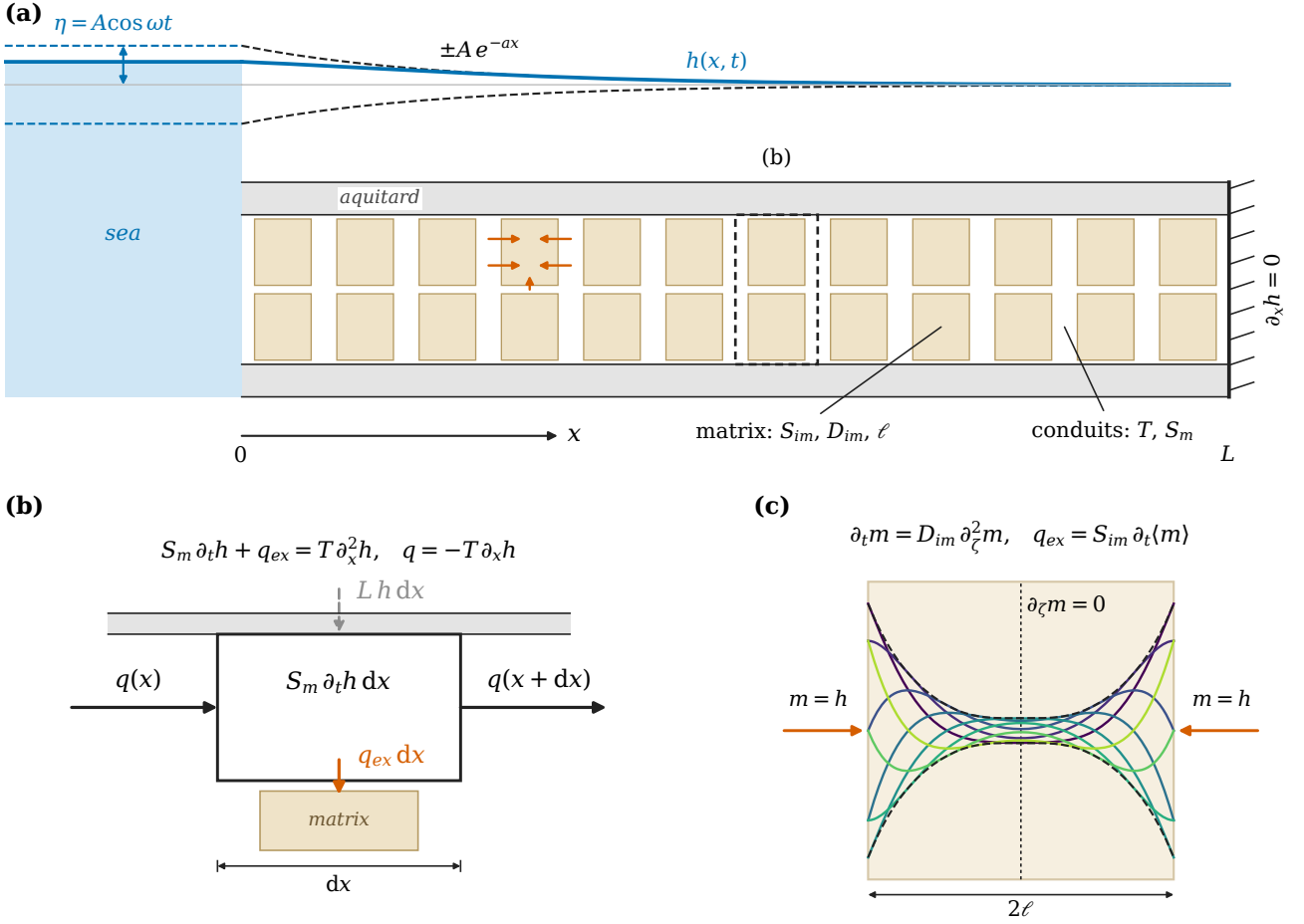


Figure 1: Model geometry and balances. (a) Idealized cross-section: the sea imposes the head $A \cos \omega t$ at $x = 0$, a confined layer carries a conduit network of transmissivity T and storativity S_m around matrix blocks of storativity S_{im} , diffusivity D_{im} , and half-thickness ℓ , and no flux crosses $x = L$. The dashed curves are the envelope $\pm A e^{-ax}$ and the solid curve the head at one instant, both computed for the M2 constituent of the reference aquifer. The dashed box is the control volume of (b). (b) Control-volume diagram of a slice of width dx : Darcy fluxes at both faces, storage in the conduits, and exchange with the matrix. The aquitard leakage $L_a h dx$ (gray, dashed) belongs to the comparison model only. (c) One matrix block: the faces follow the conduit head, the central plane carries no flux, and the curves are the computed head profiles at eight phases of the M2 cycle.

The model rests on seven assumptions, each chosen so that storage exchange is isolated from the other mechanisms known to alter tidal propagation. First, flow in the conduit network is horizontal and is described by the head averaged over the thickness b of the layer. This Dupuit approximation holds when b is small compared with the length over which the tidal signal decays, $1/a$ [44]. For the reference aquifer introduced below, $1/a$ exceeds 1.2 km for every constituent, whereas confined coastal layers are typically tens of meters thick. Second, the conduits and the matrix are treated as two overlapping continua, each with averaged properties. Homogenization shows that this double-porosity structure is the correct limit when the blocks are small compared with the scale over which the conduit head varies [1], and here $\ell a \leq 4 \times 10^{-4}$. Third, water moves within a block by linear diffusion normal to its faces, and the head is continuous across each face. A low-permeability coating on the blocks would add a resistance at the face [31], which is not included. Fourth, storage is elastic and linear, so that T , S_m , S_{im} , and D_{im} do not depend on the head. Tidal head changes of a few decimeters are small compared with the effective stress in a confined layer, which justifies this approximation. Fifth, the coastline is vertical, the head at $x = 0$ equals the sea level, and no outlet resistance, seepage face, or inland tidal loading is present. These boundary effects modify the amplitude and phase near the shore [39, 9] and would confound the comparison. Earth tides and barometric fluctuations, which add forcing at tidal frequencies through the response of the well and the formation [21], are likewise excluded. Sixth, density differences between seawater and groundwater are neglected, and heads are expressed as equivalent freshwater heads [48]. The pressure signal propagates through the whole layer, whereas density effects are confined to the saltwater wedge near the coast. Seventh, the forcing has acted long enough for the response to be periodic, apart from the switch-on transient examined separately below.

Under the first assumption, Darcy's law integrated over the layer gives the discharge per unit width $q = -T \partial h / \partial x$. Within a block, the head $m(\zeta, x, t)$ varies with the coordinate ζ normal to the faces, measured from

the central plane. Conservation of mass and Darcy's law in the block give the diffusion equation

$$\frac{\partial m}{\partial t} = D_{im} \frac{\partial^2 m}{\partial \zeta^2}, \quad m(\pm \ell, x, t) = h(x, t), \quad \frac{\partial m}{\partial \zeta}(0, x, t) = 0, \quad (1)$$

where the face condition follows from the third assumption and the central condition from symmetry. Integrating Eq. (1) across the block shows that the rate of change of the block-averaged head $\langle m \rangle \equiv \ell^{-1} \int_0^\ell m d\zeta$ equals the flux through the faces, $\partial_t \langle m \rangle = (D_{im}/\ell) \partial_\zeta m|_{\zeta=\ell}$. The volume of water that leaves the conduits and enters the matrix, per unit aquifer area and unit time, is therefore

$$q_{ex} = S_{im} \frac{\partial \langle m \rangle}{\partial t}. \quad (2)$$

A water balance on a slice of width dx (Fig. 1b) equates the net inflow $q(x) - q(x+dx) = -\partial_x q dx$ to the rate of storage in the conduits, $S_m \partial_t h dx$, plus the exchange, $q_{ex} dx$. With Darcy's law,

$$S_m \frac{\partial h}{\partial t} + q_{ex} = T \frac{\partial^2 h}{\partial x^2}. \quad (3)$$

Equations (1) to (3) form the diffusive double-porosity model [31]. In first-order models, the block is replaced by a single storage element that relaxes toward h at a fixed rate [3, 47].

For a constituent of angular frequency ω , we write $h = \text{Re}[H(x) e^{i\omega t}]$ and $m = \text{Re}[M(\zeta, x) e^{i\omega t}]$. Equation (1) becomes $M'' = k^2 M$ with $k^2 = i\omega/D_{im}$, whose solution that is even in ζ and equals H at the faces is $M = H \cosh(k\zeta)/\cosh(k\ell)$. Its block average is $\langle M \rangle = H \tanh(k\ell)/(k\ell)$. We define the matrix diffusion time and the dimensionless variables

$$\tau \equiv \frac{\ell^2}{D_{im}}, \quad \Omega \equiv \omega\tau, \quad z \equiv \sqrt{i\Omega}, \quad \beta \equiv \frac{S_{im}}{S_m}, \quad x_* \equiv \sqrt{\frac{T\tau}{S_m}}, \quad (4)$$

so that $k\ell = z$, and the ratio of the block-averaged head to the conduit head is the matrix transfer function

$$G(\Omega) \equiv \frac{\langle M \rangle}{H} = \frac{\tanh z}{z}. \quad (5)$$

The partial-fraction expansion $\tanh z = \sum_{n \geq 0} 2z/(z^2 + \mu_n)$ with $\mu_n \equiv (n + \frac{1}{2})^2 \pi^2$ gives

$$G(\Omega) = \sum_{n=0}^{\infty} \frac{w_n}{1 + i\Omega/\mu_n}, \quad w_n = \frac{2}{\mu_n} > 0, \quad \sum_{n=0}^{\infty} w_n = 1. \quad (6)$$

The block thus responds as a sum of relaxations with positive weights, one for each eigenmode of Eq. (1), and its response to a unit step in h approaches the final state monotonically. The weights sum to one because $\sum_n (n + \frac{1}{2})^{-2} = \pi^2/2$.

Substituting $q_{ex} = \text{Re}[i\omega S_{im} G H e^{i\omega t}]$ into Eq. (3) and scaling x by x_* gives

$$\frac{d^2 H}{dx^2} = \kappa^2 H, \quad \kappa^2 = i\Omega c(\Omega), \quad c(\Omega) \equiv 1 + \beta G(\Omega), \quad (7)$$

where c is the complex storage capacity in units of S_m . The solution that equals A at the coast and has zero gradient at $x = L$ is $H(x) = A \cosh[\kappa(L-x)]/\cosh(\kappa L)$. It differs from $A e^{-\kappa x}$ by a reflected wave of relative amplitude $e^{-2\kappa(L-x)}$, which is negligible when L exceeds a few attenuation lengths. We choose the root $\kappa \equiv a + ib$ with $a > 0$, so that the signal decays inland. Expanding $(a + ib)^2 = a^2 - b^2 + 2iab$ and equating real and imaginary parts with $i\Omega(\text{Re } c + i \text{Im } c)$ gives two exact identities,

$$a^2 - b^2 = -\Omega \text{Im } c, \quad 2ab = \Omega \text{Re } c, \quad (8)$$

and writing $c = |c|e^{i \arg c}$ gives the argument of κ and the ratio of phase lag to attenuation,

$$r \equiv \frac{b}{a} = \tan \theta, \quad \theta \equiv \arg \kappa = \frac{\pi}{4} + \frac{1}{2} \arg c. \quad (9)$$

In the two classical limits the capacity is real. Without matrix storage, $c = 1$, and with the matrix in full equilibrium with the conduits, $c = 1 + \beta$. In both cases $\theta = \pi/4$, $a = b$, and the diffusivity is T/S_m or $T/(S_m + S_{im})$. Equation (8) shows that the attenuation and phase lag measured for one constituent determine the complex capacity at that frequency, apart from the scale T , and contain no further information. The classical estimates $D_a \equiv \Omega/(2a^2)$ and $D_b \equiv \Omega/(2b^2)$ coincide only when c is real, and their ratio $D_a/D_b = r^2$ is the slope factor of the tidal-method literature [9]. Estimates of diffusivity that differ between constituents have been reported for island aquifers [45].

Blocks of other shapes and other exchange laws enter only through G . For cylinders and spheres of radius ℓ and for first-order exchange,

$$G_c = \frac{2 I_1(z)}{z I_0(z)} = \sum_{n=1}^{\infty} \frac{4}{z^2 + j_{0,n}^2}, \quad G_s = \frac{3(z \coth z - 1)}{z^2} = \sum_{n=1}^{\infty} \frac{6}{z^2 + n^2 \pi^2}, \quad G_1 = \frac{1}{1 + i\Omega}, \quad (10)$$

where I_0 and I_1 are modified Bessel functions and $j_{0,n}$ are the zeros of J_0 . Each expansion follows from the eigenfunctions of the diffusion equation in the corresponding geometry, and each is again a sum of relaxations with positive weights that add to one. Such a function is the Fourier transform of a positive, decaying kernel and therefore satisfies the Kramers-Kronig relations of a causal response [43]. We also consider slabs whose half-

thickness is distributed. If the fraction of matrix volume held in slabs of relative half-thickness $s \equiv \ell'/\ell \in (0, 1]$ has density $q s^{q-1}$ with $q > 0$, each slab has diffusion time τs^2 , and

$$G_q(\Omega) = \int_0^1 q s^{q-1} \frac{\tanh(zs)}{zs} ds. \quad (11)$$

A small q places most of the matrix volume in thin slabs.

Two comparison models are used. In a leaky aquifer, water also crosses the aquitard at the rate $L_a h$ per unit area, where L_a is the leakance, so that the balance of Eq. (3) acquires a term $L_a h$ on its left side, and $\kappa^2 = i\Omega + \lambda$ with $\lambda \equiv L_a \tau / S_m$ [25]. In the notation of Eq. (7) this is $c = 1 - i\lambda/\Omega$, a loss at zero frequency rather than a storage relaxation. A time-fractional aquifer obeys $D_t^\gamma h = \partial_x^2 h$ in dimensionless form, with a Caputo derivative of order $\gamma \in (0, 1]$ [8, 12]. When the derivative is taken from $t = -\infty$, as is appropriate for a periodic state, it acts on $e^{i\Omega t}$ as multiplication by $(i\Omega)^\gamma$ [29], so that $\kappa^2 = (i\Omega)^\gamma$, $\theta = \gamma\pi/4$, and $r = \tan(\gamma\pi/4)$ at every frequency. Fractional derivatives of this kind have also been used for pressure transients in fractured reservoirs [38]. For any capacity, we define the local fractional order $\gamma_{\text{eff}} \equiv 4\theta/\pi$, the order of the fractional medium that reproduces the measured ratio at that frequency.

The ratio is bounded by the structure of c . Each relaxation term $w_n/(1+i\Omega/\mu_n)$ has its argument in $(-\pi/2, 0]$, and a sum of complex numbers whose arguments lie in a sector narrower than π remains in that sector. The capacity $c = 1 + \beta G$ adds a positive real number, so $\arg c \in (-\pi/2, 0]$, and Eq. (9) gives $\theta \in (0, \pi/4]$ and $r \leq 1$. Leakage gives the same bound, since $\text{Re } c = 1 > 0$ and $\text{Im } c < 0$. A ratio above one therefore cannot arise from storage exchange or from leakage.

Diffusive exchange restricts the argument further. With $z = (1+i)u$ and $u \equiv \sqrt{\Omega/2}$, the identity $\tanh(x+iy) = (\sinh 2x + i \sin 2y)/(\cosh 2x + \cos 2y)$ gives $\arg \tanh z = \arctan(\sin 2u/\sinh 2u)$, and with $\arg(1/z) = -\pi/4$,

$$\arg G + \frac{\pi}{4} = \arctan \frac{\sin 2u}{\sinh 2u}. \quad (12)$$

The right-hand side is positive for $u < \pi/2$, negative for $\pi/2 < u < \pi$, and decays as e^{-2u} for large u . Its minimum, δ_s , is found numerically. The slab terms of c for any block size, which differ only in Ω , and the constant one all lie in the sector $[-\pi/4 + \delta_s, 0]$, and so does their sum, for any combination of slab sizes and diffusivities. Hence

$$r \geq \tan\left(\frac{\pi}{8} + \frac{\delta_s}{2}\right). \quad (13)$$

For cylinders and spheres, the large- Ω expansions of Eq. (10) are $G_c = 2z^{-1} - z^{-2} + O(z^{-3})$ and $G_s = 3z^{-1} - 3z^{-2} + O(e^{-2z})$. The second term, $-z^{-2} = i/\Omega$ times a positive constant, lifts the argument above $-\pi/4$, so their phase approaches the half-order value from above and $r > \tan(\pi/8)$. For a single slab with $\beta \gg 1$, the matrix term dominates c when $\beta|G| \gg 1$, and $G \simeq (i\Omega)^{-1/2}$ when $\Omega \gg 1$. In the band $1 \ll \Omega \ll \beta^2$, therefore, $\kappa^2 \simeq \beta(i\Omega)^{1/2}$, and the aquifer behaves as a time-fractional medium of order one half.

For the power-law distribution, the substitution $w = zs$ in Eq. (11) gives $G_q = q z^{-q} \int_0^z w^{q-2} \tanh w dw$. The integrand behaves as w^{q-1} near the origin, which is integrable for $q > 0$, and as w^{q-2} for large $|w|$, which is integrable for $q < 1$. For $0 < q < 1$ and large $|z|$, we write the integral as the integral to infinity minus the remainder from z to infinity. The path along the ray $\arg w = \pi/4$ may be rotated onto the real axis because $\tanh w$ has no poles in the sector $0 \leq \arg w \leq \pi/4$ and the arc at infinity contributes nothing when $q < 1$. In the remainder, $\tanh w = 1 + O(e^{-2w})$. Hence

$$G_q = q I_q (i\Omega)^{-q/2} - \frac{q}{1-q} (i\Omega)^{-1/2} + O(e^{-2\sqrt{\Omega/2}}), \quad I_q \equiv \int_0^\infty w^{q-2} \tanh w dw. \quad (14)$$

The first term dominates the second when $\Omega^{(1-q)/2} \gg 1/[(1-q)I_q]$, and dominates the conduit storage when $\beta q I_q \Omega^{-q/2} \gg 1$. In that band, $\kappa^2 \simeq \beta q I_q (i\Omega)^{1-q/2}$. Returning to dimensional variables, the conduit head obeys

$$S_{\text{eff}} D_t^\gamma h = T \frac{\partial^2 h}{\partial x^2}, \quad \gamma = 1 - \frac{q}{2} \in \left(\frac{1}{2}, 1\right), \quad S_{\text{eff}} = q I_q S_{im} \tau^{\gamma-1}, \quad (15)$$

which identifies the fractional order with the exponent of the block-size distribution and the fractional storage coefficient with the matrix storage and the diffusion time of the largest block. For $q > 1$, the integral $\int_0^z w^{q-2} \tanh w dw$ grows as $z^{q-1}/(q-1)$, $G_q \simeq [q/(q-1)] (i\Omega)^{-1/2}$, and the order returns to one half. Diffusive exchange with matrix blocks thus produces fractional orders between one half and one. Exchange through a broad distribution of first-order rates is not restricted in this way, because each relaxation term may approach the phase $-\pi/2$, and such distributions produce orders between zero and one in solute transport [14, 40, 11].

The fractional limit has a consequence in time. Consider $D_t^\gamma h = \partial_x^2 h$ on $x > 0$, with $h = 0$ at $t = 0$, $h(0, t) = \sin \Omega t$, and the Caputo derivative taken from $t = 0$. The Laplace transform of the Caputo derivative is $s^\gamma \hat{h} - s^{\gamma-1} h(x, 0) = s^\gamma \hat{h}$, so that $\hat{h}'' = s^\gamma \hat{h}$, and the solution that remains bounded as $x \rightarrow \infty$ is $\hat{h} = \Omega(s^2 + \Omega^2)^{-1} \exp(-s^{\gamma/2} x)$, with the branch cut of $s^{\gamma/2}$ along the negative real axis. The Bromwich contour is closed to the left around the poles at $s = \pm i\Omega$ and the cut. The residues at the poles give the periodic state, and on the two sides of the cut, $s = \rho e^{\pm i\pi}$, $s^{\gamma/2} = \rho^{\gamma/2} e^{\pm i\pi\gamma/2}$. Combining the two sides gives the exact decomposition

$$h = \text{Im } e^{i\Omega t - (i\Omega)^{\gamma/2} x} - \frac{1}{\pi} \int_0^\infty e^{-\rho t} \frac{\Omega}{\rho^2 + \Omega^2} \text{Im } e^{-\rho^{\gamma/2} e^{i\pi\gamma/2} x} d\rho. \quad (16)$$

The second term is the memory of the switch-on. For large t , the integral is dominated by small ρ , where $\Omega/(\rho^2 + \Omega^2) \simeq 1/\Omega$ and $\text{Im } e^{-\rho^{1/2} e^{i\pi\gamma/2} x} \simeq -x\rho^{\gamma/2} \sin(\pi\gamma/2)$. Watson's lemma [5] then gives

$$h - h_{\text{per}} \simeq \frac{x \sin(\pi\gamma/2) \Gamma(1 + \gamma/2)}{\pi \Omega} t^{-1-\gamma/2}, \quad (17)$$

which decays more slowly than the classical $t^{-3/2}$ whenever $\gamma < 1$. A record that begins shortly after a change in the forcing regime therefore retains the memory of that change for longer in a fractional medium.

For an unconfined aquifer, the conduit head is the water-table elevation above an impermeable base, and the transmissivity becomes Kh , with K the hydraulic conductivity. Under the Dupuit approximation, the discharge is $q = -Kh \partial_x h = -\frac{1}{2} K \partial_x (h^2)$, and the storage in the conduits is the specific yield S_y . We scale heads by the mean saturated thickness D , time by τ , and x by $x_* = \sqrt{KD\tau/S_y}$, and set $\beta = S_{im}/S_y$. The balance of Eq. (3) becomes the Boussinesq equation with matrix exchange,

$$\frac{\partial H}{\partial t} + \beta \frac{\partial \langle m \rangle}{\partial t} = \frac{1}{2} \frac{\partial^2 (H^2)}{\partial x^2}, \quad H(0, t) = 1 + \varepsilon \cos \Omega t, \quad \partial_x (H^2)|_{x=L} = 0, \quad (18)$$

with $\varepsilon \equiv A/D$ and the block law of Eq. (1) written for H . The Dupuit approximation requires $Da \ll 1$, the shallow-aquifer limit; outside it, vertical flow alters both rates [20]. The exchange remains linear in H , which assumes that the blocks lie below the lowest water table, and S_y is constant, which neglects the capillary fringe. In any state that is periodic in time, the period means of $\partial_t H$ and $\partial_t \langle m \rangle$ vanish, so $\partial_x^2 \overline{H^2} = 0$, where the overbar denotes the period mean. The no-flux condition then makes $\overline{H^2}$ uniform, and its value at the coast, $(1 + \varepsilon \cos \Omega t)^2 = 1 + \varepsilon^2/2$, gives

$$\overline{H^2}(x) = 1 + \frac{\varepsilon^2}{2} \quad \text{for all } x, \quad (19)$$

independently of β , of the block geometry, and of the amplitude. This extends a known integral relation for periodic nonlinear diffusion [36] to an aquifer with storage memory. Expanding $H = 1 + \varepsilon H_1 + \varepsilon^2 H_2 + \dots$ gives $H^2 = 1 + 2\varepsilon H_1 + \varepsilon^2 (H_1^2 + 2H_2) + \dots$. At first order, H_1 obeys the linear problem with $c = 1 + \beta G$, so $H_1 = \text{Re}[e^{-\kappa x + i\Omega t}]$ on a long domain and $\overline{H_1} = 0$, $\overline{H_1^2} = \frac{1}{2} e^{-2ax}$. At second order, Eq. (19) gives $\overline{H_2} + \frac{1}{2} \overline{H_1^2} = \frac{1}{4}$, so that

$$\overline{H} - 1 \simeq \frac{\varepsilon^2}{4} (1 - e^{-2ax}), \quad (20)$$

where a is the linear attenuation rate. The mean water-table rise, a classical result for beaches [33], depends on the attenuation alone and carries no information on the phase lag or on the memory beyond what a already contains. Far inland, Eq. (19) gives $\overline{H} - 1 = \sqrt{1 + \varepsilon^2/2} - 1 = \varepsilon^2/4 - \varepsilon^4/32 + \dots$, the second term of which measures the departure from Eq. (20).

To quantify how well a record of finite length separates memory from leakage, we consider a well at dimensionless distance $X = x/x_*$ and a record of N samples covering an integer number of cycles of every constituent. Harmonic analysis fits $\text{Re}[Z_k e^{i\omega_k t}]$ for each constituent k by least squares [35], and for white noise of standard deviation σ_k the cosine and sine coefficients of Z_k are uncorrelated with variance $2\sigma_k^2/N$. To first order in the noise, the log-amplitude $\ln |Z_k|$ and the phase $\arg Z_k$ respond to the components of the error parallel and perpendicular to Z_k , each divided by $|Z_k|$. With a well amplitude $|Z_k| = A_k e^{-a_k X}$, the log-amplitude ratio $a_k X$ and the phase lag $b_k X$ relative to the coast are therefore unbiased, with standard deviation

$$s_k = \sqrt{\frac{2}{N}} \frac{\sigma_k}{A_k e^{-a_k X}}, \quad (21)$$

and are independent of each other and of other constituents. The coastal record is treated as exact, on the grounds that a tide gauge records the sea level with much less noise than a well records the aquifer head; this is an optimistic simplification. Constituents are resolved only if the record is longer than the inverse of their frequency separation, the Rayleigh criterion [35]. If the data are generated by a memory model and fitted by the leaky model, whose two parameters are chosen by weighted least squares, the expected value of the likelihood-ratio statistic in excess of its null value is the noncentrality

$$\Lambda = \min_{s, l} \sum_k \frac{(a_k X - a_k^{\text{lk}} X)^2 + (b_k X - b_k^{\text{lk}} X)^2}{s_k^2}, \quad (\kappa^{\text{lk}} X)^2 = i\Omega_k s + l. \quad (22)$$

The statistic then follows a noncentral chi-square distribution with $2K - 2$ degrees of freedom for K constituents, from which the power of a test of size 0.05 follows. With one constituent, $\Lambda = 0$ exactly, since the leaky model then has as many parameters as data. Because every s_k^2 is proportional to $1/N$, Λ is proportional to the number of samples. The precision with which X , β , and τ can be estimated is measured by the Cramér-Rao bound, the inverse of the Fisher matrix $\mathbf{F} = \mathbf{J}^T \Sigma^{-1} \mathbf{J}$, where $\mathbf{J}$ is the Jacobian of the $2K$ observations with respect to the logarithms of the three parameters and Σ is the diagonal matrix of the s_k^2 . White noise and an exact coastal record make these figures optimistic.

The dimensionless results depend only on β , Ω , and x/x_* . To place them on a physical scale, we use a reference aquifer whose round parameter values lie in a plausible range for a karstic limestone coastal aquifer of the kind common on uplifted islands of the Indonesian Maritime Continent (Table 1). The values are illustrative and represent no particular site. Constituent periods are the standard astronomical values, and amplitudes are

illustrative. Head noise is set to 10 mm in the tidal bands and to 50 mm for periods above two days, where weather and river stage contribute, and heads are logged every 10 minutes. The well used for the information analysis lies where the M2 amplitude has fallen to $e^{-1.5}$ of its coastal value.

Table 1: Reference aquifer and tidal constituents. All values are illustrative. Ω is the dimensionless frequency of Eq. (4), and a , b , and r are computed from Eqs. (7) to (9) for the slab model, with a and b in units of $1/x_*$.

Quantity	Symbol	Value
Transmissivity of the conduit network	T	$5 \times 10^{-2} \text{ m}^2 \text{ s}^{-1}$
Storativity of the conduits	S_m	5×10^{-5}
Storativity of the matrix	S_{im}	$1 \times 10^{-3} \text{ } (\beta = 20)$
Largest block half-thickness	ℓ	0.5 m
Matrix hydraulic diffusivity	D_{im}	$1.93 \times 10^{-6} \text{ m}^2 \text{ s}^{-1}$
Matrix diffusion time	τ	1.50 d
Length scale	x_*	11.4 km

Constituent	Period (h)	Amplitude (m)	Ω	a	b	r	$1/a \text{ (m)}$
M2	12.42	0.50	18.20	8.910	4.388	0.4925	1277
S2	12.00	0.20	18.84	8.994	4.443	0.4940	1265
K1	23.93	0.30	9.446	7.544	3.463	0.4590	1509
O1	25.82	0.20	8.756	7.417	3.379	0.4555	1535
MSf	354.4	0.02	0.638	2.788	2.286	0.8200	4082
Mm	661.3	0.02	0.342	1.985	1.782	0.8977	5734
Ssa	4383	0.03	0.052	0.742	0.730	0.9838	15341
Sa	8766	0.08	0.026	0.523	0.518	0.9918	21782

2.2 Numerical implementation

The closed forms are checked against time-domain solvers that share no code path with them, so that an error in the frequency-domain algebra would appear as a disagreement. The confined double-porosity model of Eqs. (1) to (3) is solved in dimensionless form. The conduit heads h_i are stored at $x_i = i \Delta x$, $i = 1, \dots, N_x$, with $\Delta x = L/N_x$, and the matrix heads $m_{i,j}$ at $\zeta_j = j \Delta \zeta$, $j = 0, \dots, N_\zeta - 1$, with $\Delta \zeta = 1/N_\zeta$ and $m_{i,N_\zeta} \equiv h_i$ at the face. Second derivatives are replaced by central differences, which are second-order accurate for smooth solutions. The coastal head enters through $h_0 = h(0, t)$, and the no-flux conditions at $x = L$ and at the block center are imposed with ghost values $h_{N_x+1} = h_{N_x-1}$ and $m_{i,-1} = m_{i,1}$, which make the center equation $\dot{m}_{i,0} = 2(m_{i,1} - m_{i,0})/\Delta \zeta^2$.

The exchange term requires care, because a discretization that does not balance the water leaving the conduits against the water entering the matrix would create or destroy storage. We evaluate the block average with the trapezoidal rule including the face node, $\langle m \rangle_i = \Delta \zeta [\frac{1}{2} m_{i,0} + \sum_{j=1}^{N_\zeta-1} m_{i,j} + \frac{1}{2} h_i]$. Differentiating in time and substituting the discrete block equations, the sum of second differences telescopes, $\Delta \zeta [\frac{1}{2} \dot{m}_{i,0} + \sum_{j=1}^{N_\zeta-1} \dot{m}_{i,j}] = (h_i - m_{i,N_\zeta-1})/\Delta \zeta$, so that

$$\left(1 + \frac{\beta \Delta \zeta}{2}\right) \frac{dh_i}{dt} = \frac{h_{i+1} - 2h_i + h_{i-1}}{\Delta x^2} - \frac{\beta}{\Delta \zeta} (h_i - m_{i,N_\zeta-1}). \quad (23)$$

The last term is the discrete face flux, the same quantity that enters the outermost matrix node, so the discretization conserves water exactly. The factor $1 + \beta \Delta \zeta / 2$ is the storage of the half cell at the face, which the trapezoidal rule assigns to the conduit node.

Equations (23) and the block equations form the linear system $\mathbf{M} \dot{\mathbf{u}} = \mathbf{A} \mathbf{u} + \mathbf{f}(t)$, with $\mathbf{M}$ diagonal and positive and $\mathbf{f}$ carrying the coastal head. The eigenvalues of $\mathbf{M}^{-1} \mathbf{A}$ are real and negative, as expected for a discretized diffusion problem, and range over several orders of magnitude, so the system is stiff. We advance it with the second-order backward differentiation formula (BDF2),

$$\mathbf{M} \frac{3\mathbf{u}^{n+1} - 4\mathbf{u}^n + \mathbf{u}^{n-1}}{2\Delta t} = \mathbf{A} \mathbf{u}^{n+1} + \mathbf{f}(t_{n+1}), \quad (24)$$

which is A-stable [10, 15], so that the time step is chosen for accuracy alone. The matrix $\frac{3}{2} \Delta t^{-1} \mathbf{M} - \mathbf{A}$ is factorized once by sparse LU decomposition. The periodic orbit of the semi-discrete system is obtained exactly by solving $(i\Omega \mathbf{M} - \mathbf{A}) \mathbf{U} = \mathbf{F}$ for each constituent, which separates the spatial error from the temporal error. The periodic orbit of the fully discrete BDF2 system, used below to isolate nonlinear effects, follows by replacing $i\Omega$ with $(3 - 4\xi^{-1} + \xi^{-2})/(2\Delta t)$, $\xi \equiv e^{i\Omega \Delta t}$. The time-domain runs use $L = 3$, for which the reflected wave of the M2 constituent is reduced by $e^{-2aL} \approx 10^{-23}$, $N_x = 600$, $N_\zeta = 24$, and 100 steps per M2 period. The forcing is multiplied by $1 - e^{-t}$ to limit the switch-on transient, and the runs cover 45 days, of which the last 15 are analyzed.

The time-fractional equation is discretized by the L1 scheme. The Caputo derivative $D_t^\gamma u(t_n) = \Gamma(1 - \gamma)^{-1} \int_0^{t_n} (t_n - s)^{-\gamma} u'(s) ds$ is evaluated by replacing u on each step by its linear interpolant, whose derivative is

constant, and integrating the kernel exactly over each step. This gives

$$D_t^\gamma u(t_n) \approx \frac{\Delta t^{-\gamma}}{\Gamma(2-\gamma)} \sum_{j=0}^{n-1} b_j (u^{n-j} - u^{n-j-1}), \quad b_j = (j+1)^{1-\gamma} - j^{1-\gamma}, \quad (25)$$

whose truncation error is of order $\Delta t^{2-\gamma}$ for smooth solutions, and the implicit scheme built on it is unconditionally stable [28]. Solutions forced from rest have a weak singularity at $t = 0$, and the error there is of first order in Δt [41, 26]. We therefore compare the switch-on transient with Eq. (16) only after three time units, and we remove the leading error by Richardson extrapolation, $u_R = (2^p u_{\Delta t/2} - u_{\Delta t}) / (2^p - 1)$ with $p = 2 - \gamma$, applied to runs with 128 and 256 steps per period. These runs use $L = 16$ with 320 nodes and the observation point at $x = 1$; the reflected wave there is reduced by $\exp(-2L \operatorname{Re} i^{\gamma/2}) \approx 3 \times 10^{-12}$. The branch-cut integral of Eq. (16) is evaluated in $\ln \rho \in [-60, 40]$ with 1200 panels of 48-point Gauss-Legendre quadrature, and its value at $t = 0$, where the two terms of Eq. (16) must cancel, serves as a check.

The nonlinear model of Eq. (18) is discretized in the same way, with the flux written as the second difference of H^2 and the matrix exchange of Eq. (23). Over one period of a periodic discrete solution, the BDF2 differences sum to zero, the matrix equations then force the period mean of every matrix node to equal that of the adjacent conduit node, and the second difference of $\overline{H^2}$ vanishes. The discrete analogue of Eq. (19) is therefore exact, which makes it a stringent test of the implementation. Each BDF2 step is solved by Newton iteration, with the Jacobian $\frac{3}{2} \Delta t^{-1} \mathbf{M} - \partial \mathbf{N} / \partial \mathbf{u}$ factorized once per step at the extrapolated state $2\mathbf{u}^n - \mathbf{u}^{n-1}$ and the iteration continued until the increment falls below 10^{-12} . The integration starts from the periodic orbit of the linearized discrete system plus the second-order mean rise of Eq. (20), so that the remaining transient is of order ε^3 and decays within the 120 periods integrated. The runs use $L = 7/a$, $N_x = 140$, $N_\zeta = 12$, and 80 steps per period.

The transfer function of Eq. (11) is evaluated in $\ln s$ with 400 panels of 64-point Gauss-Legendre quadrature, truncated where the weight s^q falls below 10^{-17} , and compared with Eq. (14), whose constant I_q is computed by adaptive quadrature. The Kramers-Kronig relation for a transfer function that vanishes at infinite frequency,

$$\operatorname{Re} G(\Omega) = -\frac{2}{\pi} \operatorname{P} \int_0^\infty \frac{\Omega' \operatorname{Im} G(\Omega')}{\Omega'^2 - \Omega^2} d\Omega', \quad (26)$$

is evaluated with the trapezoidal rule in $\ln \Omega'$ over $10^{-12} < \Omega' < 10^{14}$. The principal value is removed by subtracting $\Omega \operatorname{Im} G(\Omega)$ from the numerator and adding its integral over the truncated range in closed form, and the contribution above the upper limit is added assuming that $\operatorname{Im} G$ decays as a power of Ω' there. A function with a pole at zero frequency, such as the leaky capacity, does not satisfy Eq. (26).

Harmonic analysis of simulated records uses least squares with a mean, a linear trend, and the cosine and sine of each constituent. Rates are obtained from the ratio of well and coastal amplitudes as $a = -\ln |Z_{\text{well}}/Z_{\text{coast}}|/X$ and $b = -\arg(Z_{\text{well}}/Z_{\text{coast}})/X$, which requires $bX < \pi$. The leaky fit of Eq. (22) uses the Levenberg-Marquardt method in the logarithms of s and l , started from the exact single-constituent inversions $l = (a_k X)^2 - (b_k X)^2$ and $s = 2(a_k X)(b_k X)/\Omega_k$ for each constituent, and the smallest misfit is retained. The Jacobian of the Fisher matrix is computed by central differences with a step of 10^{-6} in the logarithms.

Six tests are used: spatial convergence of the semi-discrete periodic orbit to the exact solution, for the conduit head and for the matrix head inside the blocks; temporal convergence of BDF2 started on the discrete periodic orbit; convergence of the L1 scheme to a manufactured solution of the semi-discrete system for three orders; Kramers-Kronig consistency of every transfer function; agreement of the power-law quadrature with Eq. (14); and the scaling of the nonlinear correction to the fundamental harmonic. The approach follows the practice adopted for related idealized solvers [18, 23, 24, 19]. The code is written in Python with NumPy [16], SciPy [46], and Matplotlib [22], and is distributed as the package `pasang`.

3 Results

The solvers pass every verification test (Fig. 2, Table 2). The periodic orbit of the dual-porosity solver converges to the exact solution at observed orders between 1.96 and 2.00 for both the conduit head and the matrix head, and BDF2 converges at orders between 1.98 and 2.00. The L1 scheme converges at 1.72, 1.49, and 1.25 for $\gamma = 0.25, 0.5$, and 0.75 , approaching the theoretical $2 - \gamma$ from below. The Kramers-Kronig reconstruction reproduces $\operatorname{Re} G$ to about 10^{-11} for slab, cylinder, sphere, and first-order exchange, and to 9×10^{-8} for the power-law distribution, which is evaluated on a coarser grid. The power-law quadrature agrees with Eq. (14) to 7×10^{-16} to 6×10^{-15} for $\Omega > 10^3$. The departure of the nonlinear fundamental from the linear orbit, divided by ε , scales with slope 2.000, as expected for a cubic correction to the fundamental.

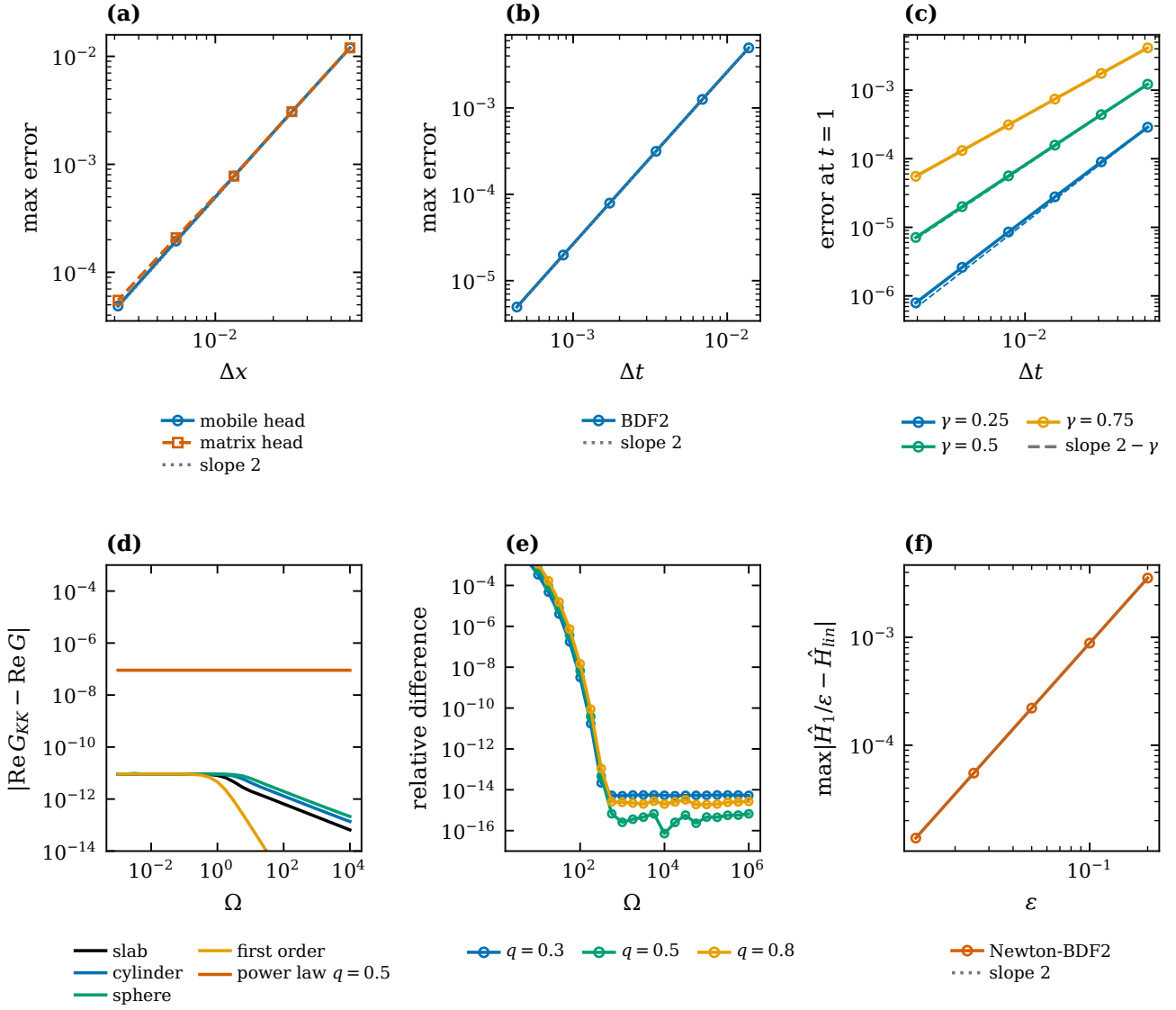


Figure 2: Verification. (a) Maximum error of the semi-discrete periodic orbit against the exact solution for the conduit and matrix heads, reference M2 constituent, with Δx and $\Delta \zeta$ refined together. (b) BDF2 started on the discrete periodic orbit and integrated for six M2 periods with M2 and K1. (c) L1 scheme on a manufactured solution of the semi-discrete system at $t = 1$, with reference slopes $2 - \gamma$. (d) Kramers-Kronig reconstruction of $\text{Re } G$ from $\text{Im } G$. (e) Power-law quadrature against Eq. (14). (f) Departure of the first harmonic of the nonlinear solution, divided by ε , from the periodic orbit of the linearized discrete system.

Table 2: Measured residuals of the verification tests.

Test	Result
Dual porosity, periodic orbit, observed orders	1.96, 1.98, 2.00, 2.00
Dual porosity, finest grid, conduit / matrix error	4.9×10^{-5} / 5.5×10^{-5}
BDF2, observed orders	1.98 to 2.00
L1, observed order, $\gamma = 0.25, 0.5, 0.75$	1.72, 1.49, 1.25
Kramers-Kronig, slab / cylinder / sphere	9.2×10^{-12} / 9.2×10^{-12} / 9.7×10^{-12}
Kramers-Kronig, first order / power law	9.2×10^{-12} / 9.0×10^{-8}
Power-law quadrature against closed form	6.7×10^{-16} to 5.6×10^{-15}
Nonlinear correction to the fundamental, slope	2.000 to 2.003
Dual porosity from rest, wells, last 4 days	4.7×10^{-4} m
Harmonic analysis, error in a plus error in b	1.9×10^{-2}
Caputo from rest, first six periods	2.0×10^{-3}
Switch-on transient, relative, $3 \leq t \leq 60$	4.0×10^{-2}
Invariant $\bar{H}^2$, classical / memory, $\varepsilon = 0.2$	5.7×10^{-13} / 6.4×10^{-8}

The time-domain solutions reproduce the frequency-domain theory (Fig. 3). Forced from rest by M2 and K1 together, the dual-porosity aquifer reaches the exact periodic state, and over the last four of 45 simulated days the heads at wells 0.64, 1.28, and 2.55 km from the coast differ from it by at most 0.47 mm. Harmonic analysis of the last 15 days recovers a and b at every node to within 1.9×10^{-2} in units of $1/x_*$, about 0.2 percent of a . Both rates lie between the classical values for the conduit storage alone and for the total storage, and they differ from each other, as Eq. (8) requires. For the Caputo medium with $\gamma = 0.75$, the L1 solution agrees with Eq. (16) to 2.0×10^{-3} over the first six periods, and the exact solution vanishes at $t = 0$ to round-off. The switch-on transient computed from the L1 runs agrees with the branch-cut integral to 4 percent between $t = 3$ and 60 and follows the $t^{-1.375}$ law of Eq. (17), whereas the classical transient decays as $t^{-3/2}$.

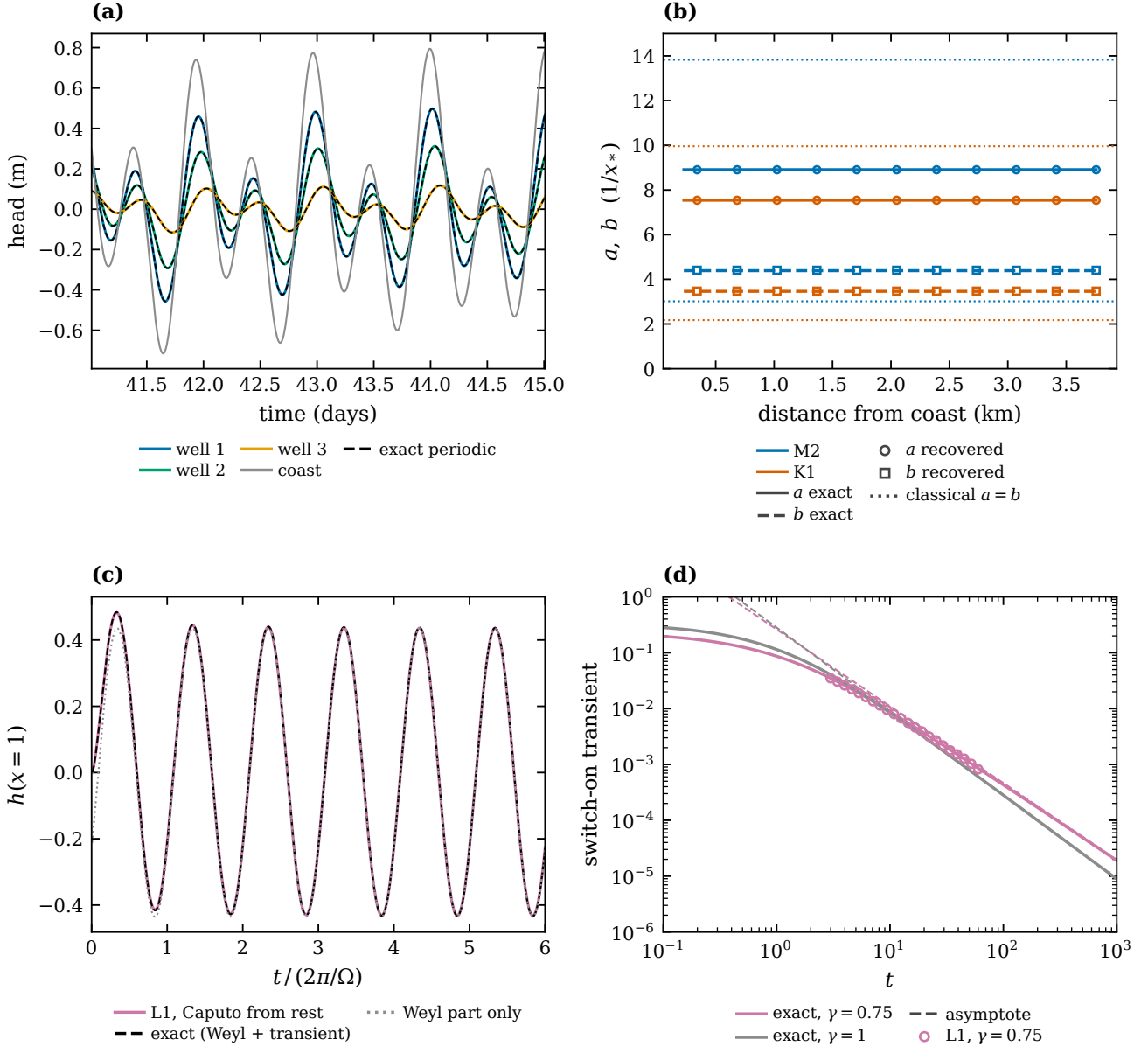


Figure 3: Time-domain solutions. (a) Heads at three wells in the reference aquifer forced from rest by M2 and K1, over the last four of 45 days, with the exact periodic solution. (b) Attenuation and phase-lag rates recovered by harmonic analysis of the last 15 days (markers) and exact rates (lines); dotted lines are the classical rates $a = b$ for the conduit storage alone (lower) and for the total storage (upper). (c) Caputo medium, $\gamma = 0.75$, forced from rest by $\sin t$: L1 solution at $x = 1$, exact solution, and its periodic part. (d) Magnitude of the switch-on transient at $x = 1$: exact branch-cut integral, large-time asymptote, and L1 solution after Richardson extrapolation, with the classical case $\gamma = 1$.

The matrix transfer functions share one structure (Fig. 4). At low frequency, every block fills and drains in phase with the conduits, $G \rightarrow 1$, and the full matrix storage is active. At high frequency, only a skin of thickness $\sqrt{D_{im}/\omega}$ responds, and G decays as $(i\Omega)^{-1/2}$ for blocks of any shape and as $(i\Omega)^{-1}$ for first-order exchange. The loss part, $-\text{Im } G$, peaks at $\Omega = 2.5$ for slabs, 6.3 for cylinders, 11.6 for spheres, and 1.0 for first-order exchange. The power-law distributions spread the transition over several decades of frequency. For the reference aquifer,

the diurnal and semidiurnal constituents fall above the slab peak, and the fortnightly to annual constituents fall below it.

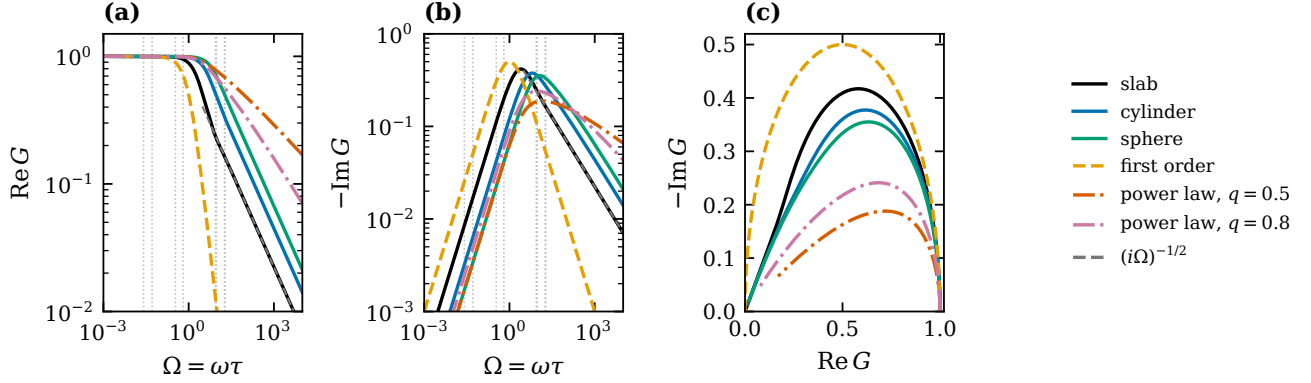


Figure 4: Matrix transfer functions. (a) In-phase part $\text{Re } G$, the fraction of matrix storage that acts with the conduits; (b) quadrature part $-\text{Im } G$, the loss; (c) Cole-Cole plane. Slab, cylinder, and sphere blocks, first-order exchange, and power-law slab distributions with $q = 0.5$ and 0.8 . Dashed gray: the half-order asymptote $(i\Omega)^{-1/2}$. Dotted verticals in (a) and (b): the eight constituents of the reference aquifer.

The ratio $r = b/a$ inherits this structure (Fig. 5). For a single slab, r is one at low and high frequency and dips in between. The dip deepens with β , reaching 0.763 for $\beta = 1$, 0.488 for $\beta = 10$, 0.409 for $\beta = 100$, and 0.399 for $\beta = 1000$, close to the floor of Eq. (13), and for large β a plateau near $\tan(\pi/8)$ extends over $1 \ll \Omega \ll \beta^2$. For the reference aquifer, r is 0.49 at M2 and S2, 0.46 at K1 and O1, 0.82 at MSf, and above 0.98 at the annual and semiannual constituents (Table 1). The comparison mechanisms differ in a way that a single constituent cannot reveal. Leakage drives r toward zero at low frequency, where the head decays without lag, and toward one at high frequency. Storage memory returns r to one at both ends. A Weyl fractional medium holds r constant at $\tan(\gamma\pi/4)$, and the power-law distribution approaches such a constant over several decades.

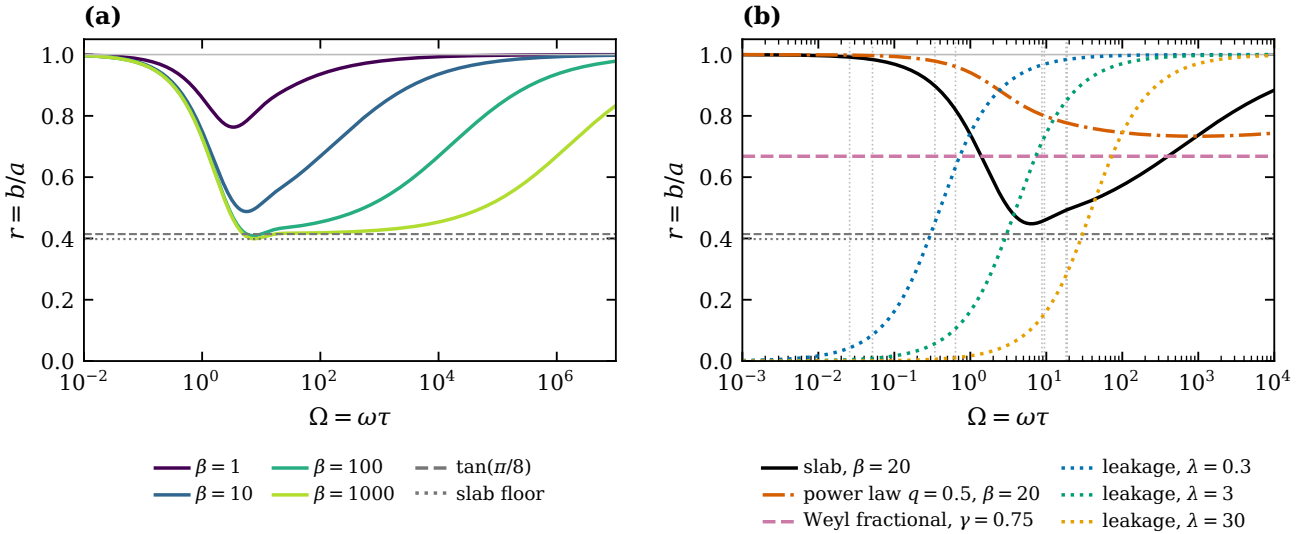


Figure 5: The ratio $r = b/a$ against frequency. (a) Single slab with $\beta = 1$ to 1000; dashed: $\tan(\pi/8)$; dotted: the floor of Eq. (13). (b) Slab memory and power-law memory for the reference $\beta = 20$, a Weyl fractional medium with $\gamma = 0.75$, and leaky aquifers with $\lambda = 0.3$, 3, and 30. Dotted verticals in (b): the reference constituents.

The two quantities that each constituent determines exactly separate the mechanisms more clearly (Fig. 6). The loss part of the capacity, $(a^2 - b^2)/\Omega$, decreases as $1/\Omega$ for leakage at all frequencies, because $a^2 - b^2 = \lambda$ is constant, whereas for storage memory it rises, peaks, and falls. The storage part, $2ab/\Omega$, is one for leakage and steps from $1 + \beta$ to one for memory. The two classical diffusivities differ accordingly. For leakage, D_a falls and D_b rises without bound toward low frequency, while for memory both approach $T/[S_m(1 + \beta)]$ at low frequency and T/S_m at high frequency, with $D_a < D_b$ in between.

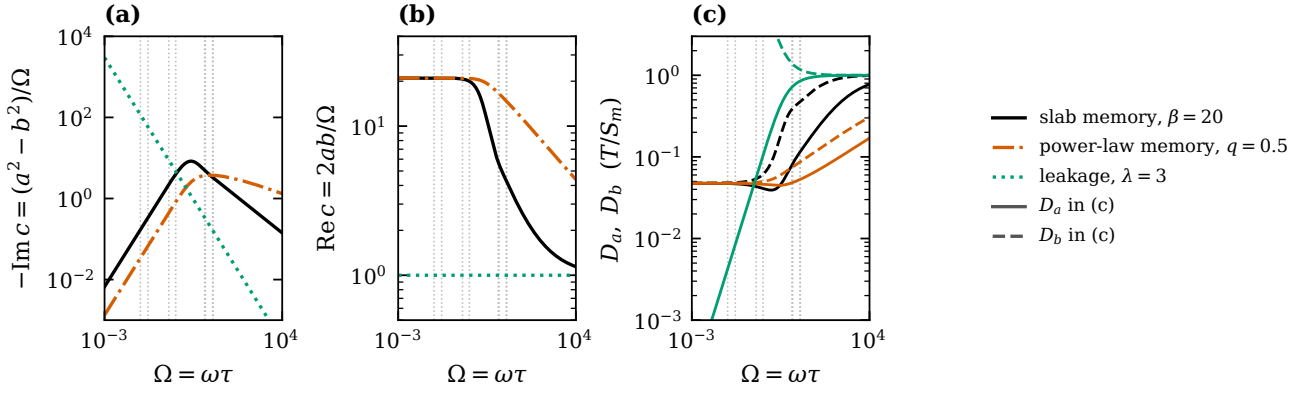


Figure 6: What each constituent measures. (a) Loss part of the capacity, $-\text{Im } c = (a^2 - b^2)/\Omega$; (b) storage part, $\text{Re } c = 2ab/\Omega$; (c) diffusivities inferred by the classical formulas from attenuation, D_a , and from phase lag, D_b , in units of T/S_m . Slab memory and power-law memory with $\beta = 20$, and leakage with $\lambda = 3$. Dotted verticals: the reference constituents.

The bound of Eq. (13) is reached and not exceeded (Fig. 7). The phase excess of Eq. (12) has its minimum $\delta_s = -0.0278677$ at $u = 1.96330$, or $\Omega = 7.7091$, so the floor is $\tan(\pi/8 + \delta_s/2) = 0.397982$, 4 percent below $\tan(\pi/8) = 0.414214$. For cylinders and spheres the excess stays positive, and their minimum ratio approaches $\tan(\pi/8)$ from above, reaching 0.4155 and 0.4157 at $\beta = 10^5$, while the slab minimum reaches 0.39799. For power-law distributions with $\beta = 10^6$, the local order γ_{eff} settles on the plateau $1 - q/2$ of Eq. (15) for $q = 0.2, 0.4, 0.6$, and 0.8 , and on one half for $q = 1.5$. The approach is slowest for $q = 0.8$, for which the second term of Eq. (14) decays only as $\Omega^{-0.1}$ relative to the first.

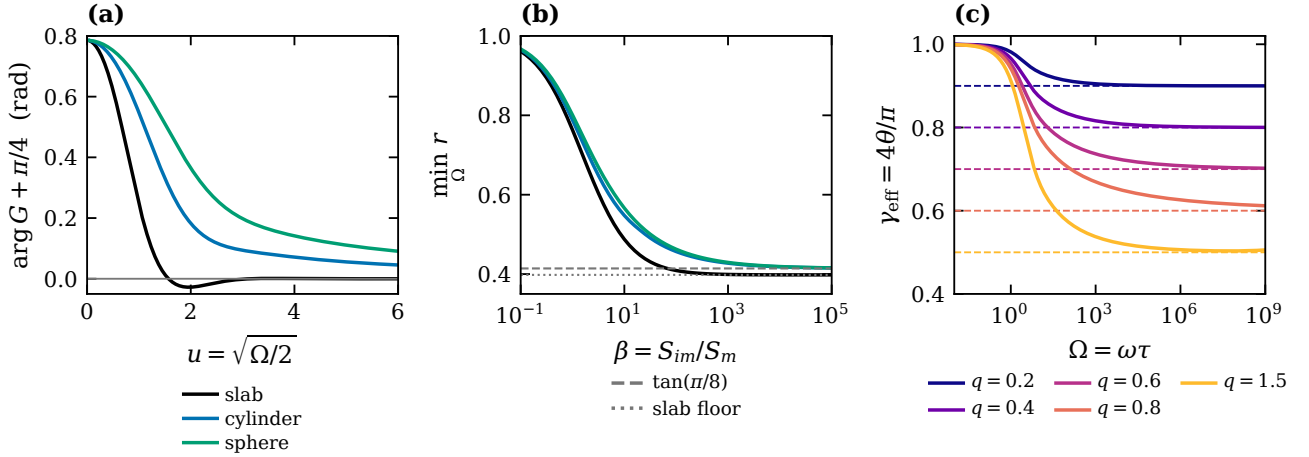


Figure 7: The half-order bound and the fractional plateau. (a) Phase excess $\arg G + \pi/4$ against $u = \sqrt{\Omega}/2$ for slab, cylinder, and sphere blocks. (b) Minimum over frequency of r for $c = 1 + \beta G$ against β . (c) Local fractional order $\gamma_{\text{eff}} = 4\theta/\pi$ for power-law slab distributions with $\beta = 10^6$; dashed: the plateau $1 - q/2$ for $q < 1$ and one half for $q > 1$.

The unconfined runs confirm the invariant and the overheight law (Fig. 8). With $\varepsilon = 0.2$, memory steepens the inland decay of the tidal envelope, while the mean water table rises in both aquifers toward the same far-field value. Plotted against $2ax$, with a the linear attenuation rate of each aquifer, the normalized mean rise of the memory runs with $\varepsilon = 0.1, 0.2$, and 0.3 and of the classical run collapses onto $1 - e^{-2ax}$. The departure from this law, divided by ε^2 , approaches $-1/8$ far inland, the fourth-order term of the exact far-field value. The period mean of H^2 equals $1 + \varepsilon^2/2$ to 5.7×10^{-13} in the classical run and to 6.4×10^{-8} in the memory run after 120 periods. The residual decays exponentially in both, more slowly with memory, because the matrix relaxes on its own time scale.

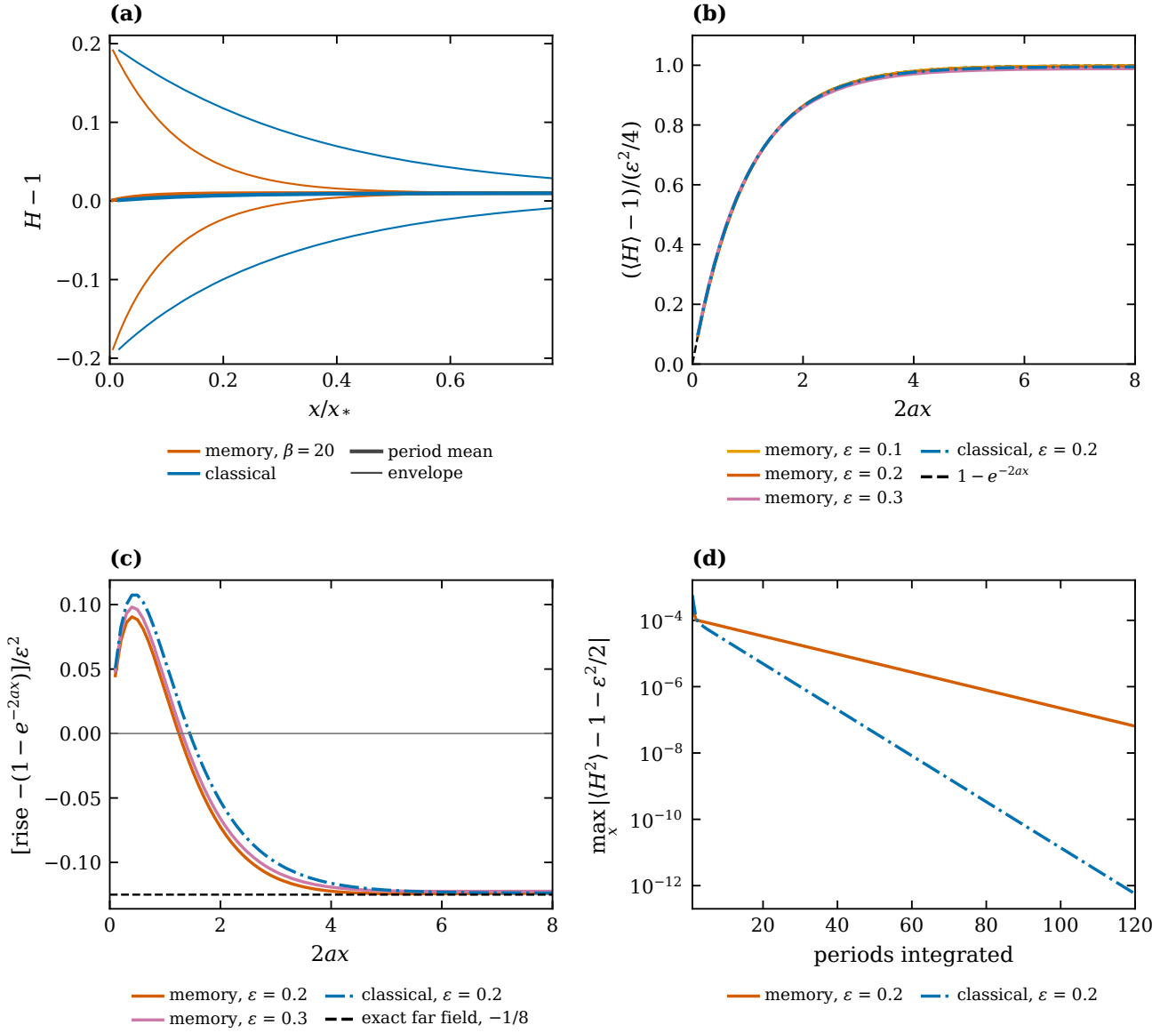


Figure 8: Nonlinear unconfined aquifer. (a) Envelope and period mean of the water table for the memory aquifer ($\beta = 20$) and the classical aquifer, $\epsilon = 0.2$. (b) Mean rise normalized by $\epsilon^2/4$ against $2ax$, with the second-order law of Eq. (20). (c) Departure from that law divided by ϵ^2 , with the exact far-field value $-1/8$. (d) Departure of $\overline{H^2}$ from Eq. (19), maximum over x , against the number of periods integrated.

A tidal record can separate memory from leakage over a wide range of aquifers (Fig. 9). With M2, S2, K1, and O1 from a 30-day record at a well where the M2 amplitude has fallen to $e^{-1.5}$ of its coastal value, the noncentrality exceeds the value for power 0.95, $\Lambda = 20.9$, for every Ω_{M2} above about 0.04 in the range computed when $\beta \geq 10$. The detectable range narrows as β decreases, to $0.1 < \Omega_{M2} < 750$ for $\beta = 1$ and $0.6 < \Omega_{M2} < 10$ for $\beta = 0.1$. For the reference aquifer, $\Lambda = 5.5 \times 10^3$. For a weak-memory case with the same τ and $\beta = 0.2$, the ratio at the four constituents lies between 0.954 and 0.969, close to the classical value, yet $\Lambda = 34.5$ and the power is 0.998. Because Λ grows linearly with the number of samples, the record that noise alone requires for power 0.95 is 0.1 d for the reference aquifer and 18.8 d for the weak case. In practice, the four constituents need about 15 d to be resolved, so noise becomes the limit only for $\beta < 0.22$. Adding the fortnightly and monthly constituents changes little, because they carry larger noise, whereas adding the annual and semiannual constituents shortens the record required by noise by a factor of about 9 to 13 for β between 0.01 and 0.2. They take a year to resolve, however. With the four tidal constituents and 30 d, the Cramér-Rao bounds for the reference aquifer are 0.8 percent for X , 4.0 percent for β , and 5.2 percent for τ . The precision on β is best near $\beta = 10$, where the constituents straddle the transition of G .

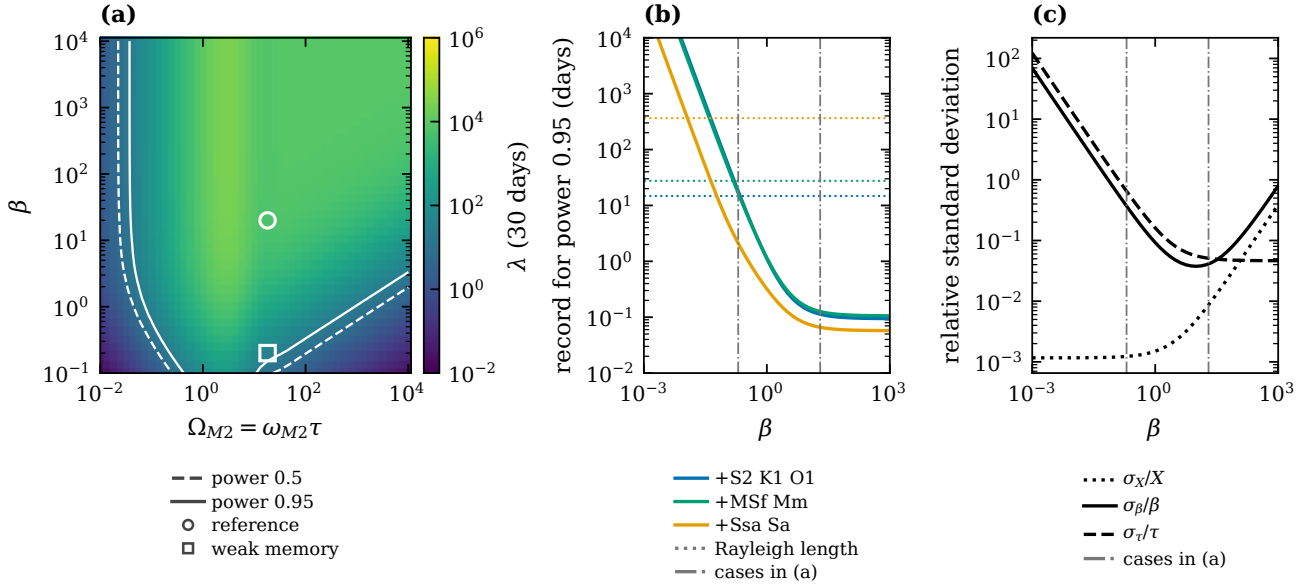


Figure 9: Information in a tidal record. (a) Noncentrality Λ of the leaky-model misfit for M2, S2, K1, and O1 from a 30-day record, over the M2 frequency and β , with contours for power 0.5 and 0.95 of the size-0.05 test; markers: reference and weak-memory cases. (b) Record length required by noise alone for power 0.95 against β , for three constituent sets; dotted: the Rayleigh length that each set requires. (c) Cramér-Rao relative standard deviations of X , β , and τ for M2, S2, K1, and O1 over 30 days. Dash-dotted verticals in (b) and (c): the two cases marked in (a).

4 Discussion

The analysis gives the time-fractional description of a coastal aquifer a physical origin and a limited range. Diffusive exchange with matrix blocks produces fractional behavior when the matrix storage greatly exceeds the conduit storage and the forcing period is short compared with the diffusion time of the largest blocks. A single block size gives order one half over the band $1 \ll \Omega \ll \beta^2$, and a power-law distribution of sizes gives order $1 - q/2$. Outside these bands the aquifer is classical, with the total storage at low frequency and the conduit storage at high frequency. A fractional order fitted to tidal data is therefore a property of the storage structure in a frequency band, not a constant of the medium, which is consistent with reports of constituent-dependent diffusivities [45] and of period-dependent hydraulic properties from oscillatory tests [37]. The bound $1/2 \leq \gamma_{\text{eff}}$, or $r \geq 0.398$, applies to diffusive exchange. An order fitted below one half would point to exchange with a broad distribution of first-order rates, as in multirate mass transfer [14, 40], or to a mechanism other than storage exchange.

The identities of Eq. (8) change how a tidal record is read. One constituent fixes the complex capacity at one frequency and no more, so any model with two free parameters, leaky or fractional, fits it exactly, and the ratio alone cannot identify the mechanism. The frequency dependence can. Leakage makes $a^2 - b^2$ the same for every constituent, while storage memory makes it a relaxation that vanishes at both ends. The separation is strongest at long periods, where leakage drives the phase lag to zero and memory restores the classical equality. The geometric mechanisms raised in earlier work, including variable thickness [9], dual tides [39], and finite aquifer depth [20], are not represented here, and they can produce ratios above one. Since storage exchange and leakage cannot, a measured ratio above one points to geometry rather than to memory. The observation that storm surges of several days show equal rates while semidiurnal tides do not [20] is qualitatively consistent with the return of r toward one at low frequency, although the authors of that study attributed it to finite aquifer depth, and the present model cannot separate the two.

The nonlinear analysis shows where memory does not appear. The invariant of Eq. (19) holds with or without matrix exchange, and the mean water-table rise depends on the attenuation rate alone. Memory steepens the inland decay of the tidal envelope and thereby compresses the zone over which the water table rises, but it does not change the far-field rise, which is fixed by the tidal amplitude. Estimates of recharge or of submarine discharge that rely on the overheight would thus be affected by memory only through a .

The information analysis indicates that memory should be detectable in practice when it is present. A month of semidiurnal and diurnal constituents suffices for the reference aquifer and for much weaker memory, and the Cramér-Rao bound on β is about four percent. These figures assume white noise, an exact coastal record, correctly specified constituents, and a model without leakage, and they are therefore optimistic. Records in which weather, river stage, or pumping contaminate the long-period bands would lose much of the information that those bands carry. The analysis also assumes that leakage and memory do not act together. A model that contains both has four parameters and would require at least three constituents.

The conclusions are subject to several further limitations. The model is one-dimensional, with a vertical coastline and a prescribed head at the shore, and it omits seepage faces, capillarity, density differences, the loading of the aquifer by the tide, and the response of wells to earth tides [21]. The double-porosity description assumes a separation of scales between the blocks and the tidal wavelength [1], which may fail in aquifers dominated by a few large conduits. The matrix geometry is uniform along the aquifer, and the exchange is linear even in the nonlinear runs. The parameter values are illustrative and represent no site. The classical tidal method was developed for confined aquifers of uniform properties [44], and the present results do not replace it; they indicate which departures from it storage memory can and cannot produce. A site-specific assessment would require measured tidal records at several distances, constituent-resolved analysis with error estimates, and independent information on the storage of the matrix.

5 Conclusions

We derived the tidal response of a confined coastal aquifer whose conduits exchange water by diffusion with the matrix blocks they surround, and we examined when this exchange makes the aquifer behave as a fractional medium in time. The decay and the lag of each tidal constituent together fix the complex storage capacity of the aquifer at the frequency of that constituent and carry no further information. The ratio of lag to decay measured for one constituent therefore cannot distinguish storage memory from leakage, and the distinction rests on how the capacity changes from one constituent to another. Passive storage exchange and leakage never allow the lag to grow faster than the amplitude decays, so a ratio above the classical value points to the geometry of the aquifer rather than to its storage. Diffusive exchange with blocks of any sizes also places a lower limit on the ratio, slightly below the value of a medium of half order. When the block sizes follow a power law, the aquifer behaves as a fractional medium over a band of frequencies, and its order is set by the exponent of the size distribution and lies between one half and one. A fractional medium started from rest retains the memory of its start longer than a classical one. In the unconfined aquifer, the period mean of the squared head is conserved whether or not memory is present, and the mean rise of the water table depends on the decay rate alone. For a karstic reference aquifer, a month of diurnal and semidiurnal records separates memory from leakage with high confidence, even when the matrix holds much less water than the conduits. Every closed form was verified against independent numerical solvers. Testing these predictions against tidal records from fractured and karstic coastal aquifers, combining leakage and memory in one model, and extending the analysis to two dimensions and to flow with variable density would show how far the fractional description carries over to real coastal aquifers.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this article.

Declaration of generative AI use

During the preparation of this work, the authors used Claude Sonnet 5 solely for the purposes of English grammar, vocabulary refinement, and improving the overall readability of the manuscript. The authors maintain full responsibility for the conceptualization, model development, mathematical derivations, computational execution, and the analysis and interpretation presented in this study.

Code and data availability

This study uses no external data. The analysis code and all of its outputs, including CSV tables of the data behind each figure panel, the nine figures in vector and raster form, plain-text computation reports, animations, and the scripts that regenerate them, are openly available at https://github.com/sandyherho/fractional_tidal_aquifer under the MIT License.

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