



Peer review status:

This is a non-peer-reviewed preprint submitted to EarthArXiv.

Semi-analytic reference solutions for 3D controlled-source electromagnetics: a conductive sphere beneath Gaussian topography in a layered earth

Klaus Lippert*

Independent researcher

Abstract

Three-dimensional forward codes for controlled-source and transient electromagnetics are verified mostly by comparison with one another when the model contains topography and a compact body in a layered host. We present `terrasphere_csem`, an open-source semi-analytic reference for this coupled problem. A Gaussian hill or depression forms the surface of a horizontally layered earth, and a conductive, optionally magnetically permeable sphere lies in any layer below. Grounded wires, grounded electric dipoles and current loops on the flat ground serve as sources; all components of the electric and magnetic fields are computed on the terrain, in the frequency domain and as transients.

The formulation combines plane-wave representations of the layered earth with vector spherical-wave expansions of arbitrary degree and order for the sphere, connected through identities from scattering theory. Topography enters through an expansion in the terrain height formulated entirely in the magnetic field: at each order the problem reduces to a flat-surface layered-earth solution with a low-rank correction for the sphere.

Particular emphasis is placed on independent verification and reproducibility. Each part is tested against a solution it was not derived from: closed-form limits, an independent direct-current solution, a one-dimensional code and electromagnetic reciprocity. Coupled terrain–sphere responses are compared with a three-dimensional finite-element code in the frequency domain; the transform to transients is validated separately. Numerical examples quantify the non-additive interaction of topography and the buried sphere.

For terrain of moderate slope, the framework provides an independently verified, reproducible reference for assessing the accuracy and convergence of three-dimensional CSEM forward solvers in models that combine topography, layering and a compact conductive target, and of transient solvers for such targets in a layered earth without topography. The software is available under the MIT licence.

1 Introduction

Topography distorts controlled-source and transient electromagnetic (EM) data: with grounded wires or loops its effect is largest at early times and decays afterwards (Hördt and Müller, 2000; Zhu et al., 2026), and a one-dimensional interpretation of transients over a modest Gaussian hill already produces resistive artefacts beneath the summit (Falk et al., 2026). Exploration targets are compact bodies in a layered host, with the terrain on top of them.

Three-dimensional forward codes that honour topography and arbitrary conductivity exist for controlled-source EM (CSEM) and transient EM (TEM): tetrahedral finite elements (Rochlitz et al., 2019; Rulff et al., 2021), spectral elements (Weiss et al., 2023) and finite-difference time-domain schemes (e.g. Liu et al., 2019; Zhu et al., 2026). Their verification rests mostly on comparisons between codes. Semi-analytic references exist for parts of the problem only.

*ORCID <https://orcid.org/0000-0002-7662-9548>; e-mail klaus.lippert.geo@gmail.com.

Compact bodies. A conductive sphere in a conductive host is one of the few three-dimensional targets for which the induction problem separates. Wait (1951) and Wait and Spies (1969) solved the permeable sphere in a uniform transient field, Nabighian (1970) added a dipole source and all multipoles in free space, and Singh (1973) a conductive full space. Lee (1975) gave the first solution with all degrees for a sphere in a half-space or below an overburden, excited by a coaxial loop; Lee (1981) formulated a sphere in a layered earth with arbitrary sources but evaluated it for the lowest degree with static kernels, and Lee (1983) turned this into a late-time formula for coaxial and offset loops. In electromagnetic scattering theory, Kristensson (1980) treated a scatterer below a plane interface with its transition (T) matrix, Bobbert and Vlieger (1986) the sphere on a substrate, and Wang and Chew (2004) the buried sphere with all multipoles and the multiple reflections between sphere and surface; Frezza et al. (2013) and Frezza and Mangini (2016) treat plane-wave incidence on a sphere in a lossy half-space. Desmarais and Smith (2016) treated a sphere under a conductive overburden with a dipole series, Vallée and Moussaoui (2023) a sphere in a layered earth for airborne dipole sources, with cylindrical-wave expansions and a reflection matrix as here but a numerically projected incident field and six degrees; their open code MTSphere applies it to magnetotellurics (Vallée and Moussaoui, 2026); and Zhao and Hong (2026) a sphere beneath a multilayered formation at high frequencies. Among these semi-analytic sphere formulations, Zamani et al. (2018) also consider a deformed interface: a dielectric sphere below a slightly rough surface of a lossless half-space, illuminated by a plane wave and solved to first order in the roughness with the small-perturbation method, all interactions between sphere and surface summed into linear systems. Three-dimensional objects below a locally rough surface have also been treated numerically, by a method of moments with a numerically computed Green’s function of the rough half-space (Altuncu, 2015). None of them treats the quasi-static regime of a controlled source over a conductive layered earth with topography.

Topography. Closed-form or semi-analytic references with topography are confined to plane waves over periodic two-dimensional surfaces (Ohshiman and Rikitake, 1985; Schwalenberg and Edwards, 2004), Rayleigh–Fourier expansions (Jiracek et al., 1989; Martinelli and Osella, 1997), a local curvature expansion of the magnetotelluric impedance (Vallianatos, 2002) and special direct-current (DC) geometries (Van Nostrand and Cook, 1966). For a controlled source over a localised three-dimensional terrain, and for any transient response with topography, we found none; the closest work is a coil above a conductor with a localised surface deformation in eddy-current testing (Caire et al., 2013; Prémel and Granet, 2018). Boundary-perturbation methods supply the mathematics: they expand the field in powers of the height of a deformed interface (Craig and Sulem, 1993; Bruno and Reitich, 1993a; Nicholls and Reitich, 2001); the expansion is analytic in the height (Nicholls and Reitich, 2003), converges for slopes below one on unbounded surfaces (Hu and Nicholls, 2010), and extends to vector Maxwell problems in layered media (Bruno and Reitich, 1993b; Hong and Nicholls, 2018), so far for periodic gratings and water waves; to first order (the small-perturbation method) it has been combined with a buried sphere (Zamani et al., 2018).

This paper combines the two lines into one reference for controlled sources in the quasi-static regime, to high order in the terrain height: a Gaussian hill or depression, $h(\mathbf{x}) = H e^{-|\mathbf{x}|^2/2s^2}$, forms the surface of the top layer of a horizontally layered earth, and a conductive, optionally permeable sphere lies in any layer below the terrain. It contributes:

1. a formulation in which the sphere is described by spherical vector waves of all degrees and orders and the layered earth by plane waves, connected by four identities from the scattering literature (Chang and Mei, 1980; Pogorzelski and Lun, 1976; Han et al., 2008; Devaney and Wolf, 1974);
2. a field expansion of the terrain posed in the magnetic field alone, in which each order is a flat-surface problem over the layered earth with the sphere, solved per horizontal wavenumber in closed form plus a correction of low rank for the sphere;
3. all components of $\mathbf{E}$, $\mathbf{H}$ and $\partial_t \mathbf{B}$ on the terrain for grounded wires, grounded dipoles and loops,

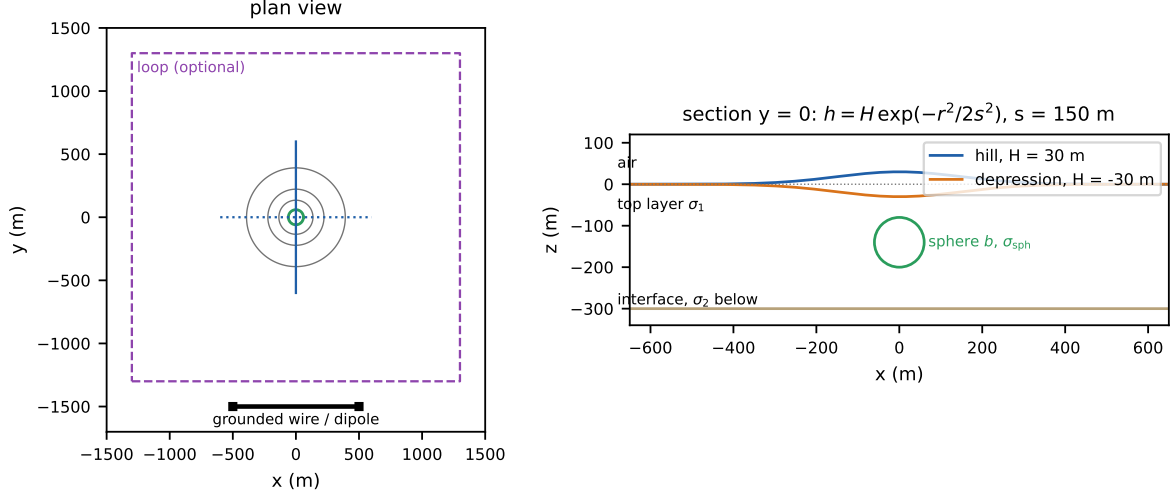


Figure 1: Model: a Gaussian hill (or depression) of height H and width s forms the surface of the top layer of a horizontally layered earth; a sphere of radius b lies in any layer below; transmitters (grounded wire, grounded dipole, rectangular or circular loop) lie on the flat ground off the hill; receivers on the terrain.

in the frequency domain and as transients, with a scaling that keeps the sphere computation finite from the DC to the skin-depth limit and a terrain expansion kept stable by an analytic window, a spectral low-pass and a box extrapolation;

4. a verification chain in which each part is checked against a solution it was not derived from, given in full in Appendix A, and an open implementation, `terrasphere_csem`, with a test suite that reproduces every number of it.

We work in the quasi-static approximation, with horizontal isotropic layers, a sphere entirely inside one layer and entirely below the level of the flat ground, and terrain whose slopes stay below about 20° . Hill and depression follow from the same coefficients, and the first order of the expansion is the sensitivity of every component to the terrain.

Section 2 gives the formulation, Section 3 its numerical realisation and Section 4 a summary of the verification. Sections 5 and 6 show results for a sphere in a layered earth and a sphere below a hill. Section 7 compares the series with independent codes, `empymod` for the flat layered earth in the time domain and the three-dimensional finite-element code `elfe3D` for terrain and sphere, and Section 8 discusses limits and extensions. Appendix A documents every test of the verification chain; Appendices B–F give derivations and numerical details.

2 Formulation

2.1 Model and conventions

Time dependence is $e^{-i\omega t}$, displacement currents are neglected, and the permeability is μ_0 everywhere except in the sphere. We use a right-handed frame (x, y, z) with z pointing up and horizontal position $\mathbf{x} = (x, y)$; depths are $d = -z$. The ground surface is $z = h(\mathbf{x}) = Hf(\mathbf{x})$ with $f = e^{-|\mathbf{x}|^2/2s^2}$, a hill for $H > 0$ and a depression for $H < 0$ (Fig. 1). Below it lie layers $j = 1, \dots, J$ with conductivities σ_j , separated by horizontal interfaces at depths $d_1 = 0 < d_2 < \dots < d_J$; layer J is the basement, and the air (layer 0) is an insulator. The terrain belongs to the top layer: all interfaces lie deeper than the bottom of the depression. The sphere has radius b , conductivity σ_{sph} and permeability $\mu_{\text{sph}} = \mu_r \mu_0$; its centre lies at depth d_c inside layer ℓ , the sphere touches no interface, and it lies entirely below $z = 0$. Wavenumbers are $k_j^2 = i\omega\mu_0\sigma_j$ with $\text{Im } k_j \geq 0$, $k = k_\ell$ is that of the sphere

layer and $k_{\text{sph}}^2 = i\omega\mu_{\text{sph}}\sigma_{\text{sph}}$ that of the sphere. For a horizontal wavenumber vector $\boldsymbol{\xi}$, $\lambda = |\boldsymbol{\xi}|$, the vertical wavenumber in layer j is $u_j = (\lambda^2 - k_j^2)^{1/2}$ with $\text{Re } u_j > 0$. Fields are $\mathbf{E}$ (V/m) and $\mathbf{H}$ (A/m), $\partial_t \mathbf{B} = -i\omega\mu_0 \mathbf{H}$; in every conductor

$$\nabla \times \mathbf{E} = i\omega\mu_0 \mathbf{H}, \quad \nabla \times \mathbf{H} = \sigma \mathbf{E} \quad \Rightarrow \quad (\nabla^2 + k^2) \mathbf{H} = 0, \quad \nabla \cdot \mathbf{H} = 0, \quad (1)$$

and in the air, away from the sources, $\nabla \times \mathbf{H} = 0$ and $\nabla \cdot \mathbf{H} = 0$.

The transmitters lie on the flat ground, far enough from the hill that the terrain height is negligible along them (below 10^{-13} m in all our examples). All of them are sums of horizontal electric dipoles (HED): a grounded wire is a Gauss–Legendre sum of HEDs along the wire (the electrode terms are contained), a grounded dipole is a single HED of moment IL or a short wire, and a rectangular or circular loop is a closed polygon of wires; for the circle we use an area-preserving polygon with 256 corners. Receivers lie on the terrain, on the earth side, and record all three components of $\mathbf{E}$, $\mathbf{H}$ and $\partial_t \mathbf{B}$.

2.2 The layered earth

Outside the sphere the field in every layer is a superposition of plane waves. Horizontal interfaces do not mix the TE_z and TM_z families, and each family is governed by one scalar: H_z for TE_z and $J_z = \sigma E_z$ for TM_z , both continuous across interfaces. From Maxwell's equations the remaining components of a mode $\phi J_m(\lambda\rho)e^{im\varphi}$, $\phi = \phi(z)$, are

$$\begin{aligned} \text{TE}_z : \quad \mathbf{E} &= -\frac{i\omega\mu_0}{\lambda^2} \hat{\mathbf{z}} \times \nabla_h H_z, & \mathbf{H}_h &= \frac{1}{\lambda^2} \nabla_h \partial_z H_z, \\ \text{TM}_z : \quad \mathbf{H} &= -\frac{1}{\lambda^2} \hat{\mathbf{z}} \times \nabla_h J_z, & \mathbf{E}_h &= \frac{1}{\sigma\lambda^2} \nabla_h \partial_z J_z, & E_z &= J_z/\sigma. \end{aligned} \quad (2)$$

Continuity of the tangential fields requires ϕ and $\xi_j \phi'$ continuous, $\xi_j = 1$ (TE) or $1/\sigma_j$ (TM). For a wave in layer i meeting layer j the Fresnel coefficient is

$$r_{ij} = \frac{\xi_i u_i - \xi_j u_j}{\xi_i u_i + \xi_j u_j}, \quad r_{j0}^{\text{TM}} = -1 \text{ (insulating air: } J_z = 0), \quad (3)$$

and the reflection coefficients of the stacks above and below layer j , seen from inside the layer at its top resp. bottom interface, follow from the recursions (Ward and Hohmann, 1988)

$$R_j^\uparrow = \frac{r_{j,j-1} + R_{j-1}^\uparrow e^{-2u_{j-1}D_{j-1}}}{1 + r_{j,j-1} R_{j-1}^\uparrow e^{-2u_{j-1}D_{j-1}}}, \quad R_j^\downarrow = \frac{r_{j,j+1} + R_{j+1}^\downarrow e^{-2u_{j+1}D_{j+1}}}{1 + r_{j,j+1} R_{j+1}^\downarrow e^{-2u_{j+1}D_{j+1}}}, \quad (4)$$

with $R_0^\uparrow = 0$, $R_j^\downarrow = 0$ and $D_j = d_{j+1} - d_j$. A source at depth d_{tx} in layer j with direct amplitudes A^+ (towards the surface) and A^- (towards depth) produces, with $\tilde{R}^\uparrow = R_j^\uparrow e^{-2u_j h_t}$ and $\tilde{R}^\downarrow = R_j^\downarrow e^{-2u_j h_b}$ (h_t , h_b the distances from the source to the top and bottom of its layer), the total up- and down-going amplitudes at the source depth

$$U = \frac{A^+ + \tilde{R}^\downarrow A^-}{1 - \tilde{R}^\uparrow \tilde{R}^\downarrow}, \quad D = \frac{A^- + \tilde{R}^\uparrow A^+}{1 - \tilde{R}^\uparrow \tilde{R}^\downarrow}, \quad (5)$$

from which the field anywhere follows by transmission layer by layer. In the top layer near the surface every field that is excited at or above the surface is a down-going wave plus its reflection from the layers below,

$$\phi(z) \propto e^{u_1 z} + \rho e^{-u_1 z}, \quad \rho = R_1^\downarrow e^{-2u_1 d_2}, \quad (6)$$

with ρ_{TE} and ρ_{TM} for the two families; for a homogeneous ground $\rho = 0$.

2.3 Sources and the flat-earth surface field

With $G = e^{ikR}/(4\pi R) = \int_0^\infty (\lambda/4\pi u) J_0(\lambda\rho') e^{-u|z-z_{tx}|} d\lambda$ the scalars of point dipoles in a homogeneous medium are, for a horizontal electric dipole of moment $p = I dl$ and direction $\hat{\mathbf{d}}$ at azimuth ψ , for a vertical and a horizontal magnetic dipole,

$$\begin{aligned} \text{HED: } H_z &= p(d_y \partial_x - d_x \partial_y)G, & J_z &= p \partial_z(\hat{\mathbf{d}} \cdot \nabla)G, \\ \text{VMD: } H_z &= m_z \lambda^2 G, & J_z &= 0, \\ \text{HMD: } H_z &= m_h \partial_z(\hat{\mathbf{d}} \cdot \nabla)G, & J_z &= k^2 m_h (d_y \partial_x - d_x \partial_y)G, \end{aligned} \quad (7)$$

where $\lambda^2 G$ stands for the kernel $\lambda^3/(4\pi u)$ and the derivatives act on $J_0(\lambda\rho')e^{-u|z-z_{tx}|}$. A horizontal circular loop of radius a carrying I is purely TE_z with

$$H_z = \frac{Ia}{2} \int_0^\infty \frac{\lambda^2}{u} J_1(\lambda a) J_0(\lambda\rho') e^{-u|z-z_{tx}|} d\lambda.$$

To couple a source to the sphere, these scalars are referred to the vertical axis through the sphere centre by the addition theorems of Neumann and Graf ([Abramowitz and Stegun, 1964](#)) (Appendix B); one λ -integral per azimuthal order m remains.

The terrain expansion needs the flat-earth field of the sources on the surface itself, where the spectral kernels do not decay and a digital Hankel filter integrates the galvanic part poorly. For a homogeneous ground we use closed forms of [Ward and Hohmann \(1988\)](#) for E_x , E_y and H_z and two smooth Hankel integrals for the horizontal magnetic field (Appendix D). For a layered ground we add the field reflected by the layers: the difference between the layered and the half-space kernels contains only the reflected waves and decays like $e^{-2u_1 d_2}$, so the digital filter integrates it accurately, and the non-decaying source terms cancel exactly when both kernels are evaluated at the same abscissae. By symmetry the surface fields of a unit x -directed HED have the form

$$E_x = a_0 + a_2 \cos 2\varphi, \quad E_y = a_2 \sin 2\varphi, \quad H_x = c_2 \sin 2\varphi, \quad H_y = c_0 - c_2 \cos 2\varphi, \quad H_z = d \sin \varphi \quad (8)$$

in polar coordinates (r, φ) of the receiver, so the five radial functions of the layer correction are tabulated from receivers on two axes and splined in $\log r$. Wires, dipoles and loops are sums over their HEDs, rotated by ψ .

2.4 Spherical vector waves

The sphere is described in a frame with origin at its centre and z pointing up as in the global frame, with spherical coordinates (r, θ, φ) or cylindrical coordinates (ρ, φ, z) about the vertical axis through the centre; a point at depth d has $z = d_c - d$ there.

With the orthonormal spherical harmonics

$$Y_{nm}(\theta, \varphi) = N_{nm} P_n^m(\cos \theta) e^{im\varphi}, \quad N_{nm} = \sqrt{\frac{(2n+1)(n-m)!}{4\pi(n+m)!}},$$

with the factor $(-1)^m$ included in P_n^m (Condon–Shortley phase), and $\psi_{nm} = z_n(kr)Y_{nm}$ with $z_n = j_n$ (regular, superscript (1)) or $z_n = h_n^{(1)}$ (outgoing, superscript (3)), the vector spherical wave functions are ([Stratton, 1941](#); [Bohren and Huffman, 1983](#))

$$\mathbf{M}_{nm} = \nabla \times (\mathbf{r} \psi_{nm}) = z_n(kr) \left[\hat{\boldsymbol{\theta}} \frac{im}{\sin \theta} Y_{nm} - \hat{\boldsymbol{\varphi}} \partial_\theta Y_{nm} \right], \quad (9)$$

$$\mathbf{N}_{nm} = \frac{1}{k} \nabla \times \mathbf{M}_{nm} = \hat{\mathbf{r}} \frac{n(n+1)z_n(kr)}{kr} Y_{nm} + \frac{[kr z_n(kr)]'}{kr} \left[\hat{\boldsymbol{\theta}} \partial_\theta Y_{nm} + \hat{\boldsymbol{\varphi}} \frac{im}{\sin \theta} Y_{nm} \right]. \quad (10)$$

$\mathbf{M}$ has no radial electric component (TE with respect to r , magnetic multipoles), $\mathbf{N}$ has (TM, electric multipoles). For $\mathbf{E} = \sum a_{nm} \mathbf{M}_{nm} + b_{nm} \mathbf{N}_{nm}$ the magnetic field is $\mathbf{H} = (k/i\omega\mu_0) \sum a_{nm} \mathbf{N}_{nm} +$

$b_{nm}\mathbf{M}_{nm}$. Two projections will be used repeatedly: because $\mathbf{r} \cdot \mathbf{M}_{nm} = 0$ and $\mathbf{r} \cdot \mathbf{N}_{nm} = n(n+1)z_n(kr)Y_{nm}/k$,

$$b_{nm} = \frac{k \langle \mathbf{r} \cdot \mathbf{E}, Y_{nm} \rangle}{n(n+1)j_n(kr)}, \quad a_{nm} = \frac{i\omega\mu_0 \langle \mathbf{r} \cdot \mathbf{H}, Y_{nm} \rangle}{n(n+1)j_n(kr)}, \quad (11)$$

where $\langle \cdot, Y_{nm} \rangle$ is the projection on the sphere of radius r .

2.5 Plane-wave spectra: four identities

The sphere is described in spherical waves about its centre, the layered earth in plane waves. Four identities translate between the two: (I) and (IV) write the outgoing scalar and vector spherical waves of the sphere as spectra of plane waves, so that the layers can reflect them and the receivers can evaluate them; (II) and (III) write a scalar and a vector plane wave as a series of regular spherical waves, so that the field of the sources and the field reflected by the layers can be matched at the sphere. Each identity is exact and taken from the literature cited with it.

A plane wave in the sphere layer has wave vector $\mathbf{k} = k\hat{\mathbf{k}}$ with $\hat{\mathbf{k}} = (\sin \alpha \cos \beta, \sin \alpha \sin \beta, \cos \alpha)$, horizontal wavenumber $\lambda = k \sin \alpha$ and vertical wavenumber $k_z = k \cos \alpha = \pm iu$ (upper sign: up-going, $e^{ik_z z} = e^{-uz}$ for $z > 0$). The “angle” α is complex; we fix the branch by $\sin \alpha = \lambda/k$ and $\cos \alpha = \pm iu/k$, and define the unit vectors $\hat{\boldsymbol{\theta}}(\alpha, \beta) = (\cos \alpha \cos \beta, \cos \alpha \sin \beta, -\sin \alpha)$, $\hat{\boldsymbol{\varphi}}(\beta) = (-\sin \beta, \cos \beta, 0)$ as the analytic continuation of the real ones. Cylindrical vector plane waves of azimuthal order m with TE_{*z*} (electric field horizontal) and TM_{*z*} polarisation are

$$\mathbf{W}_m^{\text{TE}}(\lambda, \pm) = \frac{1}{2\pi i^m} \int_0^{2\pi} e^{im\beta} \hat{\boldsymbol{\varphi}}(\beta) e^{i\mathbf{k} \cdot \mathbf{r}} d\beta, \quad \mathbf{W}_m^{\text{TM}}(\lambda, \pm) = \frac{1}{2\pi i^m} \int_0^{2\pi} e^{im\beta} \hat{\boldsymbol{\theta}}(\alpha, \beta) e^{i\mathbf{k} \cdot \mathbf{r}} d\beta. \quad (12)$$

The β -integrals are elementary (Jacobi–Anger) and give, with $J_m = J_m(\lambda\rho)$, J'_m its derivative with respect to the argument and $\mathcal{E} = e^{im\varphi} e^{ik_z z}$,

$$\mathbf{W}_m^{\text{TE}} = \left[-\frac{m}{\lambda\rho} J_m \hat{\boldsymbol{\rho}} - iJ'_m \hat{\boldsymbol{\varphi}} \right] \mathcal{E}, \quad \mathbf{W}_m^{\text{TM}} = \left[-i \cos \alpha J'_m \hat{\boldsymbol{\rho}} + \cos \alpha \frac{m}{\lambda\rho} J_m \hat{\boldsymbol{\varphi}} - \sin \alpha J_m \hat{\mathbf{z}} \right] \mathcal{E}, \quad (13)$$

$$H_z[\mathbf{W}_m^{\text{TE}}] = \frac{\lambda}{\omega\mu_0} J_m \mathcal{E}, \quad E_z[\mathbf{W}_m^{\text{TM}}] = -\frac{\lambda}{k} J_m \mathcal{E}. \quad (14)$$

The last line is what couples the vector waves to the scalar description of the layered earth (Section 2.2). The four identities are:

(I) Weyl representation of the outgoing scalar wave. For $z \gtrless 0$,

$$h_n^{(1)}(kr) P_n^m(\cos \theta) e^{im\varphi} = \frac{i^{m-n-1}}{k} \int_0^\infty \frac{\lambda}{u} P_n^m\left(\pm \frac{iu}{k}\right) J_m(\lambda\rho) e^{im\varphi} e^{-u|z|} d\lambda. \quad (15)$$

Its angular-spectrum form is eq. (A13) of [Devaney and Wolf \(1974\)](#). With the β -integral done it is eq. (11) with the closed amplitude (21) of [Chang and Mei \(1980\)](#), converted to our time convention; their factor $[\text{sgn}(z)]^{n-m}$ is the sign change of the argument of P_n^m in the lower half-space.

(II) Scalar cylindrical wave in regular spherical waves.

$$J_m(\lambda\rho) e^{im\varphi} e^{ik_z z} = \sum_{n \geq |m|} (2n+1) i^{n-m} \frac{(n-m)!}{(n+m)!} P_n^m\left(\frac{k_z}{k}\right) j_n(kr) P_n^m(\cos \theta) e^{im\varphi}, \quad (16)$$

which is eq. (27) of [Pogorzelski and Lun \(1976\)](#) with $\cos \alpha = k_z/k$. For $m = 1$ it is also Appendix 8 of [Lee \(1981\)](#).

(III) **Vector cylindrical waves in regular spherical vector waves.** This relation was given for complex k by [Pogorzelski and Lun \(1976\)](#), eqs. (28)–(30), and in complex form by [Han et al. \(2008\)](#), eqs. (15)–(19) and (35); in our normalisation $\mathbf{W}_m^q = \sum_{n \geq \max(|m|, 1)} [A_{nm}^q \mathbf{M}_{nm}^{(1)} + B_{nm}^q \mathbf{N}_{nm}^{(1)}]$, $q = \text{TE, TM}$, with

$$\begin{aligned} A_{nm}^{\text{TE}} &= -\frac{4\pi i^{n-m} N_{nm}}{n(n+1)} \partial_\theta P_n^m(\alpha), & B_{nm}^{\text{TE}} &= -\frac{4\pi i^{n-m} N_{nm}}{n(n+1)} \frac{m P_n^m(\cos \alpha)}{\sin \alpha}, \\ A_{nm}^{\text{TM}} &= -\frac{4\pi i^{n+1-m} N_{nm}}{n(n+1)} \frac{m P_n^m(\cos \alpha)}{\sin \alpha}, & B_{nm}^{\text{TM}} &= -\frac{4\pi i^{n+1-m} N_{nm}}{n(n+1)} \partial_\theta P_n^m(\alpha), \end{aligned} \quad (17)$$

where $\partial_\theta P_n^m(\alpha) = [n \cos \alpha P_n^m - (n+m) P_{n-1}^m] / \sin \alpha$ is the θ -derivative at the complex angle; Appendix C gives the conversion to the notation of [Han et al. \(2008\)](#).

[Zhao and Hong \(2026\)](#) give the same transformation for down-going propagating waves (their eqs. 26–27); [Vallée and Moussaoui \(2023\)](#) obtain the coefficients numerically. The coefficients are analytic in k_z , and the relation holds unchanged for the evanescent waves of the layered earth, $\cos \alpha = \pm i u / k$ with complex k ; Appendix A.1.5 checks both. For $m = 0$ and TE the result reduces to (II) with $m = 1$ (the coaxial case of [Lee, 1981](#)); for $m \neq 0$ a TE_z plane wave excites both magnetic ($\mathbf{M}$) and electric ($\mathbf{N}$) multipoles, which is the origin of Lee’s four coefficient families.

(IV) **Plane-wave spectra of the outgoing spherical vector waves.** For $z \geq 0$,

$$\mathbf{M}_{nm}^{(3)} = \int_0^\infty [a_{\text{TE}}^M \mathbf{W}_m^{\text{TE}} + a_{\text{TM}}^M \mathbf{W}_m^{\text{TM}}] d\lambda, \quad \mathbf{N}_{nm}^{(3)} = \int_0^\infty [a_{\text{TE}}^N \mathbf{W}_m^{\text{TE}} + a_{\text{TM}}^N \mathbf{W}_m^{\text{TM}}] d\lambda, \quad (18)$$

$$\begin{aligned} a_{\text{TE}}^M &= -i^{m-n} N_{nm} \frac{\lambda}{iku} \partial_\theta P_n^m(\alpha_\pm), & a_{\text{TM}}^M &= +i^{m-n} N_{nm} \frac{\lambda}{iku} \frac{im P_n^m(\cos \alpha_\pm)}{\sin \alpha}, \\ a_{\text{TE}}^N &= i^{1-n+m} N_{nm} \frac{\lambda}{iku} \frac{im P_n^m(\cos \alpha_\pm)}{\sin \alpha}, & a_{\text{TM}}^N &= i^{1-n+m} N_{nm} \frac{\lambda}{iku} \partial_\theta P_n^m(\alpha_\pm), \end{aligned} \quad (19)$$

with $\cos \alpha_\pm = \pm i u / k$. This is the angular-spectrum representation of the outgoing electromagnetic multipole fields given by [Devaney and Wolf \(1974\)](#), their eqs. (4.14), (B1a) and (B1b), and used for a buried scatterer by [Kristensson \(1980\)](#), his eqs. (29) and (30), with the β -integral of (12) carried out; [Vallée and Moussaoui \(2023\)](#), eq. (11), give an equivalent expansion obtained from derivative relations.

All four identities are verified numerically to 10^{-14} (Appendix A.1.5).

2.6 The sphere

For an incident regular field $\sum a_{nm} \mathbf{M}_{nm}^{(1)} + b_{nm} \mathbf{N}_{nm}^{(1)}$ the sphere scatters $\sum (-b_{nm} a_{nm}) \mathbf{M}_{nm}^{(3)} + (-a_{nm} b_{nm}) \mathbf{N}_{nm}^{(3)}$ with the Mie coefficients for unequal permeabilities ([Bohren and Huffman, 1983](#), eqs. 4.56–4.57), written with the Riccati–Bessel functions $\psi_n(z) = z j_n(z)$, $\xi_n(z) = z h_n^{(1)}(z)$, $x = kb$, $m = k_{\text{sph}} / k$:

$$a_n = \frac{m \psi_n(mx) \psi'_n(x) - \mu_r \psi_n(x) \psi'_n(mx)}{m \psi_n(mx) \xi'_n(x) - \mu_r \xi_n(x) \psi'_n(mx)}, \quad b_n = \frac{\mu_r \psi_n(mx) \psi'_n(x) - m \psi_n(x) \psi'_n(mx)}{\mu_r \psi_n(mx) \xi'_n(x) - m \xi_n(x) \psi'_n(mx)}. \quad (20)$$

Both vanish identically for $\sigma_{\text{sph}} = \sigma_h$, $\mu_r = 1$, which is the basis of the null test of Appendix A.1.1. Numerically they are evaluated as $a_n = G_n^a J_n$, $b_n = G_n^b J_n$ with $J_n = j_n(x) / h_n^{(1)}(x)$ and

$$G_n^a = \frac{m q_\psi - \mu_r D_n}{m q_\xi - \mu_r D_n}, \quad G_n^b = \frac{\mu_r q_\psi - m D_n}{\mu_r q_\xi - m D_n}, \quad q_\psi = \frac{\psi'_n(x)}{\psi_n(x)}, \quad q_\xi = \frac{\xi'_n(x)}{\xi_n(x)}, \quad D_n = \frac{\psi'_n(mx)}{\psi_n(mx)}, \quad (21)$$

where D_n is obtained by downward recurrence, q_ψ and q_ξ from the ratios j_n / j_{n-1} (downward continued fraction) and h_n / h_{n-1} (upward), and J_n / x^{2n+1} from the same ratios, so that nothing over- or underflows ([Bohren and Huffman, 1983](#), ch. 4).

2.7 Assembly

Let $\mathbf{c}$ be the vector of regular coefficients (a_{nm}, b_{nm}) at the sphere, $\mathbf{s}$ that of the outgoing coefficients, ordered $[(n = 1..N, \mathbf{M}), (n = 1..N, \mathbf{N})]$ for a fixed order m . Since the layering is invariant under rotations about the sphere axis, the orders decouple and

$$\mathbf{c} = \mathbf{c}^p + \mathbf{R} \mathbf{s}, \quad \mathbf{s} = -\mathbf{T} \mathbf{c} \quad \implies \quad (\mathbf{I} + \mathbf{T} \mathbf{R}) \mathbf{s} = -\mathbf{T} \mathbf{c}^p, \quad (22)$$

with $\mathbf{T} = \text{diag}(b_1..b_N, a_1..a_N)$. Solving (22) directly corresponds to the inverse $[\mathbf{I} - \mathbf{Q}]^{-1}$ of Wang and Chew (2004) (their eqs. 28 and 29), whereas Zhao and Hong (2026) iterate the multiple reflections (their eq. 55); here the system has dimension $2(N - \max(|m|, 1) + 1) \leq 2N$ per order and is always solved directly. The three ingredients are built from the identities:

Primary coefficients. For each source, the amplitudes of Table 3 are propagated with (5) to the sphere centre depth, giving the up- and down-going scalars $\Phi_q^\pm(\lambda)$ of the field $\Phi_q^+ e^{-uz} + \Phi_q^- e^{uz}$ in the sphere layer. Each is a cylindrical vector wave with amplitude Φ/c_q , where by (14) $c_{\text{TE}} = \lambda/\omega\mu_0$ and $c_{\text{TM}} = -\sigma_h \lambda/k$; with (17),

$$c_{nm}^p|_{\mathbf{M}} = \int_0^\infty \sum_q \frac{1}{c_q} [\Phi_q^+ A_{nm}^q(\alpha_+) + \Phi_q^- A_{nm}^q(\alpha_-)] d\lambda, \quad c_{nm}^p|_{\mathbf{N}} = \text{same with } B_{nm}^q. \quad (23)$$

Reflection operator. A unit outgoing wave (n', X) , $X \in \{\mathbf{M}, \mathbf{N}\}$, has by (18) the direct scalars $A_q^\pm = c_q a_q^X(n', \alpha_\pm)$. The stacks above and below return, by (5) applied to the sphere as a source at its own centre ($\tilde{R}^\uparrow = R_\ell^\uparrow e^{-2uh_\uparrow}$, $\tilde{R}^\downarrow = R_\ell^\downarrow e^{-2uh_\downarrow}$ with the distances $h_{\uparrow, \downarrow}$ from the centre to the interfaces), the incoming waves

$$D_q = \frac{\tilde{R}_q^\uparrow (A_q^+ + \tilde{R}_q^\downarrow A_q^-)}{1 - \tilde{R}_q^\uparrow \tilde{R}_q^\downarrow} \text{ (down-going),} \quad U_q = \frac{\tilde{R}_q^\downarrow (A_q^- + \tilde{R}_q^\uparrow A_q^+)}{1 - \tilde{R}_q^\uparrow \tilde{R}_q^\downarrow} \text{ (up-going),} \quad (24)$$

which include all multiple reflections between the two stacks, and

$$\mathbf{R}_{(n, \mathbf{M}), (n', X)} = \int_0^\infty \sum_q \frac{1}{c_q} [D_q A_{nm}^q(\alpha_-) + U_q A_{nm}^q(\alpha_+)] d\lambda, \quad \mathbf{R}_{(n, \mathbf{N}), (n', X)} = \text{same with } B_{nm}^q. \quad (25)$$

For $m = 0$ only the TE family couples the $\mathbf{M}$ waves and only TM the $\mathbf{N}$ waves; for $m \neq 0$ all four blocks are non-zero. Unlike the corresponding operators of Vallée and Moussaoui (2023) (their C), Wang and Chew (2004) (a single interface, their eq. 29) and Zhao and Hong (2026) (the stack above the sphere, their eqs. 43, 44 and 65), $\mathbf{R}$ contains the stacks above and below the sphere layer and the multiple reflections between them, so the sphere may lie in any layer.

Receivers. The scattered field at a receiver is obtained by treating the sphere as a source: its direct scalars $\sum_{n', X} s_{n'X} c_q a_q^X(n', \alpha_\pm)$ are propagated with (5) to the receiver depth (in any layer) and evaluated with (2). Point receivers give $\mathbf{E} \cdot \hat{\mathbf{d}}$, $\mathbf{H} \cdot \hat{\mathbf{n}}$ or $-i\omega\mu_0 \mathbf{H} \cdot \hat{\mathbf{n}}$; wires and polygonal loops are Gauss–Legendre sums of $\mathbf{E} \cdot d\mathbf{l}$, and a horizontal circular receiver loop of radius a centred at (ρ_0, φ_0) uses Faraday’s law with the disk integral $\iint J_m(\lambda\rho) e^{im\varphi} dA = (2\pi a/\lambda) J_1(\lambda a) J_m(\lambda\rho_0) e^{im\varphi_0}$ of a mode. Receivers in the depth band of the sphere ($|z| \leq b$), where the spectral representation does not converge, are evaluated directly with (9)–(10) as $\sum s[\mathbf{M}^{(3)}, \mathbf{N}^{(3)}] + \sum (\mathbf{R} \mathbf{s})[\mathbf{M}^{(1)}, \mathbf{N}^{(1)}]$, valid for $b < r < \min(h_\uparrow, h_\downarrow)$. The 1-D (primary) field at the receivers is computed from the same kernels with the modes referred to each source’s own axis ($m = 0, \pm 1$) and Key’s 401-point Hankel filter (Key, 2009).

2.8 Reciprocity

The scattered response obeys reciprocity: for electric dipoles at A and B with directions $\hat{\mathbf{d}}_A$, $\hat{\mathbf{d}}_B$, $\hat{\mathbf{d}}_B \cdot \mathbf{E}_B(p_A) = \hat{\mathbf{d}}_A \cdot \mathbf{E}_A(p_B)$, and for a magnetic dipole at B , $\hat{\mathbf{d}}_A \cdot \mathbf{E}_A(m_B) = i\omega\mu_0 \hat{\mathbf{n}}_B \cdot \mathbf{H}_B(p_A)$. In our construction the receiver map (sphere $\rightarrow$ receiver) and the source map (source $\rightarrow$ sphere) are built from the dual identities (IV) and (III); reciprocity is therefore a hard test of both and of the operator $\mathbf{R}$ (Appendix A.1.5).

2.9 Derivatives

The sphere parameters σ_{sph} , b , μ_r enter only $\mathbf{T}$, so from (22)

$$\partial \mathbf{s} = -(\mathbf{I} + \mathbf{T}\mathbf{R})^{-1} \partial \mathbf{T} (\mathbf{c}^p + \mathbf{R}\mathbf{s}), \quad (26)$$

and the receiver map is linear in $\mathbf{s}$. The derivatives of the Mie coefficients (21) follow in closed form from the Riccati–Bessel equation, and those with respect to the horizontal position of the sphere from a shift of the azimuthal orders (Appendix E). The depth d_c enters every exponential factor; its derivative is evaluated by Richardson-extrapolated central differences (closed forms are straightforward but not implemented).

2.10 Time constants

In free space ($k \rightarrow 0$) the denominator of b_n reduces to $z j_{n-1}(z) + (\mu_r - 1) n j_n(z) = 0$, $z = k_{\text{sph}} b$, whose roots $z_{n,\nu}$ give the time constants $\tau_{n,\nu} = \mu_0 \mu_r \sigma_{\text{sph}} b^2 / z_{n,\nu}^2$; for $\mu_r = 1$ these are the zeros of j_{n-1} of Nabighian (1970) ($\pi, 2\pi, \dots$ for $n = 1$). In a conductive host the exterior wave $h_n^{(1)}(kb)$ with $k = \sqrt{i\omega\mu_0\sigma_h}$ is not analytic across the negative imaginary ω -axis, where the time-decaying exponentials live: the response has a branch cut there, and Lee (1981) found no pole-type responses for the degree-one term. In Section 5 we follow the zero of the modal denominator along this axis from free space into the host: for the conductivities scanned it ceases to exist and the late time is governed by the cut. We therefore treat the time domain by integral transforms rather than by a sum of exponentials; the absence of poles off this axis is not proven here.

2.11 Terrain: a boundary-value problem in the magnetic field alone

Because μ is the same on both sides of the ground surface S , all three components of $\mathbf{H}$ are continuous across it. This is the complete set of interface conditions for $\mathbf{H}$: the condition that no current flows into the air, $\mathbf{n} \cdot \mathbf{J} = 0$ on S , follows from it. For any field the normal component of the curl on a surface is the negative surface divergence of the tangential trace, $\mathbf{n} \cdot (\nabla \times \mathbf{H}) = -\nabla_S \cdot (\mathbf{n} \times \mathbf{H})$. The tangential traces of $\mathbf{H}$ on both sides coincide and $\nabla \times \mathbf{H} = 0$ on the air side, hence $\mathbf{n} \cdot \sigma_1 \mathbf{E} = \mathbf{n} \cdot \nabla \times \mathbf{H} = 0$ on the earth side. Counting unknowns confirms the closure: per horizontal wavenumber the earth carries two divergence-free modes and the air one scalar potential, matched by three continuity conditions.

We split the field into a reference field and the field due to the terrain. The reference field $\mathbf{H}_0$ is the field of the sources over the *flat* layered earth, including the sphere (Sections 2.3–2.7). It is continuous across $z = 0$ but has two analytic continuations off the plane: the earth continuation $\mathbf{H}_0^e$, which satisfies (1) with $k = k_1$, and the air continuation $\mathbf{H}_0^a$, which is harmonic. Both are smooth near the hill because the sources are far from it and the sphere lies below $z = 0$. The terrain field is $\mathbf{H}_s^e$ in the earth and $-\nabla\Phi$ in the air, with $\nabla^2\Phi = 0$ and $\Phi \rightarrow 0$ for $z \rightarrow \infty$. The interface condition reads

$$\mathbf{H}_s^e(\mathbf{x}, h) + \nabla\Phi(\mathbf{x}, h) = -[\mathbf{H}_0^e(\mathbf{x}, h) - \mathbf{H}_0^a(\mathbf{x}, h)]. \quad (27)$$

The right-hand side vanishes on the flat surface and is of first order in H .

2.12 Field expansion in the terrain height

Following Bruno and Reitich (1993a) we expand $\mathbf{H}_s^e = \sum_{n \geq 1} H^n \mathbf{h}_n$ and $\Phi = \sum_{n \geq 1} H^n \phi_n$, represent each order by plane waves,

$$\mathbf{h}_n(\mathbf{x}, z) = \int [\hat{\mathbf{h}}_n^\downarrow(\boldsymbol{\xi}) e^{u_1 z} + \hat{\mathbf{h}}_n^\uparrow(\boldsymbol{\xi}) e^{-u_1 z}] e^{i\boldsymbol{\xi} \cdot \mathbf{x}} d^2 \boldsymbol{\xi}, \quad \phi_n(\mathbf{x}, z) = \int \hat{b}_n(\boldsymbol{\xi}) e^{i\boldsymbol{\xi} \cdot \mathbf{x} - \lambda z} d^2 \boldsymbol{\xi}, \quad (28)$$

with $u_1 = (\lambda^2 - k_1^2)^{1/2}$. Here and below $\mathcal{F}[g](\boldsymbol{\xi}) = \int g(\mathbf{x}) e^{-i\boldsymbol{\xi} \cdot \mathbf{x}} d^2 x$, and the hatted amplitudes are spectral densities, $\hat{g} = (2\pi)^{-2} \mathcal{F}[g]$, so that $g = \int \hat{g} e^{i\boldsymbol{\xi} \cdot \mathbf{x}} d^2 \boldsymbol{\xi}$; the up-going part $\hat{\mathbf{h}}_n^\uparrow$ is the reflection

of the down-going one by the layers below the top layer (6) and by the sphere (Section 2.14). Taylor-expanding (27) about $z = 0$ with $F_j = f^j/j!$ gives the order- n condition

$$\mathbf{h}_n(\mathbf{x}, 0) + \nabla \phi_n(\mathbf{x}, 0) = \mathbf{R}_n(\mathbf{x}) = - \sum_{j=1}^n F_j(\mathbf{x}) \left[\partial_z^j \mathbf{h}_{n-j} + \partial_z^j \nabla \phi_{n-j} \right](\mathbf{x}, 0), \quad (29)$$

where order zero stands for the reference field, $\mathbf{h}_0 \equiv \mathbf{H}_0^e$ and $-\nabla \phi_0 \equiv \mathbf{H}_0^a$. For $n \geq 1$ the z -derivatives are multiplications in the wavenumber domain, $\partial_z^j \mathbf{h}_n \rightarrow u_1^j (\hat{\mathbf{h}}_n^\downarrow + (-1)^j \hat{\mathbf{h}}_n^\uparrow)$ and $\partial_z^j \phi_n \rightarrow (-\lambda)^j \hat{b}_n$. Each order is a problem with a flat interface, driven by a known source that is confined to the hill because every term carries a factor f^j .

The derivatives of the reference field follow from the five surface quantities $E_{0x}, E_{0y}, \mathbf{H}_0$ alone, whatever the field below (layers, sphere): on the earth side $E_{0z} = 0$, and the curl and divergence conditions in the top layer give

$$\partial_z \mathbf{H}_0^e = (\partial_x H_{0z} + \sigma_1 E_{0y}, \partial_y H_{0z} - \sigma_1 E_{0x}, -\partial_x H_{0y} - \partial_y H_{0x}), \quad \partial_z^{j+2} \mathbf{H}_0^e = -(\nabla_h^2 + k_1^2) \partial_z^j \mathbf{H}_0^e, \quad (30)$$

and the air continuation is the same without the σ_1 terms and without k_1^2 . The first-order source is thus $\mathbf{R}_1 = -\sigma_1 f(E_{0y}, -E_{0x}, 0)$: to first order the hill acts as a thin sheet of conductance $\sigma_1 h$ carrying the current $\sigma_1 h \mathbf{E}_0$, and the depression as a sheet of negative conductance.

2.13 Solution per wavenumber

Decompose the Fourier transform $\hat{\mathbf{R}}_n$ into the components along $\hat{\boldsymbol{\xi}} = \boldsymbol{\xi}/\lambda$, along $\hat{\boldsymbol{\xi}}_\perp = \hat{\mathbf{z}} \times \hat{\boldsymbol{\xi}}$ and along $\hat{\mathbf{z}}$. The divergence-free earth modes are the TM_z mode $\hat{\boldsymbol{\xi}}_\perp$ (horizontal $\mathbf{H}$, $J_z \neq 0$) and the TE_z mode $u_1 \hat{\boldsymbol{\xi}} - i\lambda \hat{\mathbf{z}}$ for the down-going part, with the up-going partners $\hat{\boldsymbol{\xi}}_\perp$ and $-u_1 \hat{\boldsymbol{\xi}} - i\lambda \hat{\mathbf{z}}$; the air mode is $\hat{b}(-i\boldsymbol{\xi} + \lambda \hat{\mathbf{z}})$. With the reflection coefficients $\rho_{\text{TE}}, \rho_{\text{TM}}$ of (6), the order- n solution of the flat-layered problem is

$$a_\perp = \frac{\hat{R}_\perp}{1 + \rho_{\text{TM}}}, \quad a_\parallel = \frac{\hat{R}_\parallel + i\hat{R}_z}{u_1(1 - \rho_{\text{TE}}) + \lambda(1 + \rho_{\text{TE}})}, \quad \hat{b} = -\frac{\hat{R}_z + i\lambda(1 + \rho_{\text{TE}})a_\parallel}{\lambda}, \quad (31)$$

$$\hat{\mathbf{h}}^\downarrow = a_\perp \hat{\boldsymbol{\xi}}_\perp + a_\parallel (u_1 \hat{\boldsymbol{\xi}} - i\lambda \hat{\mathbf{z}}), \quad \hat{\mathbf{h}}^\uparrow = \rho_{\text{TM}} a_\perp \hat{\boldsymbol{\xi}}_\perp + \rho_{\text{TE}} a_\parallel (-u_1 \hat{\boldsymbol{\xi}} - i\lambda \hat{\mathbf{z}}). \quad (32)$$

For a homogeneous ground ($\rho = 0$) the determinant is $-\lambda(u_1 + \lambda)$, which never vanishes for $\lambda > 0$ because $\text{Re } u_1 > 0$: diffusion removes the Rayleigh anomalies that complicate the wave problem. At $\lambda = 0$ only the uniform horizontal earth mode exists; it meets the layers at normal incidence and takes the TM-type coefficient, $\hat{\mathbf{h}}^\downarrow = \hat{\mathbf{R}}_h/(1 + \rho_{\text{TM}})$.

2.14 Sphere below the terrain

With the sphere present, each order is the same flat-surface problem over the layered earth containing the sphere. Its solution is the flat-layered solution (31)–(32) plus the field that the sphere radiates when it is excited by that solution, including all its interactions with the layers and the flat surface. This field satisfies the homogeneous interface conditions at $z = 0$, so (31) is not changed; only $\hat{\mathbf{h}}_n^\downarrow$, $\hat{\mathbf{h}}_n^\uparrow$ and $\hat{b}_n$ receive the sphere's contribution before they enter the next orders. The steps are those of Section 2.7 with plane waves instead of point sources:

1. The down-going part $\hat{\mathbf{h}}_n^\downarrow$ is converted into the scalars of the two families, H_z (TE) and $J_z = i\lambda \hat{\boldsymbol{\xi}}_\perp \cdot \hat{\mathbf{h}}_n^\downarrow$ (TM), and transmitted through the layers to the sphere depth with (5), without a further reflection at the surface, which (31) already contains.
2. Every plane wave $e^{i\boldsymbol{\xi} \cdot \mathbf{x}}$ is expanded about the sphere axis by Jacobi–Anger, $e^{i\boldsymbol{\xi} \cdot \mathbf{x}'} = \sum_m i^m e^{-im\beta} J_m(\lambda\rho) e^{im\varphi}$ (β the azimuth of $\boldsymbol{\xi}$), and by identity (III) into regular spherical waves; summed over the wavenumbers this gives the regular coefficients $\mathbf{c}_n$ per order m .

3. The sphere is solved per order m , $(I + TR)\mathbf{s} = -T\mathbf{c}_n$, with the same T and R as for the sources, factorised once per frequency.
4. The outgoing waves are converted by identity (IV) into plane waves at the same wavenumbers, propagated to $z = 0$ with (5), and added to $\hat{\mathbf{h}}_n^\downarrow, \hat{\mathbf{h}}_n^\uparrow$; the scalar P of a family gives $a_\parallel = iP/\lambda$ (TE) or $a_\perp = -iP/\lambda$ (TM), and the air potential grows by $\hat{H}_z(0)/\lambda$. A cylindrical wave $P_m(\lambda)J_m(\lambda\rho)e^{im\varphi}$ about the sphere axis has the two-dimensional spectral density $P_m(\lambda)e^{im\beta}/(2\pi i^m \lambda)$.

The same exchange between plane and spherical vector waves, closed by a linear system for all interactions, underlies the first-order rough-surface solution of [Zamani et al. \(2018\)](#). The correction has the rank of the sphere's T-matrix, $2N(N + 2)$ coefficients for degrees up to N , and its cost is small compared with the terrain orders. The reference field of order zero contains the sphere's response to the sources, computed with Section 2.7. The expansion of every order about $z = 0$ requires that the sphere lie entirely below $z = 0$; locally the terrain depth must stay smaller than the distance to the sphere, which allows a sphere beside a depression to lie higher than its bottom.

2.15 Fields on the terrain

Receivers sit on the terrain, on the earth side. Any earth field Q ($\mathbf{H}$ or $\mathbf{E} = \nabla \times \mathbf{H}/\sigma_1$) is evaluated by the same Taylor continuation,

$$Q(\mathbf{x}, h(\mathbf{x})) = \sum_{n \geq 0} H^n c_n(\mathbf{x}), \quad c_n = \sum_{j=0}^n F_j \partial_z^j q_{n-j}(\mathbf{x}, 0), \quad (33)$$

with q_0 the earth continuation of the reference field. The coefficients c_n depend on s, ω , the layers, the sphere and the sources, but not on H . The depression is the same series with $H \rightarrow -H$; odd orders change sign, even orders do not, so the hill-depression asymmetry is carried entirely by the even orders.

2.16 Why a Gaussian

All powers of the shape function are Gaussians, $\mathcal{F}[f^j](\boldsymbol{\xi}) = (2\pi s^2/j) e^{-\lambda^2 s^2/2j}$. The spectral width of order j grows only like $\sqrt{j}/s$, and every f^j is the lowest Hermite function of its own width ([He et al., 2014](#)). The maximum slope is $e^{-1/2}H/s \approx 0.61 H/s$. For the Laplace part of the problem, which governs the air and the DC limit, the series in H converges at least for slopes below one ([Hu and Nicholls, 2010](#)), i.e. for $H < e^{1/2}s \approx 1.65 s$, and the operator is analytic along the whole real H axis ([Nicholls and Reitich, 2003](#)), so that Padé summation can continue it beyond that radius. No such theorem is known for the diffusive part; we determine its convergence numerically.

2.17 Time domain

With $V(0)$ the direct-current value of a response (zero for inductive responses, non-zero for the galvanic part of grounded-wire responses, including the sphere's galvanic anomaly), the switch-on, switch-off and impulse responses are

$$\begin{aligned} V_{\text{on}}(t) &= \frac{2}{\pi} \int_0^\infty \text{Re } V(\omega) \frac{\sin \omega t}{\omega} d\omega, & V_{\text{imp}}(t) &= \frac{2}{\pi} \int_0^\infty \text{Re } V(\omega) \cos \omega t d\omega, \\ V_{\text{off}}(t) &= V(0) - V_{\text{on}}(t) = -\frac{2}{\pi} \int_0^\infty [\text{Re } V(\omega) - V(0)] \frac{\sin \omega t}{\omega} d\omega, \end{aligned} \quad (34)$$

evaluated with the 201-point sine/cosine filter of [Key \(2012\)](#) directly at its abscissae $\omega_i = b_i/t$. With the filter base $b_{i+1}/b_i = e^\Delta$, $\Delta = 0.139$, the times $t_k = t_{\text{max}} e^{-k\Delta}$ share their abscissae, so $K + 1$ times cost $201 + K$ frequencies, as many as an interpolation grid, but without interpolation error. A linear

ramp-off of duration T is $T^{-1} \int_t^{t+T} V_{\text{off}}$. Independently, $V(\text{is})$ at complex frequency is the Laplace transform of the impulse response, and its inversion by Talbot's method (Talbot, 1979), which uses no oscillatory integral, serves as an independent check of the time transform (Appendix A.1.4). Lee (1981) and Lee (1983) deform the Fourier path analytically around the branch cut of Section 2.10 and obtain closed-form late-time series for degree one with quasi-static kernels; Talbot's method does the corresponding deformation numerically, in the Laplace plane, for all degrees and the exact kernels.

3 Numerical realisation

Sphere: scaling. For $|k|h \ll 1$ (low frequency, or any frequency far from the skin-depth limit) the complex-angle Legendre functions grow like $P_n^m(iu/k) \sim (u/k)^n$, the regular coefficients like k^n , the outgoing ones like k^{-n-1} and the Mie coefficients like k^{2n+1} . We therefore compute throughout $S_n^m = k^n P_n^m(y/k)$, $y = \pm iu$, and $T_n^m = k^n \partial_\theta P_n^m$, which are polynomials in (y, λ, k^2) obtained from the three-term recurrence multiplied by k^{n+1} ,

$$\begin{aligned} S_m^m &= (-1)^m (2m-1)!! \lambda^m, & S_{m+1}^m &= (2m+1)y S_m^m, \\ (n-m+1)S_{n+1}^m &= (2n+1)y S_n^m - (n+m)k^2 S_{n-1}^m, & T_n^m &= \frac{ny S_n^m - (n+m)k^2 S_{n-1}^m}{\lambda}, \end{aligned} \quad (35)$$

and the scaled unknowns $\hat{\mathbf{c}} = k^n \mathbf{c}$, $\hat{\mathbf{s}} = k^{-(n+1)} \mathbf{s}$, $\hat{\mathbf{T}} = k^{-(2n+1)} \mathbf{T}$, $\hat{\mathbf{R}} = k^n \mathbf{R} k^{n'+1}$, for which (22) keeps its form. Every quantity then stays finite from the direct-current limit ($|k|b \sim 10^{-6}$ in Fig. 13) to the skin-depth limit; Zhao and Hong (2026) treat the opposite, large-argument limit (their eqs. 41 and 42). A second, exact diagonal similarity transform balances the system across degrees: the regular coefficients are multiplied by $b^n/(2n+1)!!$ and the outgoing ones by $(2n-1)!!/b^{n+1}$ (the natural sizes of j_n and $h_n^{(1)}$ at $r=b$), so that all entries of $\mathbf{I} + \hat{\mathbf{T}}\hat{\mathbf{R}}$ are of order one times geometric ratios (Appendix F).

Sphere: quadrature. The λ -integrals (23), (25) and those of the receiver map decay like e^{-uh} with h the smallest vertical distance involved (source to sphere centre, twice the distance to the nearest interface, sphere centre to receiver) and oscillate like $J_m(\lambda\rho_{\text{max}})$. The products of Bessel functions with different arguments call for composite Gauss–Legendre panels; the surface-to-surface primary field, which does not decay, is done with the digital filter. The azimuthal orders are added in the sequence $0, \pm 1, \pm 2, \dots$ until $\|\hat{\mathbf{T}}\hat{\mathbf{c}}_m^p\|$ drops below 10^{-12} of its maximum. Panel sizes, their tests and the computing times are given in Appendix F.

Terrain: horizontal discretisation. All orders live on a periodic box of $N_g \times N_g$ points with spacing Δx ($N_g = 512$, $\Delta x = 10$ m, box 5.12 km). The reference field is needed only where f^j is not negligible. We multiply it by the analytic window $W = \frac{1}{2} \text{erfc}((r-R)/w)$ ($R = 860$ m, $w = 40$ m) before its horizontal derivatives are taken spectrally. Where $W = 1$ (to 10^{-8} for $r < 680$ m, where $f > 10^{-5}$) the derivatives are exact. A smooth but non-analytic window made the series diverge, because its slowly decaying spectrum is amplified by the factors u_1^j ; the error function has a Gaussian spectrum.

Terrain: spectral low-pass. The field expansion multiplies by u_1^j and λ^j and therefore amplifies any spectral content above the physical band, such as round-off and table interpolation errors, roughly like $(\lambda s)^n$. We apply $\exp[-(\lambda/\lambda_c)^{16}]$ with $\lambda_c = 0.2$ rad/m to the reference data and to every order. The physical content of order n is confined to $\lambda \lesssim 3\sqrt{n}/s$, i.e. $\lambda < 0.1$ rad/m for $n \leq 24$ and $s = 150$ m.

Terrain: summation. The coefficients c_n are computed to order 20. The partial sums are truncated where $|c_n H^n|$ is smallest (optimal truncation). Diagonal Padé approximants can be formed from the same coefficients; they are implemented in `terrasphere_csem` but not used for the results of this paper.

Reference field and sphere coupling on the grid. The order-zero data are the surface fields of the sources over the flat layered earth (Section 2.3) and the sphere’s response to the sources, evaluated exactly at the grid points of the window. The plane waves of a higher order are those of the FFT grid inside the low-pass; identities (III) and (IV) are evaluated at these wavenumbers, and the sphere’s contribution is returned as FFT coefficients, periodic in the box like the terrain orders (Appendix F).

Terrain: periodic box. The FFT grid makes every order periodic in the box. The periodic images distort the inductive part of the response once the skin depth approaches the box (5 km at 1 Hz in 100 Ω m). For frequencies up to 100 Hz we therefore compute the series also in boxes of 5.12, 10.24 and 20.48 km on a 20 m grid (12 orders, low-pass edge 0.12 rad/m below the Nyquist wavenumber). The box error falls like L^{-2} (by a factor of 4.1–4.5 per doubling). We add to the production spectrum the box correction $\Delta V_\infty - \Delta V_{5\text{ km}}$, all on the same coarse grid so that truncation and grid errors cancel, with $\Delta V_\infty = (L_b^2 \Delta V_b - L_a^2 \Delta V_a) / (L_b^2 - L_a^2)$ from the 10 and 20 km boxes. Transients with terrain follow from these spectra with the filter of Section 2.17, and `terrasphere_csem` provides them (Appendix F); this paper verifies the terrain in the frequency domain and the transform to the time domain separately, against `empymod` transients over the flat layered earth (Section 7.2).

4 Verification

A reference solution is only as useful as the evidence that it is right. Each part of the formulation is tested against a solution it was not derived from (Table 1; every test with its configuration and figures in Appendix A, all of them part of the software). Three kinds of evidence are distinguished. Closed-form limits and internal consistency (null contrast, symmetry, reciprocity, independent single-order implementations) check the mathematics. Independent formulations check the new elements: the 1-D code `empymod` (Werthmüller, 2017) (run quasi-statically, without displacement currents, and with sources and receivers on the surface moved 1 μ m into the ground, where `terrasphere_csem` places them, because `empymod` assigns a point on an interface to the layer above) checks the sphere at null contrast and in first order, and an independent scalar-potential expansion at DC, which shares only the grid with the magnetic-field formulation, checks the terrain, also over layers. These agree to the level of round-off or of the reference itself. Complete models, finally, are compared with independent codes in Section 7: `empymod` for the flat layered earth in the time domain and the three-dimensional finite-element code `elfe3D` for terrain and sphere in the frequency domain, where the agreement is at the per-cent level.

Table 1: Verification chain (details in Appendix A). Agreement as maximal relative deviation unless stated; “of the effect”: relative to the terrain or sphere effect.

Part tested	Reference	Agreement
<i>Sphere in the layered earth</i>		
sources, propagators, receivers (null contrast)	empymod , 1-D	10^{-8} (magn.), 10^{-6} (HED)
sphere to first order	Born integral of empymod fields	10^{-11} – 10^{-15}
reflection operator R	empymod (layered – full space)	10^{-11} – 10^{-13}
DC limit, galvanic path	point-source series	3×10^{-7}
free-space limit	Nabighian (1970)	3×10^{-5} – 3×10^{-4}
time transform	Talbot (Laplace)	10^{-14} – 5×10^{-8}
identities, Mie, kernels, derivatives, reciprocity	quadrature, boundary conditions, empymod , finite differences, Lorentz	10^{-8} – 10^{-16}
<i>Terrain over a half-space</i>		
flat-earth surface field	empymod , 1-D	10^{-6}
series truncation	smallest term at optimal truncation (indicator, not a bound)	10^{-12} (7°), 10^{-6} (20°)
terrain at DC	independent potential expansion	10^{-7}
grid, low-pass, window, box ($H = 30$ m)	parameter variation	$< 3 \times 10^{-5}$, box 1.2×10^{-3} of the effect; $H = 90$ m: Appendix A.2.4
<i>Unified code and coupling</i>		
rotated HED, loops (closed form)	layered 1-D code	10^{-5} (E), 10^{-7} (H)
layer correction of the surface field	layered 1-D code	5×10^{-6} (E), 10^{-11} (H)
grounded dipole over terrain	potential expansion at DC; 90° rotation	3×10^{-8} ; 3×10^{-15}
terrain over layers at DC	potential expansion with layers	3×10^{-8} – 10^{-6}
sphere response on the surface (order 0)	sphere model, direct receiver map	10^{-10} – 6×10^{-8}
sphere coupling of the orders (grid route)	sphere model, smooth incident field	periodic box only: 2×10^{-4} (5 km) $\rightarrow 3 \times 10^{-5}$ (10 km)
coupled series	null contrast; 90° rotation	9×10^{-17} ; 4×10^{-15}
transients, flat layered earth (wires, loops)	empymod , time domain (Section 7.2)	3×10^{-7} (E), 4×10^{-8} ($\partial_t \mathbf{B}$)
terrain over layers; sphere below the hill	elfe3D, finite elements (Section 7)	Table 2

5 A sphere in a layered earth

5.1 Validity of the classical approximations

[Lee \(1981\)](#) concluded that “for most of the time range used in prospecting, only the lowest order vector wave function is significant”, and [Lee \(1983\)](#) built the practical formula on that, with static kernels valid for $\sigma_h \mu_0 l^2 / t \lesssim 0.2$. Fig. 2 quantifies both restrictions with the full solution. With l the distance from the sphere centre to the centre of the coaxial loop (as in [Lee, 1983](#); the depth of the centre below the loop plane) and c the distance to the loop wire, for a sphere with $b/l = 0.45$ ($b = 225$ m, $l = 500$ m, loop radius 400 m, host $10^4 \Omega\text{m}$, $\tau_1 = 5.6$ ms), the degrees $n \geq 2$ carry 26 % of the response at $0.1\tau_1$ (35 % of the dipole term), 13 % at $0.85\tau_1$, 2.5 % at $2.6\tau_1$ and less than 1 % only after $3.5\tau_1$, because the primary field varies by a factor of five across the sphere and the higher degrees have shorter time constants ($\tau_{n,1}/\tau_{1,1} = \pi^2/z_{n,1}^2$). In the 1:100 design ($b/l = 0.72$, host $100 \Omega\text{m}$) the host interaction is strong: the early response changes sign between 0.1 and $0.2\tau_1$ (the shares are undefined there), and afterwards the degrees $n \geq 2$ carry 45 % at $0.5\tau_1$, 24 % at $1.1\tau_1$, 5.6 % at $2.6\tau_1$ and 2.2 % at $3.5\tau_1$. The full solution exceeds the free-space multipole response by

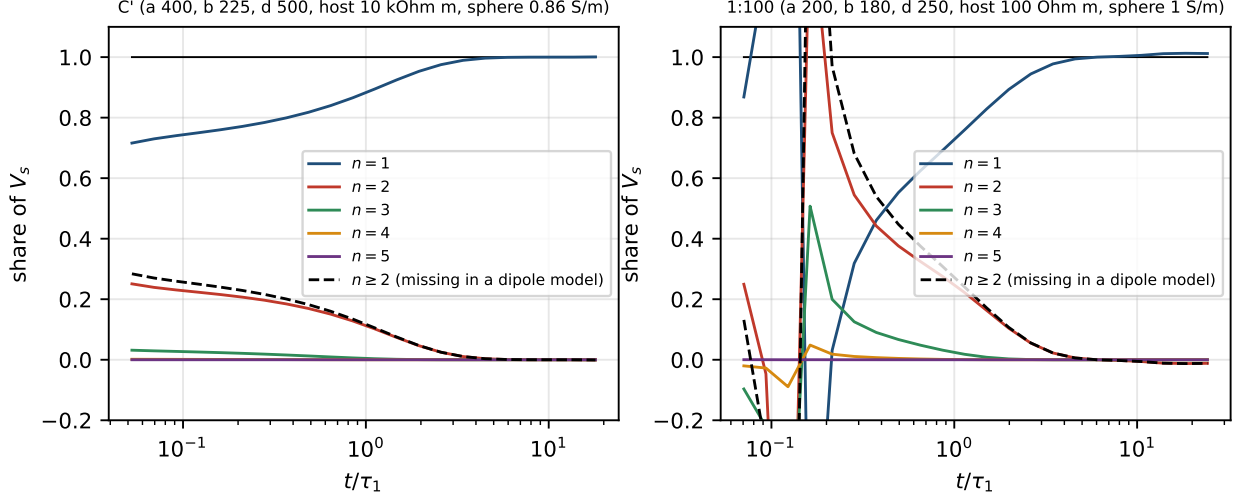


Figure 2: Degree-by-degree content of the coaxial switch-off response for a large sphere (left, $b/l = 0.45$) and the 1:100 design (right, $b/l = 0.72$), l the distance from the sphere centre to the loop centre: shares of the degrees $n = 1 \dots 5$ and of everything a dipole model misses ($n \geq 2$). In the 1:100 design the response changes sign near $0.15\tau_1$, where the shares diverge (clipped).

factors 0.4–2.8, and the degree-one formula of Lee (1983) recovers only 60–92 % of the signal between τ_1 and $9\tau_1$, inside its own range of validity ($t \geq 5\sigma_h\mu_0 l^2 = 3.9$ ms), and overestimates it by 12 % at $12\tau_1$ (Fig. 3). The time constants of Section 2.10 explain why: in both designs a dipole model becomes adequate only after about $3\tau_1$; before that, and whenever $\sigma_h\mu_0 l^2/t > 0.2$, the full solution is required.

Fig. 4 shows the free-space time constants as functions of the permeability (the characteristic equation of Section 2.10; $\mu_r = 3$ lengthens τ_1 by a factor 2.3) and the modulus of the TE denominator along the negative imaginary frequency axis: in free space it vanishes at the time constants, in a host of 10^{-4} S/m already no zero remains although the minima still mark the free-space values, and at 10^{-2} S/m they are gone. In the cases scanned the response of a sphere in a conductive host is therefore not governed by a discrete set of pole-type decay constants on this axis, and its transient is a branch-cut integral.

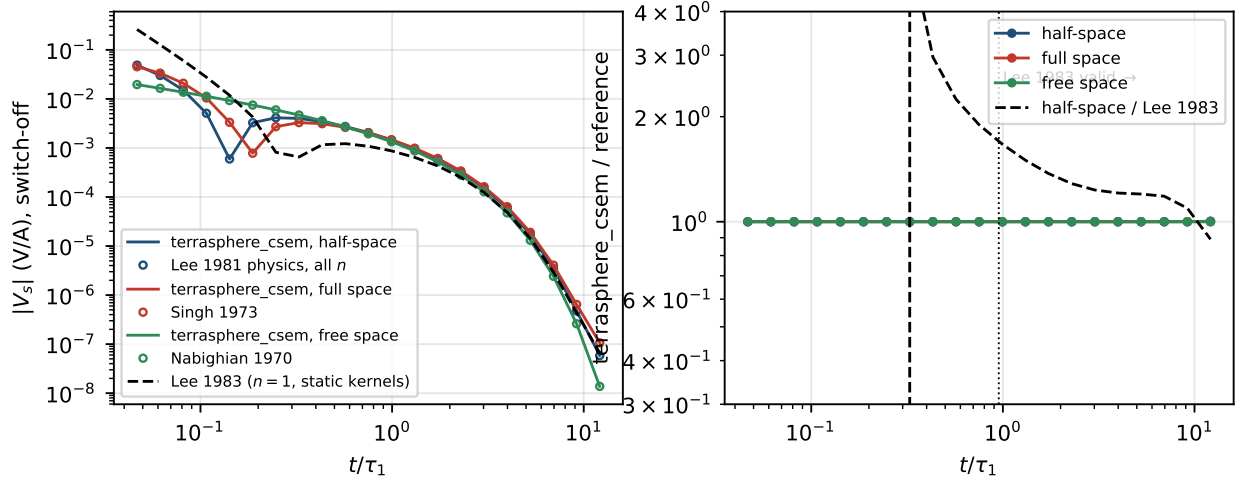


Figure 3: Switch-off response of the coaxial loop over the sphere (1:100 design) in the half-space, in the full space and in free space: **terrasphere_csem** (lines) against independent single-order references (half-space, full space; Appendix A.1.6) and the free-space solution of Nabighian (1970) (circles), and the degree-one formula of Lee (1983); right, ratios. τ_1 is the free-space time constant of the sphere.

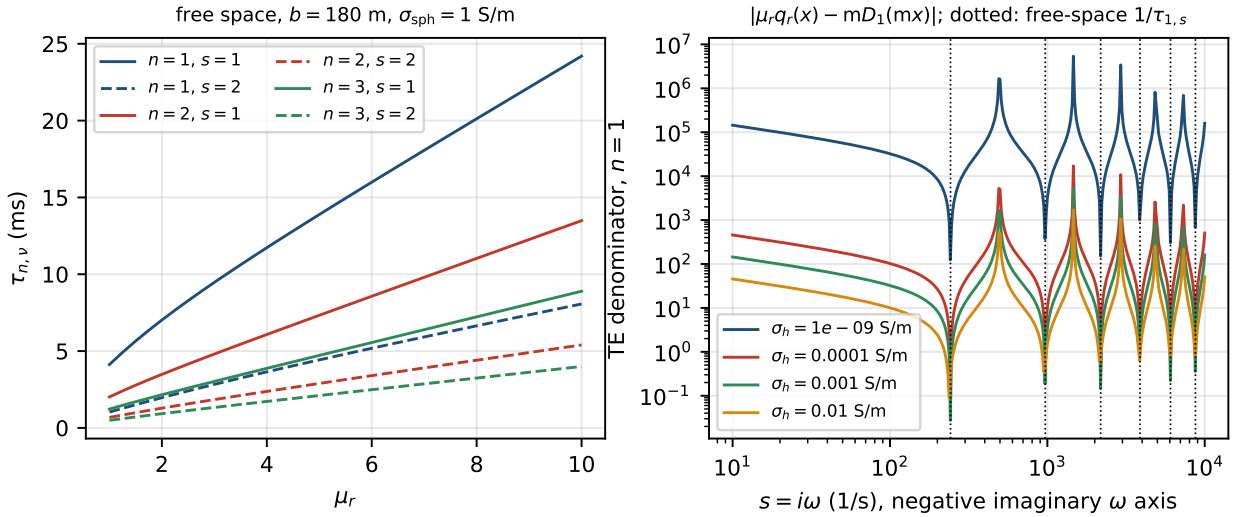


Figure 4: Left: free-space time constants $\tau_{n,\nu}$ of the sphere ($b = 180$ m, $\sigma_{\text{sph}} = 1$ S/m) versus relative permeability. Right: modulus of the TE modal denominator ($n = 1$) on the negative imaginary ω -axis for increasing host conductivity; dotted, the free-space time constants.

6 A sphere below a hill

Model. The terrain is a Gaussian of width $s = 150$ m with heights $H = \pm 30$ m (maximum slope 6.9°) and ± 90 m (20°), Fig. 5. A grounded wire runs along x from $(-500, -1500)$ to $(500, -1500)$ m and carries 1 A. Receivers lie on two profiles over the summit, along y at $x = 0$ and along x at $y = 0$, for $|x|, |y| \leq 600$ m. The same wire and terrain over a half-space of 0.01 S/m serve for the verification of the terrain expansion (Appendix A.2).

Figure 6 combines the two targets: the grounded wire of Fig. 5 over a two-layer earth (0.01 S/m to 300 m, 0.003 S/m below) with a sphere of radius 60 m and 1 S/m centred at $z = -140$ m, 140 m below the level of the flat ground, so that its top lies at $z = -80$ m. The coefficients were computed at $3, 30, 300$ and 3000 Hz on the production grid with 20 orders.

Convergence. Over the layered earth without the sphere the series behaves as over the half-space: at optimal truncation the smallest term is 2×10^{-13} of the flat field for $|H| = 30$ m and $3-9 \times 10^{-7}$ for 90 m. With the sphere the terms for $H = 30$ m still decrease up to order 20 , to 2×10^{-8} . For 60 and 90 m they reach a minimum near order $9-12$ and grow again beyond it (Fig. 6a): the series is then truncated at its smallest term, 10^{-4} for 60 m and 10^{-2} for 90 m, which indicates the truncation error but does not bound it. The reason is the local condition of Section 2.14: the Taylor continuation from $z = 0$ to the terrain converges only within the distance to the nearest singularity of the continued field, and the sphere's field is singular inside the sphere, whose top is 80 m below $z = 0$. A hill of 90 m reaches beyond that distance, although it rises away from the sphere, and converges poorly. A depression of 90 m would cut into this sphere and is not an admissible model; the depression is therefore used here with 30 m only. For this sphere a convergent series is obtained for heights of 30 m, less than half the depth of its top.

Coupling. The two effects do not superpose. For $H = 30$ m the coupling term, the total change minus the changes by hill and sphere alone, amounts to $39-58\%$ of the sphere effect in E_x, E_y, H_x, H_y and H_z between 3 and 300 Hz and to $58-69\%$ at 3 kHz; for the depression of 30 m it is $77-190\%$ (up to 280% at 3 kHz). Most of it is geometric: a receiver on the summit is 30 m farther from the sphere, which reduces a dipole-like anomaly by $(140/170)^3 = 0.56$, and a receiver at the bottom of the depression 30 m closer, which increases it by $(140/110)^3 = 2.06$; both match the size of the coupling. The rest is the mutual induction between the hill and the sphere. Relative to the terrain effect the coupling is $1-4\%$ in E_y and E_z , $8-37\%$ in H_y and H_z , $42-50\%$ in E_x and $3-65\%$ in H_x for the hill of 30 m. A model that adds the anomaly of a buried body computed for a flat surface to a terrain correction computed without the body misses this term. Radar scattering shows the counterpart: the pattern of a sphere below a flat interface does not depend on its depth, below a slightly rough one it does (Zamani et al., 2018).

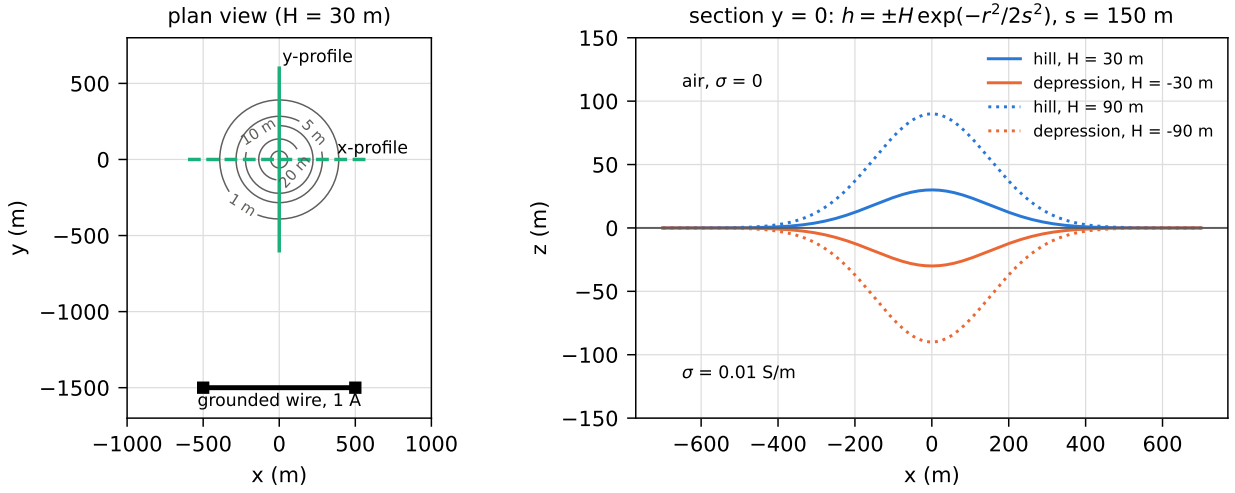


Figure 5: Geometry: grounded wire on the flat ground 1.5 km from the centre of a Gaussian hill or depression of width $s = 150$ m in a top layer of 0.01 S/m; receiver profiles over the summit (left); sections for $H = \pm 30$ and ± 90 m (right).

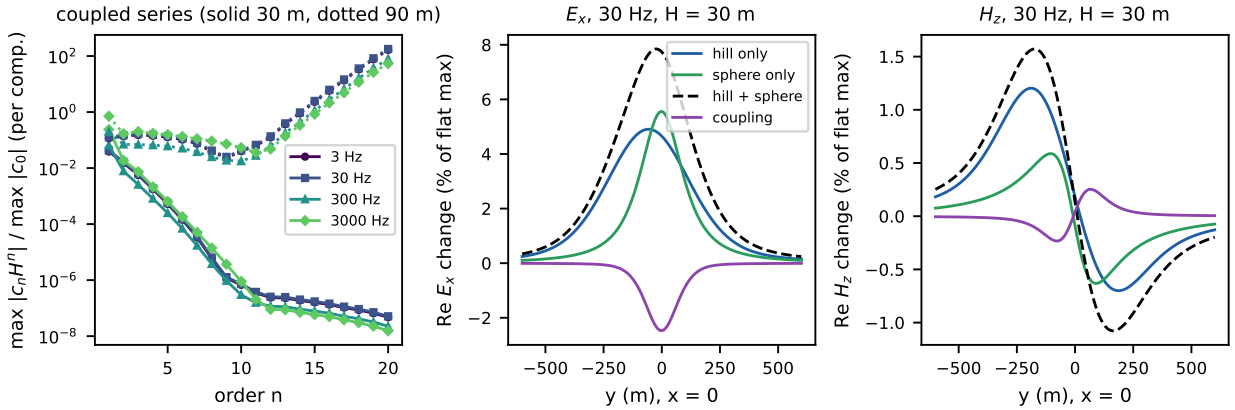


Figure 6: Sphere below a hill: grounded wire of Fig. 5, top layer 0.01 S/m to 300 m over 0.003 S/m, sphere of radius 60 m and 1 S/m centred 140 m below the flat ground ($z = 0$). (a) Terms of the coupled series on the summit profiles relative to the flat field, $H = 30$ m (solid) and 90 m (dotted). (b, c) Real parts of E_x and H_z along y at $x = 0$, 30 Hz, $H = 30$ m: change by the hill alone, by the sphere alone, by both together, and the coupling (both minus the sum of the single effects), in per cent of the profile maximum of the flat layered field.

7 Comparison with independent codes

Here the semi-analytic solution is the reference and the three-dimensional code the object of the test. Three-dimensional EM codes are routinely checked against analytic solutions, against other formulations and against each other (Newman et al., 1986; Wang and Hohmann, 1993; Heagy et al., 2017; Chang et al., 2024). With terrain, comparisons between codes differ by several per cent (Weiss et al., 2023); a reference that separates the terrain effect, the layering and a compact body is what these comparisons lack. Two codes that share nothing with `terrasphere_csem` serve here: the 1-D code `empymod` for the flat layered earth in the time domain (Section 7.2), and the three-dimensional finite-element code `elfe3D` for terrain and a sphere below it in the frequency domain (Sections 7.3 and 7.4). The conventions needed for the two codes (time convention, frame, normalisation of the sources) are documented at the beginning of the validation report of the repository.

7.1 Setup

Four sources excite a three-layer earth (0.01, 0.005 and 0.03 S/m, interfaces at 200 and 600 m depth) with a Gaussian hill or depression of width $s = 150$ m at the origin (Fig. 7): a grounded wire of 500 m in line with the x -profile (T1) and broadside to it (T2), both centred 1500 m from the summit, a square loop of 400 m side at the same place (T3) and a square loop of 2.6 km side centred on the summit (T4). The terrain heights are ± 30 m and ± 90 m. The receivers lie 1 cm below the surface on the x - and y -profile from -600 to 600 m every 20 m (122 receivers), at 0.1, 10 and 1000 Hz. A fifth model adds a sphere of $10 \Omega\text{m}$ and radius 180 m, centred 400 m below the origin in the second layer, under the 90 m hill. `elfe3D` (Rulff et al., 2021; Rulff, 2025b), version v1.0.1 plus 16 commits (commit 58528bd7, Rulff, 2025a), with MUMPS 5.8.2 (Amestoy et al., 2001, 2019) and TetGen 1.5.0 (Si, 2015), computes each model on a fixed mesh of a ± 15 km domain with refined boxes around the source and the receivers, one mesh for 0.1 and 10 Hz and a finer one for 1 kHz (1.6 to 3.5 million unknowns); `terrasphere_csem` uses its FFT grid of 512^2 points at 10 m, 20 orders of the height series and the box correction, and sphere degrees up to $N = 12$. Without terrain `elfe3D` is not needed: the flat models of the four sources agree with `empymod` to 2.5×10^{-5} of the profile maximum in all components, except those of the two loops at 0.1 Hz (7×10^{-5} for $\mathbf{H}$, 1.2×10^{-3} for the horizontal electric field). The deviation from `elfe3D` is given as $|V_{\text{elfe3D}} - V| / \max |V|$ per source, frequency and component, with the maximum of the same component of the series V over both profiles; its median and 90th percentile over the receivers characterise a model. A weak component is thus measured against its own small maximum, not against the dominant field. The measure refers to the total field: there are no flat-earth `elfe3D` runs, so the deviation of the terrain effect alone is not formed, and the per-cent level of Table 2 does not carry over to terrain corrections that are small against the total field. The sharper test of the terrain itself is E_z , which vanishes on a flat surface. All comparisons, with every component on both profiles, are documented in the code repository (`tests/validation`).

7.2 Flat earth in the time domain

The transform to the time domain is checked against `empymod` 2.6.0, which computes the transients with its own Hankel filter (`key_201_2009`) and Fourier filter (`key_601_2009`, frequencies chosen per time) and thus shares neither the flat-earth field nor the transform with `terrasphere_csem`. The model is the flat three-layer earth with the four sources of Fig. 7, switch-off of 1 A, receivers on both profiles every 300 m (`empymod` 1 μm below the surface) and 101 times from 9×10^{-6} to 10 s. Fig. 8 shows the strong component of each source. Relative to the maximum of each curve, E_x and E_y agree to 2×10^{-8} – 3×10^{-7} and $\partial_t \mathbf{B}$ to 3×10^{-10} – 4×10^{-8} ; the largest deviations occur at the earliest times, after 10^{-2} s they stay below 10^{-9} . Components that vanish by symmetry are exactly zero or rounding noise (10^{-16} – 10^{-15} of the largest component) in both codes. All components, both profiles and the tables of the deviations are in the validation report of the repository.

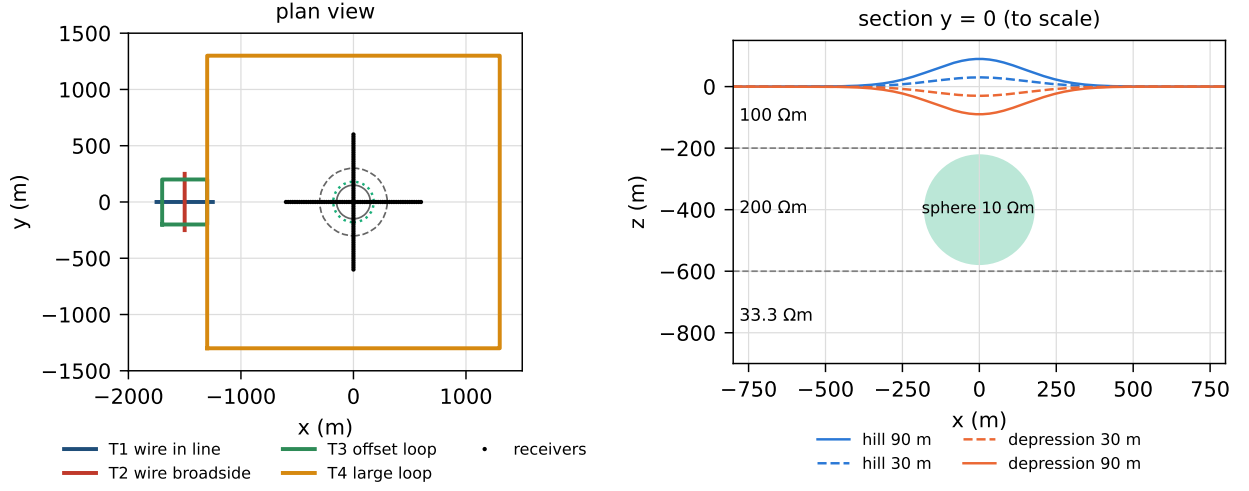


Figure 7: Setup of the comparisons with **empymod** and **elfe3D**. Left: the four sources (1 A) and the receivers on the x - and y -profile; circles: the Gaussian terrain at s and $2s$ ($s = 150$ m), dotted: the outline of the sphere. Right: section $y = 0$ to scale with the four terrain heights, the layers and the sphere, which is present only in the runs with the 90 m hill.

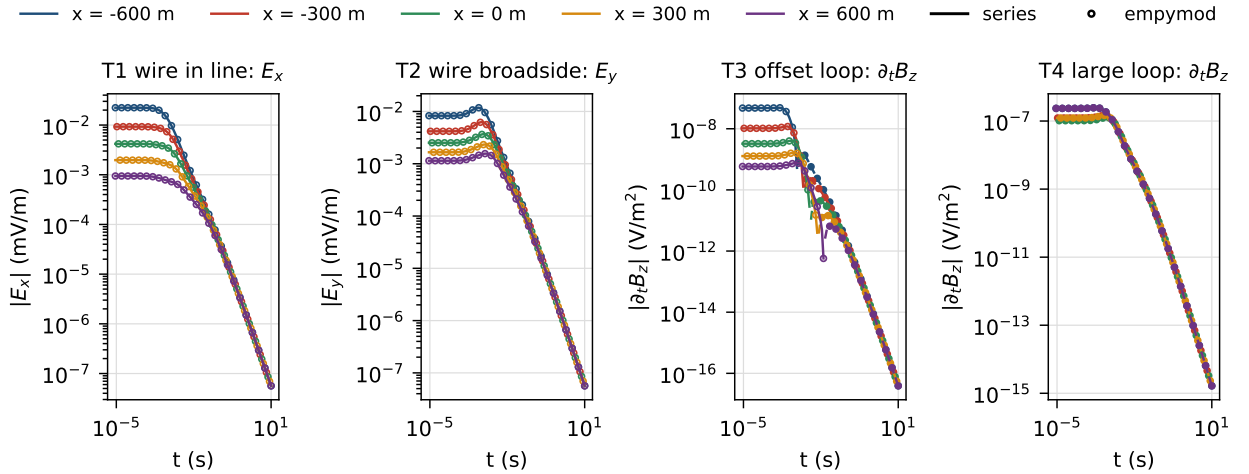


Figure 8: Switch-off transients over the flat three-layer earth, strong component of each source on the x -profile: **terrasphere_csem** (lines) against **empymod** (circles, every fourth time); solid lines and open circles $V > 0$, dashed lines and filled circles $V < 0$.

7.3 Topography

Table 2: Deviation of elfe3D from `terrasphere_csem` over the 90 m hill: median / 90th percentile over the receivers of $|V_{\text{elfe3D}} - V|/\max|V|$ in per cent. E_z of the large loop (T4) vanishes by symmetry.

source	f	E_x	E_y	E_z	H_x	H_y	H_z
T1	0.1 Hz	0.03/0.4	0.22/1.0	0.46/2.0	0.15/0.5	0.28/0.4	0.38/1.0
T1	10 Hz	0.05/0.4	0.31/1.0	0.46/2.0	0.15/0.5	0.12/0.2	0.34/1.0
T1	1000 Hz	0.22/0.9	0.84/1.3	0.45/1.2	0.69/1.3	0.22/0.6	1.94/7.7
T2	0.1 Hz	0.30/0.7	0.18/0.4	0.61/2.9	0.32/0.4	0.31/0.6	0.25/0.4
T2	10 Hz	0.34/0.7	0.15/0.4	0.45/2.4	0.06/0.2	0.27/0.6	0.21/0.4
T2	1000 Hz	0.65/1.3	0.20/0.5	0.49/1.7	0.29/0.8	0.40/1.6	0.46/2.3
T3	0.1 Hz	0.38/0.9	0.20/0.4	0.26/2.0	1.65/3.9	9.86/33.7	0.02/0.2
T3	10 Hz	0.40/0.9	0.16/0.4	0.27/1.9	0.36/1.1	0.96/3.1	0.06/0.1
T3	1000 Hz	0.61/2.3	0.05/0.4	0.56/1.9	0.09/0.4	0.30/1.4	0.14/0.9
T4	0.1 Hz	0.05/0.8	0.08/0.8	≈ 0	2.36/9.7	4.44/12.7	0.88/0.9
T4	10 Hz	0.12/0.7	0.06/0.7	≈ 0	0.24/0.8	0.44/1.0	0.78/0.9
T4	1000 Hz	0.20/0.9	0.19/0.8	≈ 0	0.13/1.0	0.06/1.0	1.26/2.0

Over the terrain the two codes agree in all components (Fig. 9), including E_z , which a flat surface does not have and which therefore tests the terrain series alone. Over the 90 m hill the median deviation is 0.02–1 % and the 90th percentile 0.1–3 % (Table 2), with two exceptions: H_z of the in-line wire at 1 kHz (7.7 %) and the horizontal magnetic field of the two loops at 0.1 Hz (4–34 %); the other terrain models give the same picture. The large values of the measure all belong to components that are small on the profiles: H_z of the in-line wire at 1 kHz (about 1 % of the largest magnetic component; 90th percentile 5–15 % over the four terrain models) and the horizontal magnetic field of the two loops at 0.1 Hz (0.07–1.4 % of the largest magnetic component; 90th percentile up to 54 %), measured against their own small maxima. E_z of the large loop vanishes on both profiles by symmetry and is not rated.

7.4 Sphere below the hill

With the sphere the total fields agree as well as without it, with the same small components as outliers. The anomaly of the sphere is at most 23 % and mostly a few per cent of the field, of the order of the deviation of the total field; the difference of two independently meshed runs would not resolve it. elfe3D therefore computes each case twice on the identical mesh, once with the sphere and once with the host resistivity in the sphere region, and the difference of the two runs is compared with the difference of the series with and without the sphere; the discretisation of the background then largely cancels. This sphere comes within 20 m of both interfaces ($b/h = 0.9$); raising its degree from $N = 12$ to 15 and 20 changes the series anomaly of the in-line wire by at most 4×10^{-7} of its maximum on both profiles (2×10^{-6} for the weak E_z anomaly at 1 kHz), so the deviations below are not caused by the truncation of the sphere part (Appendix F). Where the anomaly exceeds 1 % of the field, elfe3D reproduces it to 2.9–6.0 % of its maximum (Fig. 10: E_y of the in-line wire, an anomaly of 23 % of the field, to 3 %). Smaller anomalies, below 1 % of the field, deviate more, up to several times their own size, and are not rated; for the large loop, which encloses the sphere symmetrically, the anomaly on the profiles is below 0.5 % in all components.

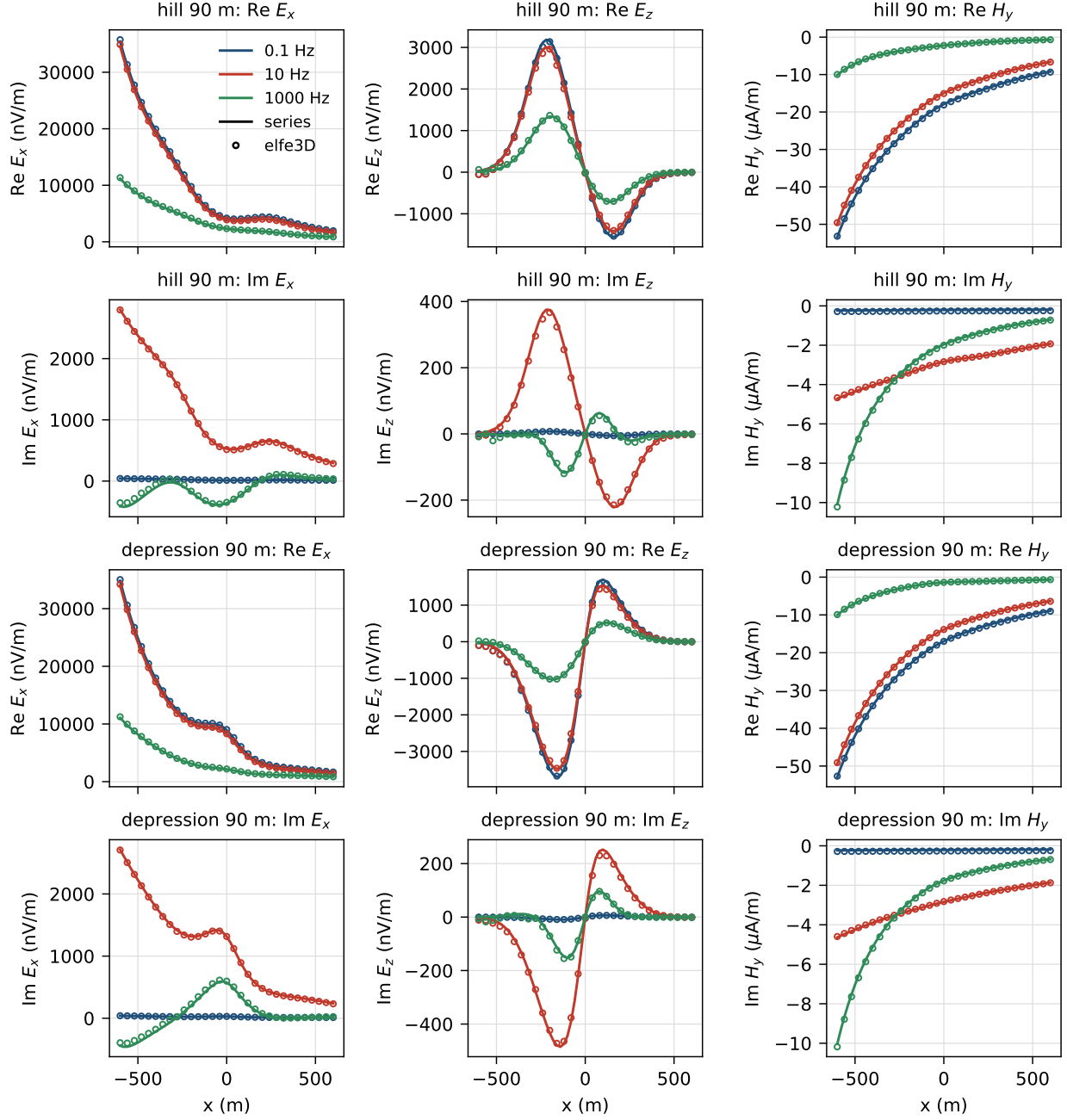


Figure 9: Grounded wire in line (T1), x -profile, hill (upper two rows) and depression (lower two rows) of 90 m: real and imaginary parts of E_x , E_z and H_y at 0.1 Hz, 10 Hz and 1 kHz. Lines: `terrasphere_csem`; circles: `elfe3D` (every second receiver). E_z vanishes on a flat surface and is the terrain effect alone.

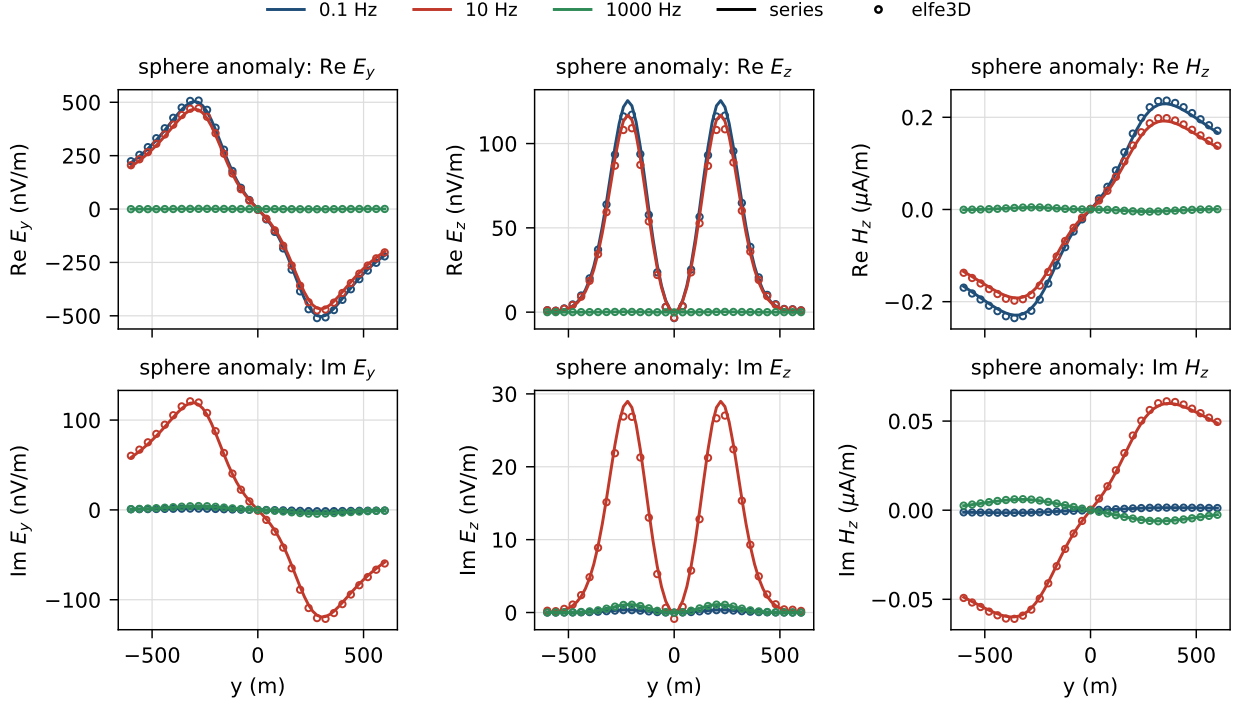


Figure 10: Sphere ($10\Omega\text{m}$, radius 180 m, centre 400 m deep) below the 90 m hill, grounded wire in line (T1), y -profile: real and imaginary parts of the sphere anomaly of E_y , E_z and H_z . Lines: `terrasphe_csem` with minus without the sphere; circles: `elfe3D` run with the sphere minus the run on the same mesh with the host resistivity in the sphere region (every second receiver).

8 Discussion

Range of validity. The sphere part is exact within quasi-statics for a sphere entirely inside one layer; if it touched an interface, the plane-wave spectra of its outgoing waves would not converge there. The number of degrees grows with $b/h_\uparrow$ and b/c (c the distance from the centre to the source wire): $N = 15$ gives 10^{-8} for $b/h_\uparrow = 0.72$. The terrain expansion is a reference for moderate terrain: for slopes up to about 20° ($H/s \leq 0.6$) optimal truncation leaves 10^{-5} – 10^{-7} between 1 Hz and 30 kHz (Appendix A.2.2); at 31° optimal truncation leaves about 10^{-3} – 10^{-4} , and Padé summation, implemented but not evaluated here, is the natural next step. These smallest terms indicate the truncation error but do not bound it, and beyond the optimal order the terms grow. For $H = 90$ m the magnetic terrain effect also depends on the numerical choices: a finer grid changes it by up to 2%, a low-pass edge widened to 0.3 rad/m by up to 40%. The edge used here is a choice between these, not the result of a convergence study, and the uncertainty of the magnetic terrain effect for $H = 90$ m is not quantified; we expect a few per cent, but this is an estimate (Appendix A.2.4). Near a sphere the expansion is limited further by the distance from $z = 0$ to the sphere (Section 6). The slope controls the radius of convergence, as for the Laplace problem (Hu and Nicholls, 2010); the skin depth enters through the factors u_1^j and limits the frequency (skin depth large against H , $f \ll 1/(\pi\mu_0\sigma_1 H^2) \approx 30$ kHz for $H = 30$ m in $100\Omega\text{m}$). The field expansion shares the ill-conditioning of its wave counterpart (Nicholls and Reitich, 2001, 2004); a transformed-field expansion would remove it at the price of a vertical discretisation. Steep slopes, overhangs and cliffs are outside any expansion in the height. The coupling requires the sphere below the level of the flat ground and the layer interfaces below the bottom of the depression; the terrain must lie in the top layer. The coupled response of terrain and sphere is compared with an independent code in the frequency domain only; its transients rest on the time transform, which is checked separately.

Sources and receivers. Transmitters lie on the flat ground off the hill; a loop may enclose the hill when its wire lies outside the window of the expansion ($W < 10^{-15}$, i.e. at least $R + 5.6w = 1.08$ km from the summit for the window of $s = 150$ m). With a sphere this holds on flat ground as well, because its coupling uses the same grid. Sources on the hill require a transformed-field formulation in which the source stays on the transformed surface. Receivers on the terrain record all components of $\mathbf{E}$ and $\mathbf{H}$; receivers in the air see the inductive part only, because the TM part of the scalar description is singular in an insulator.

Extensions. Any smooth terrain given on a grid can be treated; only the closed-form spectra of the Gaussian are lost. A sediment fill below a flat surface is the same expansion for an interface between two conductors. Displacement currents (the wave regime, treated to first order in the roughness by Zamani et al., 2018), a vertically incident plane wave (magnetotellurics, the case of MTSphere, Vallée and Moussaoui, 2026), a polarizable sphere and several spheres coupled by the translation theorems of spherical vector waves (Gumerov and Duraiswami, 2003) are possible extensions; none of them is implemented.

What the reference is good for. A three-dimensional code can be checked component by component in the frequency domain, for terrain, layering and a compact body separately and together. Transients with terrain are available as well, within the cut-off and validation limits of Appendix F. This matters most at far offsets: the electric anomaly of a sphere ($b = 70$ m, 2 S/m, 200 m deep in a three-layer earth) seen 600 m from a 1 km grounded wire is only 0.1 – 0.25% of the largest 1-D field on the profile, smaller than the typical discretisation error of a three-dimensional code for the 1-D field itself. Such a code can be checked there only by subtracting a null run on the same mesh, or against a reference that gives the sphere part separately and to many digits. The first-order term is the sensitivity of every component to the terrain, and the closed-form derivatives of the sphere part (Section 2.9) are a benchmark for numerical Jacobians.

9 Conclusions

The response of a conductive sphere in a layered earth below a Gaussian hill or depression can be computed semi-analytically for grounded wires, grounded dipoles and loops, with all components on the terrain, in the frequency domain and as transients. Spherical vector waves for the sphere and plane waves for the layered earth are connected by identities from scattering theory; the terrain is a height expansion posed in the magnetic field alone, in which every order is a flat-surface problem with a low-rank sphere correction. Terrain and sphere together do not give the sum of their single effects, so a terrain correction and the anomaly of a buried body cannot simply be added. Closed-form limits, an independent direct-current formulation, a one-dimensional code and internal consistency tests show small residuals in the configurations tested. The comparison with the finite-element code `elfe3D` shows differences at the per-cent level in the frequency domain, larger only in weak components; the time transform is checked separately. For the moderate terrain tested, the height series reaches small terms under optimal truncation; near a sphere its range is limited by the distance from the expansion plane to the sphere, and beyond it the smallest term of the series is an error indicator, not a bound. The implementation, its tests and the scripts of all figures are available with this paper.

Data and code availability

The Python package `terrasphere_csem`, the test suite reproducing every number of the verification chain, and the scripts producing every figure are available at https://github.com/klauslippert/terrasphere_csem; the version used here, v1.0.0, is archived at Zenodo (Lippert, 2026).

Acknowledgments

The author thanks Neuland Software (www.neuland-software.de) for providing computing resources.

During the preparation of this work the author used Claude Code (Anthropic) with Claude 5 models in order to improve the language of the manuscript and to support the implementation of the software. After using this tool, the author reviewed, tested and edited the content as needed and takes full responsibility for the content of the publication.

References

- Joseph Abate and Peter P. Valkó. Multi-precision Laplace transform inversion. *International Journal for Numerical Methods in Engineering*, 60:979–993, 2004. doi: 10.1002/nme.995.
- Milton Abramowitz and Irene A. Stegun. *Handbook of Mathematical Functions*. National Bureau of Standards, Washington, 1964.
- Yasemin Altuncu. A numerical method for electromagnetic scattering by 3-D dielectric objects buried under 2-D locally rough surfaces. *IEEE Transactions on Antennas and Propagation*, 63(8):3634–3643, 2015. doi: 10.1109/TAP.2015.2438859.
- Patrick R. Amestoy, Iain S. Duff, Jean-Yves L’Excellent, and Jacko Koster. A fully asynchronous multifrontal solver using distributed dynamic scheduling. *SIAM Journal on Matrix Analysis and Applications*, 23(1):15–41, 2001. doi: 10.1137/S0895479899358194.
- Patrick R. Amestoy, Alfredo Buttari, Jean-Yves L’Excellent, and Theo Mary. Performance and scalability of the block low-rank multifrontal factorization on multicore architectures. *ACM Transactions on Mathematical Software*, 45(1):2:1–2:26, 2019. doi: 10.1145/3242094.
- P. A. Bobbert and J. Vlieger. Light scattering by a sphere on a substrate. *Physica A*, 137:209–242, 1986. doi: 10.1016/0378-4371(86)90072-5.
- Craig F. Bohren and Donald R. Huffman. *Absorption and Scattering of Light by Small Particles*. Wiley, New York, 1983. doi: 10.1002/9783527618156.
- O. P. Bruno and F. Reitich. Numerical solution of diffraction problems: a method of variation of boundaries. II. Finitely conducting gratings, Padé approximants, and singularities. *Journal of the Optical Society of America A*, 10:2307–2316, 1993a. doi: 10.1364/JOSAA.10.002307.
- O. P. Bruno and F. Reitich. Numerical solution of diffraction problems: a method of variation of boundaries. III. Doubly periodic gratings. *Journal of the Optical Society of America A*, 10:2551–2562, 1993b. doi: 10.1364/JOSAA.10.002551.
- F. Caire, D. Prémel, and G. Granet. Semi-analytical computation of a quasi-static field induced by an eddy current probe in a conductor with a rough surface. *European Physical Journal: Applied Physics*, 64:24511, 2013. doi: 10.1051/epjap/2013120400.
- Jianghao Chang, Junjie Xue, Qingxin Meng, and He Li. Research progress in three-dimensional forward modeling for the transient electromagnetic method. *Journal of Geophysics and Engineering*, 21:1764–1774, 2024. doi: 10.1093/jge/gxae104.
- Shu-Kong Chang and Kenneth K. Mei. Generalized Sommerfeld integrals and field expansions in two-medium half-spaces. *IEEE Transactions on Antennas and Propagation*, 28(4):504–512, 1980. doi: 10.1109/TAP.1980.1142371.
- W. Craig and C. Sulem. Numerical simulation of gravity waves. *Journal of Computational Physics*, 108:73–83, 1993. doi: 10.1006/jcph.1993.1164.

- Jacques K. Desmarais and Richard S. Smith. Approximate semianalytical solutions for the electromagnetic response of a dipping-sphere interacting with conductive overburden. *Geophysics*, 81(4): E265–E277, 2016. doi: 10.1190/geo2015-0597.1.
- A. J. Devaney and E. Wolf. Multipole expansions and plane wave representations of the electromagnetic field. *Journal of Mathematical Physics*, 15:234–244, 1974. doi: 10.1063/1.1666629.
- F. A. Falk, A. V. Christiansen, and T. M. Hansen. Quantifying 3D modeling errors in time-domain electromagnetics: implications for deterministic and probabilistic inversions. EarthArXiv preprint 12815, 2026. doi: 10.31223/X5TB7Z.
- R. C. Fox, G. W. Hohmann, T. J. Killpack, and L. Rijo. Topographic effects in resistivity and induced-polarization surveys. *Geophysics*, 45:75–93, 1980. doi: 10.1190/1.1441041.
- Fabrizio Frezza and Fabio Mangini. Electromagnetic scattering by a buried sphere in a lossy medium of an inhomogeneous plane wave at arbitrary incidence: spectral-domain method. *Journal of the Optical Society of America A*, 33(5):947–953, 2016. doi: 10.1364/JOSAA.33.000947.
- Fabrizio Frezza, Fabio Mangini, Lara Pajewski, Giuseppe Schettini, and Nicola Tedeschi. Spectral domain method for the electromagnetic scattering by a buried sphere. *Journal of the Optical Society of America A*, 30(4):783–790, 2013. doi: 10.1364/JOSAA.30.000783.
- Nail A. Gumerov and Ramani Duraiswami. Recursions for the computation of multipole translation and rotation coefficients for the 3-D Helmholtz equation. *SIAM Journal on Scientific Computing*, 25(4):1344–1381, 2003. doi: 10.1137/S1064827501399705.
- Guoxia Han, Yiping Han, and Huayong Zhang. Relations between cylindrical and spherical vector wavefunctions. *Journal of Optics A: Pure and Applied Optics*, 10:015006, 2008. doi: 10.1088/1464-4258/10/01/015006.
- Y. He, P. Li, and J. Shen. A new spectral method for numerical solution of the unbounded rough surface scattering problem. *Journal of Computational Physics*, 275:608–625, 2014. doi: 10.1016/j.jcp.2014.07.026.
- Lindsey J. Heagy, Rowan Cockett, Seogi Kang, Gudni K. Rosenkjaer, and Douglas W. Oldenburg. A framework for simulation and inversion in electromagnetics. *Computers & Geosciences*, 107: 1–19, 2017. doi: 10.1016/j.cageo.2017.06.018.
- Y. Hong and D. P. Nicholls. A high-order perturbation of surfaces method for vector electromagnetic scattering by doubly layered periodic crossed gratings. *Journal of Computational Physics*, 372: 748–772, 2018. doi: 10.1016/j.jcp.2018.06.041.
- A. Hördt and M. Müller. Understanding LOTEM data from mountainous terrain. *Geophysics*, 65: 1113–1123, 2000. doi: 10.1190/1.1444804.
- B. Hu and D. P. Nicholls. The domain of analyticity of Dirichlet–Neumann operators. *Proceedings of the Royal Society of Edinburgh A*, 140:367–389, 2010. doi: 10.1017/S0308210509000493.
- G. R. Jiracek, R. P. Reddig, and R. K. Kojima. Application of the Rayleigh-FFT technique to magnetotelluric modeling and correction. *Physics of the Earth and Planetary Interiors*, 53:365–375, 1989. doi: 10.1016/0031-9201(89)90023-X.
- Kerry Key. 1D inversion of multicomponent, multifrequency marine CSEM data: Methodology and synthetic studies for resolving thin resistive layers. *Geophysics*, 74(2):F9–F20, 2009. doi: 10.1190/1.3058434.

- Kerry Key. Is the fast Hankel transform faster than quadrature? *Geophysics*, 77(3):F21–F30, 2012. doi: 10.1190/geo2011-0237.1.
- Gerhard Kristensson. Electromagnetic scattering from buried inhomogeneities—a general three-dimensional formalism. *Journal of Applied Physics*, 51(7):3486–3500, 1980. doi: 10.1063/1.328201.
- Terry Lee. Transient electromagnetic response of a sphere in a layered medium. *Geophysical Prospecting*, 23(3):492–512, 1975. doi: 10.1111/j.1365-2478.1975.tb01544.x.
- Terry J. Lee. Transient electromagnetic response of a sphere in a layered medium. *Pure and Applied Geophysics*, 119:309–338, 1981. doi: 10.1007/BF00877768.
- Terry J. Lee. The transient electromagnetic response of a conducting sphere in an imperfectly conducting half-space. *Geophysical Prospecting*, 31(5):766–781, 1983. doi: 10.1111/j.1365-2478.1983.tb01084.x.
- Klaus Lippert. terrasphere_csem: semi-analytic 3-D CSEM and TEM fields of a sphere below Gaussian topography in a layered earth. Zenodo, version v1.0.0, 2026. doi: 10.5281/zenodo.23434784.
- Y. Liu, C. Yin, C. Qiu, et al. 3-D inversion of transient EM data with topography using unstructured tetrahedral grids. *Geophysical Journal International*, 217:301–318, 2019. doi: 10.1093/gji/ggz014.
- P. Martinelli and A. Osella. MT forward modeling of 3-D anisotropic electrical conductivity structures using the Rayleigh–Fourier method. *Journal of Geomagnetism and Geoelectricity*, 49:1499–1518, 1997. doi: 10.5636/jgg.49.1499.
- Misac N. Nabighian. Quasi-static transient response of a conducting permeable sphere in a dipolar field. *Geophysics*, 35(2):303–309, 1970. doi: 10.1190/1.1440092.
- Gregory A. Newman, Gerald W. Hohmann, and Walter L. Anderson. Transient electromagnetic response of a three-dimensional body in a layered earth. *Geophysics*, 51(8):1608–1627, 1986. doi: 10.1190/1.1442212.
- D. P. Nicholls and F. Reitich. Stability of high-order perturbative methods for the computation of Dirichlet–Neumann operators. *Journal of Computational Physics*, 170:276–298, 2001. doi: 10.1006/jcph.2001.6737.
- D. P. Nicholls and F. Reitich. Analytic continuation of Dirichlet–Neumann operators. *Numerische Mathematik*, 94:107–146, 2003. doi: 10.1007/s002110200399.
- D. P. Nicholls and F. Reitich. Shape deformations in rough-surface scattering: cancellations, conditioning, and convergence. *Journal of the Optical Society of America A*, 21:590–605, 2004. doi: 10.1364/JOSAA.21.000590.
- N. Ohshiman and T. Rikitake. Electromagnetic induction in an irregular layer overlying the earth, 2nd paper: a semi-infinite medium of finite conductivity having an undulatory surface. *Journal of Geomagnetism and Geoelectricity*, 37:643–657, 1985. doi: 10.5636/jgg.37.643.
- Ronald J. Pogorzelski and Eugene Lun. On the expansion of cylindrical vector waves in terms of spherical vector waves. *Radio Science*, 11(10):753–761, 1976. doi: 10.1029/RS011i010p00753.
- D. Prémel and G. Granet. Development of the curvilinear coordinate method for the computation of quasi-static fields induced by an eddy current probe scanning a 3D conductor of complex shape characterized by an arbitrary 2D surface. *International Journal of Numerical Modelling: Electronic Networks, Devices and Fields*, 31(2):e2219, 2018. doi: 10.1002/jnm.2219.

- Raphael Rochlitz, Nico Skibbe, and Thomas Günther. custEM: Customizable finite-element simulation of complex controlled-source electromagnetic data. *Geophysics*, 84(2):F17–F33, 2019. doi: 10.1190/geo2018-0208.1.
- Paula Rulff. emsig/elfe3D: Modelling with the total electric field approach using finite elements in 3D. Zenodo, concept DOI for all versions; used here: v1.0.1 plus 16 commits, commit 58528bd7 (21 July 2026), 2025a. doi: 10.5281/zenodo.13309721.
- Paula Rulff. elfe3D v1.0.1: Modelling with the total electric field approach using finite elements in 3D. *Journal of Open Source Software*, 10(110):7949, 2025b. doi: 10.21105/joss.07949.
- Paula Rulff, Laura M. Buntin, and Thomas Kalscheuer. Efficient goal-oriented mesh refinement in 3-D finite-element modelling adapted for controlled source electromagnetic surveys. *Geophysical Journal International*, 227:1624–1645, 2021. doi: 10.1093/gji/ggab264.
- K. Schwalenberg and R. N. Edwards. The effect of seafloor topography on magnetotelluric fields: an analytical formulation confirmed with numerical results. *Geophysical Journal International*, 159: 607–621, 2004. doi: 10.1111/j.1365-246X.2004.02280.x.
- Hang Si. TetGen, a Delaunay-based quality tetrahedral mesh generator. *ACM Transactions on Mathematical Software*, 41(2):11:1–11:36, 2015. doi: 10.1145/2629697.
- Sarva Jit Singh. Electromagnetic transient response of a conducting sphere embedded in a conductive medium. *Geophysics*, 38(5):864–893, 1973. doi: 10.1190/1.1440381.
- Julius A. Stratton. *Electromagnetic Theory*. McGraw-Hill, New York, 1941.
- A. Talbot. The accurate numerical inversion of Laplace transforms. *IMA Journal of Applied Mathematics*, 23(1):97–120, 1979. doi: 10.1093/imamat/23.1.97.
- William M. Telford, Lloyd P. Geldart, and Robert E. Sheriff. *Applied Geophysics*. Cambridge University Press, 2nd edition, 1990. doi: 10.1017/CBO9781139167932.
- Marc A. Vallée and Mouhamed Moussaoui. Modelling the electromagnetic response of a sphere located in a layered earth. *Exploration Geophysics*, 54(4):362–375, 2023. doi: 10.1080/08123985.2022.2162382.
- Marc A. Vallée and Mouhamed Moussaoui. MTSphere: magnetotelluric response of a sphere in a layered earth. <https://github.com/marcavallee/mtsphere>, 2026. Version 2.2.1, MIT licence.
- F. Vallianatos. A note on the topographic distortion of magnetotelluric impedance. *Annals of Geophysics*, 45:313–320, 2002. doi: 10.4401/ag-3516.
- R. G. Van Nostrand and K. L. Cook. Interpretation of resistivity data. Professional Paper 499, U.S. Geological Survey, 1966. doi: 10.3133/pp499.
- James R. Wait. A conducting sphere in a time varying magnetic field. *Geophysics*, 16:666–672, 1951. doi: 10.1190/1.1437716.
- James R. Wait and Kenneth P. Spies. Quasi-static transient response of a conducting permeable sphere. *Geophysics*, 34:789–792, 1969. doi: 10.1190/1.1440051.
- G. L. Wang and W. C. Chew. Formal solution to the electromagnetic scattering by buried dielectric and metallic spheres. *Radio Science*, 39:RS5004, 2004. doi: 10.1029/2003RS002976.
- Tsili Wang and Gerald W. Hohmann. A finite-difference, time-domain solution for three-dimensional electromagnetic modeling. *Geophysics*, 58(6):797–809, 1993. doi: 10.1190/1.1443465.

- Stanley H. Ward and Gerald W. Hohmann. Electromagnetic theory for geophysical applications. In Misac N. Nabighian, editor, *Electromagnetic Methods in Applied Geophysics, Vol. 1, Theory*, pages 131–311. Society of Exploration Geophysicists, Tulsa, 1988. doi: 10.1190/1.9781560802631.ch4.
- M. Weiss, T. Kalscheuer, and Z. Ren. Spectral element method for 3-D controlled-source electromagnetic forward modelling using unstructured hexahedral meshes. *Geophysical Journal International*, 232:1427–1454, 2023. doi: 10.1093/gji/ggac358.
- Dieter Werthmüller. An open-source full 3D electromagnetic modeler for 1D VTI media in Python: empymod. *Geophysics*, 82(6):WB9–WB19, 2017. doi: 10.1190/geo2016-0626.1. Version 2.6.0 used.
- Hasan Zamani, Ahad Tavakoli, and Mojtaba Dehmollaian. Scattering by a dielectric sphere buried in a half-space with a slightly rough interface. *IEEE Transactions on Antennas and Propagation*, 66(1):347–359, 2018. doi: 10.1109/TAP.2017.2772038.
- Kai Zhao and Decheng Hong. Analytical method of scattered electromagnetic field by a buried sphere under multilayered formation. *IEEE Transactions on Geoscience and Remote Sensing*, 64: 4504915, 2026. doi: 10.1109/TGRS.2026.3713624.
- J. Zhu, L. Xiao, Z. Jiang, Z. He, M. Li, and Z. Dou. Modeling of rugged topography distortions in ground TEM surveys for grounded-wire and loop source configurations. *Journal of Geophysics and Engineering*, 23:2185–2199, 2026. doi: 10.1093/jge/gxag013.

A Verification chain

This appendix documents every link of the verification chain summarised in Table 1: the sphere in the layered earth (Appendix A.1), the terrain over a half-space (Appendix A.2) and the unified code with the new elements (Appendix A.3).

A.1 Sphere in the layered earth

A.1.1 Null test against empymod

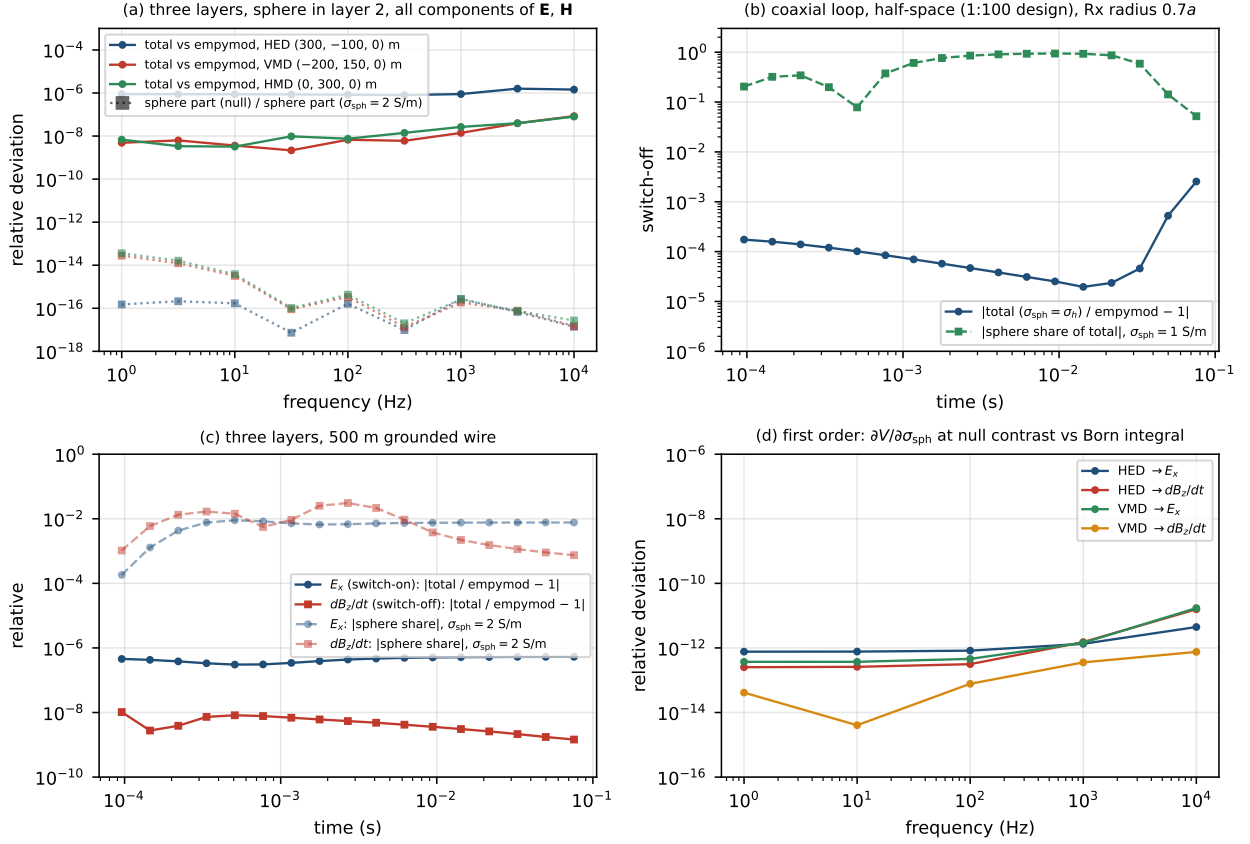


Figure 11: Null test against `empymod` (quasi-static): the sphere has the conductivity of its layer. (a) Maximum relative deviation of the total response (all components of $\mathbf{E}$ and $\mathbf{H}$ at three receivers) from `empymod` for three off-axis sources over a sphere in the middle layer of a three-layer earth; dotted, the sphere part at null contrast relative to that with $\sigma_{\text{sph}} = 2$ S/m. (b) Coaxial loop over the sphere in a half-space (1:100 design), receiver loop radius $0.7a$, switch-off; dashed, the share of the sphere in the total response for $\sigma_{\text{sph}} = 1$ S/m. (c) 500 m grounded wire in the three-layer earth: E_x (switch-on) and dB_z/dt (switch-off); dashed, the sphere share for $\sigma_{\text{sph}} = 2$ S/m. (d) First order: $\partial V / \partial \sigma_{\text{sph}}$ at null contrast from `terrasphere_csem` against the Born integral (36) of two `empymod` fields.

If the sphere has the conductivity of its layer and $\mu_r = 1$, the model is one-dimensional, and every response of `terrasphere_csem` must equal that of an independent 1-D code. We use `empymod` (Werthmüller, 2017) in quasi-static mode (Section 4). Both Mie coefficients then vanish, and the sphere part is zero to 10^{-16} of its value with contrast (Fig. 11a, dotted). The zeroth-order test therefore checks sources, Graf decomposition, propagators and receivers in a three-dimensional configuration. For off-axis HED, VMD and HMD sources over a sphere in the middle layer of a three-layer earth, all six field components agree with `empymod` from 1 Hz to 10 kHz: to 10^{-8} for the magnetic dipoles and to 10^{-6} for the grounded dipole (Fig. 11a). The grounded dipole is limited by the digital filter where source and receiver lie in the same horizontal plane, because the spectral kernel does not decay there. In the time domain the coaxial loop agrees to $2 \cdot 10^{-5}$ – $2 \cdot 10^{-4}$ from 0.1 to 33 ms (Fig. 11b). This is limited by the polygon with 256 corners with which `empymod` represents the loop. A 500 m grounded wire in the three-layer earth agrees to 3 – $5 \cdot 10^{-7}$ for E_x (switch-on) and to 10^{-7} – 10^{-9} for dB_z/dt (switch-off) between 0.1 and 76 ms (Fig. 11c). Its galvanic part needs the direct-current

term $V(0)$ of (34). In a homogeneous half-space that term equals the field of two point electrodes at the wire ends to $7 \cdot 10^{-5}$.

The zeroth-order test does not see the sphere, since $T = 0$. The *first-order* null test does: by Lorentz reciprocity the derivative of a response with respect to σ_{sph} at null contrast is the Born integral

$$\left. \frac{\partial V}{\partial \sigma_{\text{sph}}} \right|_{\sigma_{\text{sph}}=\sigma_\ell} = \int_{\text{sphere}} \mathbf{E}_{\text{tx}}(\mathbf{r}) \cdot \mathbf{E}_{\text{adj}}(\mathbf{r}) dV, \quad (36)$$

with $\mathbf{E}_{\text{tx}}$ the 1-D field of the transmitter and $\mathbf{E}_{\text{adj}}$ the 1-D field of a unit electric dipole at an electric receiver (oriented like it), or $-\mathbf{E}$ of a unit vertical magnetic dipole at a dB_z/dt receiver. Both 1-D fields come from `empymod`; the integral is evaluated with a Gauss–Legendre rule with $8 \times 12 \times 16$ points in r , θ and φ ; with $10 \times 16 \times 24$ points it changes by 10^{-15} . `terrasphere_csem` obtains the same quantity from (26), that is, from the source map with all orders m , the closed-form Mie derivative and the receiver map. For a sphere of 40 m radius in the middle layer, HED and VMD transmitters and off-axis E_x and dB_z/dt receivers, the two agree to 10^{-11} – 10^{-15} from 1 Hz to 10 kHz (Fig. 11d).

A.1.2 Reflection operator

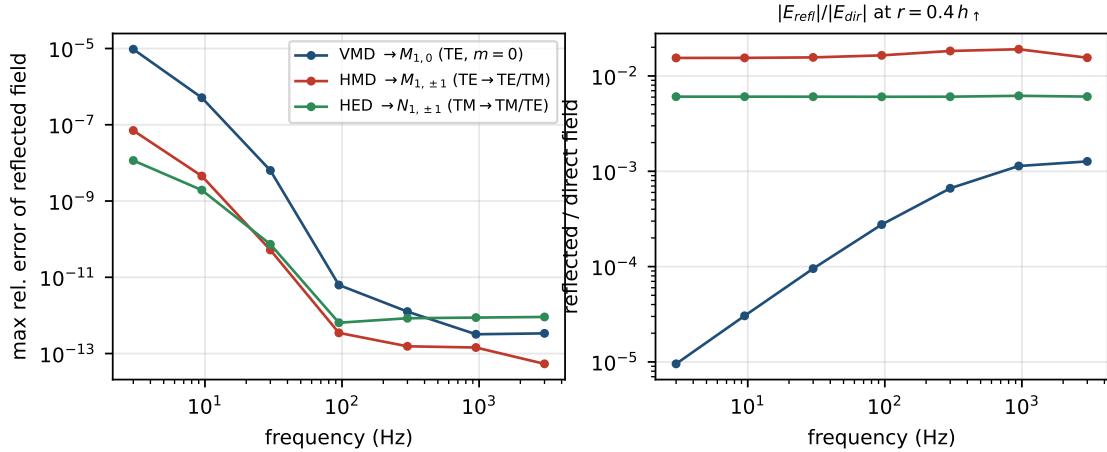


Figure 12: Reflection operator: relative error of the reflected field of a dipole at the sphere centre, expanded with R, against `empymod` (layered minus full space), and the size of the reflected field relative to the direct one at the evaluation points.

The reflection operator R couples the sphere to the layers and

is the central element of the formulation. A point dipole at the sphere centre radiates a pure degree-one outgoing wave: VMD $\rightarrow M_{1,0}$, HMD $\rightarrow M_{1,\pm 1}$, HED $\rightarrow N_{1,\pm 1}$. We place it in a two-layer earth (sphere layer = basement, $\sigma = 0.02, 0.004$ S/m, centre 110 m below the interface), re-expand its reflected field with R in regular waves up to $n = 20$ and evaluate it at points $0.4 h_\uparrow$ from the centre. The result equals `empymod`(layered) – `empymod`(full space) to $4 \cdot 10^{-13}$ (VMD), 10^{-13} (HMD) and $9 \cdot 10^{-13}$ (HED) at 400 Hz, and to 10^{-11} or better above 30 Hz (Fig. 12). Towards 3 Hz the reflected field drops to 10^{-5} of the direct one, and the subtraction in the reference limits the comparison. This checks the four coupling blocks of (25), i.e. Lee’s four coefficient families, including the TE $\leftrightarrow$ TM mixing for $m \neq 0$.

A.1.3 Closed-form limits

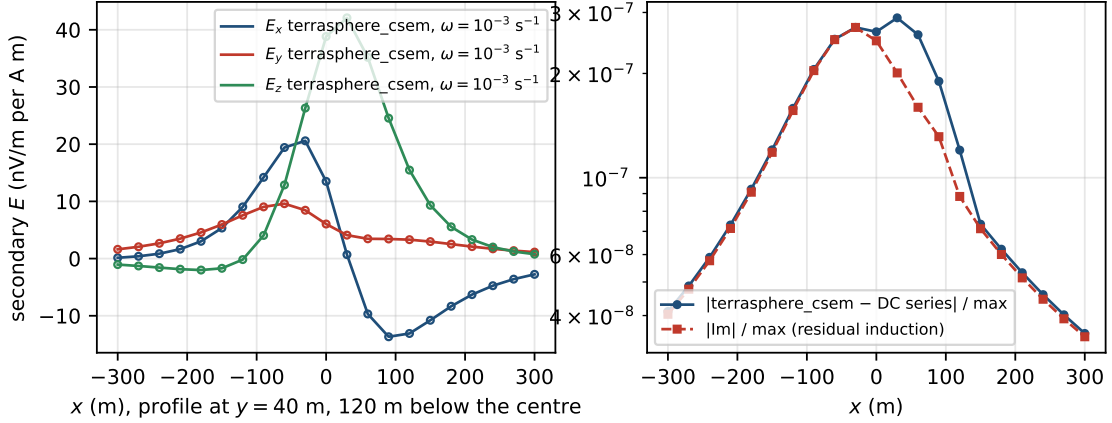


Figure 13: Direct-current limit of the grounded dipole: secondary electric field along a profile 120 m below the sphere centre from `terrasphe_csem` at $\omega = 10^{-3} \text{ s}^{-1}$ (lines) and from the point-source series (circles); right, the relative difference and the residual inductive part.

At $\omega = 10^{-3} \text{ s}^{-1}$ the secondary field (Fig. 13) of a horizontal electric dipole near a sphere in a full space ($b = 50 \text{ m}$, $\sigma_{\text{sph}}/\sigma_h = 50$) agrees with the point-source series $\phi_{\text{sec}} = (I/4\pi\sigma_h) \sum_n B_n r^{-n-1} P_n(\cos \gamma)/r_0^{n+1}$, $B_n = -b^{2n+1}n(\sigma_{\text{sph}} - \sigma_h)/(n\sigma_{\text{sph}} + (n+1)\sigma_h)$ (Telford et al., 1990), with the dipole as two electrodes, to $3 \cdot 10^{-7}$; the residual induction is of the same size. This checks the galvanic (TM) path absolutely and the scaling at $|k|b \sim 10^{-6}$. In the opposite limit, with the host conductivity set to 10^{-9} S/m , the coaxial loop over the 1:100 design reproduces the free-space multipole response of Nabighian (1970) to $3 \cdot 10^{-5}$ – $3 \cdot 10^{-4}$ (Fig. 3).

A.1.4 Time transform

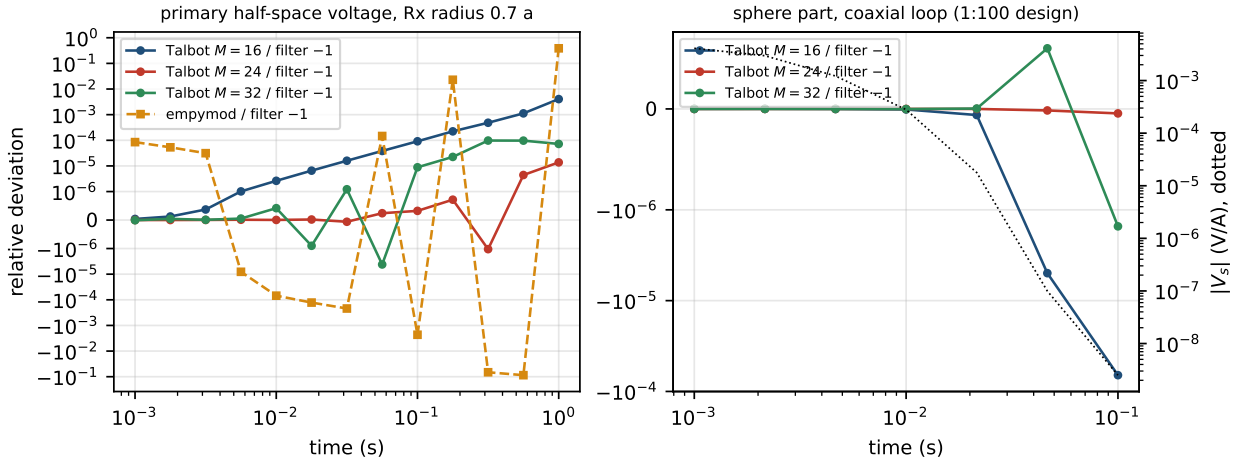


Figure 14: Laplace (Talbot) against direct sine-filter evaluation: left, primary half-space voltage (receiver loop radius $0.7a$), with `empymod` for comparison; right, sphere response of the coaxial loop (1:100 design), dotted its magnitude.

A comparison with a three-dimensional code in the frequency domain leaves the time transform to be checked; it needs its own check. We use Talbot's method in the fixed-parameter form of Abate and Valkó (2004), which replaces the Bromwich line by the contour $s(\theta) = r\theta(\cot \theta + i)$, $-\pi < \theta < \pi$,

$r = 2M/(5t)$, around the negative real s -axis (the negative imaginary ω -axis, where the branch cut lies), and invert

$$V_{\text{off}}(t) = V(0) + \mathcal{L}^{-1} \left[-\frac{V(is)}{s} - C \right](t), \quad C = \lim_{s \rightarrow \infty} \left(-\frac{V(is)}{s} \right), \quad (37)$$

with M evaluations at complex s per time. For the sphere in a conductive host $C = 0$; for the primary field of separated loops C is minus the free-space mutual inductance, which we remove analytically. For the primary half-space voltage of a 200 m loop with a receiver loop of radius 140 m, the Talbot route (37) with $M = 24$ and the direct sine-filter evaluation agree (Fig. 14, left) to 10^{-6} – 10^{-5} from 1 ms to 300 ms. **empymod**'s time-domain result deviates by 0.2 % at 100 ms and 5 % at 300 ms, where the voltage has decayed to 10^{-5} and 8×10^{-7} of its value at 1 ms; relative to the early response this is $3\text{--}4 \times 10^{-8}$, the level of the time-domain comparison with **empymod** in Table 1. The deviation does not change with **empymod**'s Fourier filter (**key_201** or **key_601**), the receiver depth (2 cm or $1 \mu\text{m}$) or the number of polygon sides (64 or 256). For the sphere response of the coaxial loop (1:100 design) the two routes agree to $3 \cdot 10^{-14}$ at 0.1 ms, 10^{-11} at 1–10 ms and $5 \cdot 10^{-8}$ at 100 ms, where the response has decayed to $3 \cdot 10^{-8}$ of its early value.

A.1.5 Consistency of the mathematics

Identities.

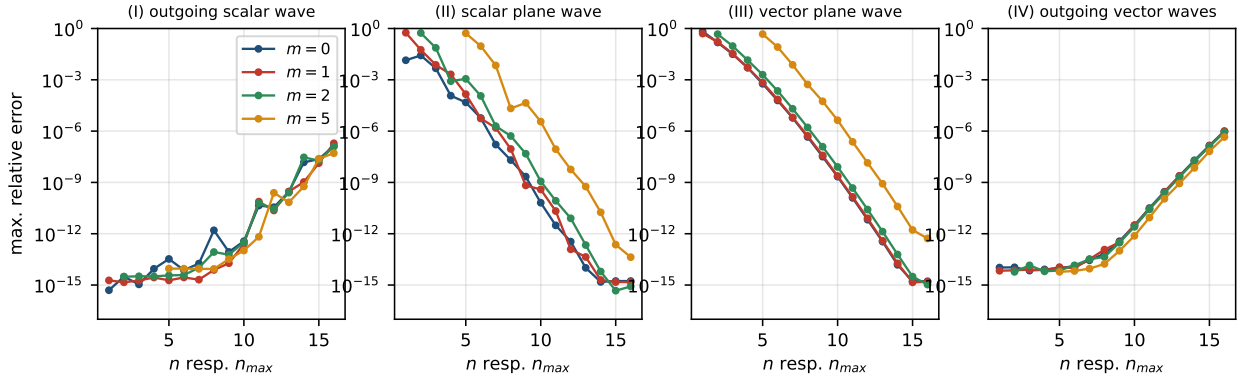


Figure 15: Numerical verification of the four identities of Section 2.5: maximal relative error over test points as a function of the degree n (I, IV) or of the truncation degree n_{max} (II, III), for several orders m ; $k = 0.9 + 0.6i$, points at $|z| = 0.8 \dots 1.5$.

(I) and (IV) are checked (Fig. 15) at points above and below the equatorial plane with complex $k = 0.9 + 0.6i$ against direct λ -quadrature; (II) and (III) against the series truncated at n_{max} , the vector case against a direct β -quadrature of the plane-wave integral (12). All reach 10^{-14} , the series converging exponentially in n_{max} . For (III) we also implemented the formulas of Han et al. (2008) exactly as printed: their series reproduces their closed cylindrical wavefunctions to $9 \cdot 10^{-15}$ for a propagating wave and for evanescent waves with complex k , and our coefficients equal theirs after the change of normalisation to $7 \cdot 10^{-16}$ (`tests/verification/test_spectral.py`). Their coefficients coincide term by term with eqs. (29) and (30) of Pogorzelski and Lun (1976). The identity (IV) is the one where a sign error is easiest to make: with the measure $\lambda/(kk_z)$ instead of $\lambda/(iku)$ the lower half-space comes out with the wrong sign.

Mie coefficients.

For each degree the incident, scattered and interior fields built with (9)–(10) and (20) are evaluated on the sphere; the residuals of the tangential $\mathbf{E}$ and $\mathbf{H}$ continuity are 10^{-15} (Fig. 16a) for $\mu_r = 1$ and 3, for both mode types and all $n \leq 30$. The scaled magnitudes $|b_n|/|k|^{2n+1}$ stay finite as $\omega \rightarrow 0$ as intended. The scaled form reproduces the direct evaluation of (20) to 10^{-16} and vanishes to 10^{-16} for zero contrast.

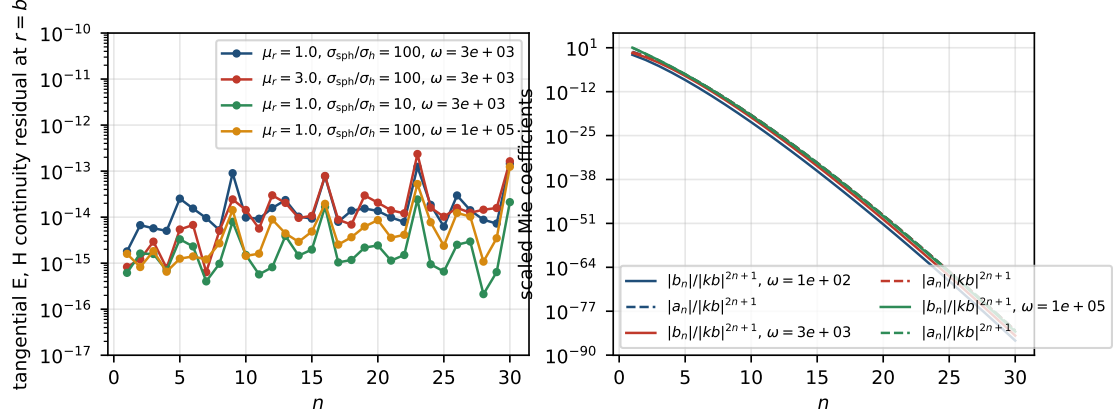


Figure 16: Mie T-matrix with permeability: (a) residual of the tangential field continuity at $r = b$ for TE and TM modes, (b) scaled coefficients of the 1:100 design ($b = 180$ m, $\sigma_{sph} = 1$ S/m, $\sigma_h = 0.01$ S/m).

Layered-earth kernels.

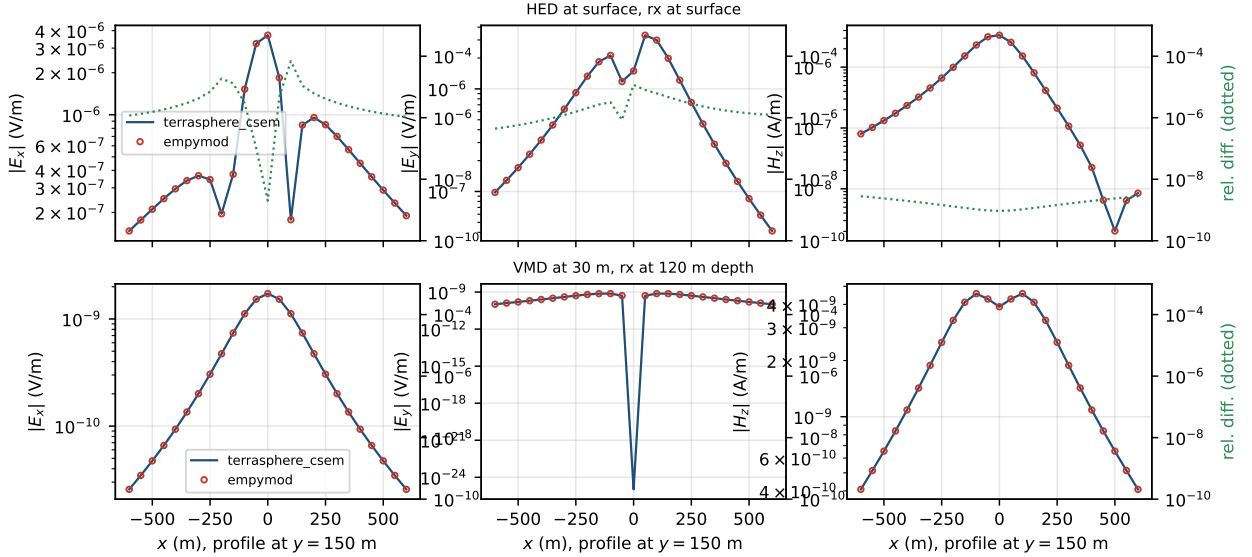


Figure 17: 1-D fields from our kernels (lines) and **empymod** (circles) along a surface profile in a three-layer earth ($\sigma = 0.02, 0.002, 0.05$ S/m, interfaces at 80 and 250 m), $f = 100$ Hz: top, HED at the surface; bottom, VMD at 30 m depth with receivers at 120 m depth. Dotted: relative difference.

The 1-D fields of HED, VMD and HMD sources in a full space, a half-space and a three-layer earth, with sources at the surface and in layers 1–3 and receivers at the surface and at depth, agree with **empymod** in all six components to 10^{-8} – 10^{-15} (Fig. 17) once **empymod** is run quasi-statically and its time, frame and sign conventions are converted (validation report of the repository; the full-space case was first checked against the closed-form dipole fields, which fixed those conventions). Only where source and receiver lie in the same horizontal plane does the spectral kernel not decay; there the digital filter limits the agreement (full space on the source plane 10^{-3} , where **empymod** uses its analytic direct field; grounded dipole surface-to-surface 10^{-5}). This validates the source scalars (7), the Graf decomposition (38), the propagator (5) and the component formulas (2).

Derivatives.

The closed-form derivatives agree with central finite differences (Fig. 18) to 10^{-8} (σ_{sph}, b, μ_r)

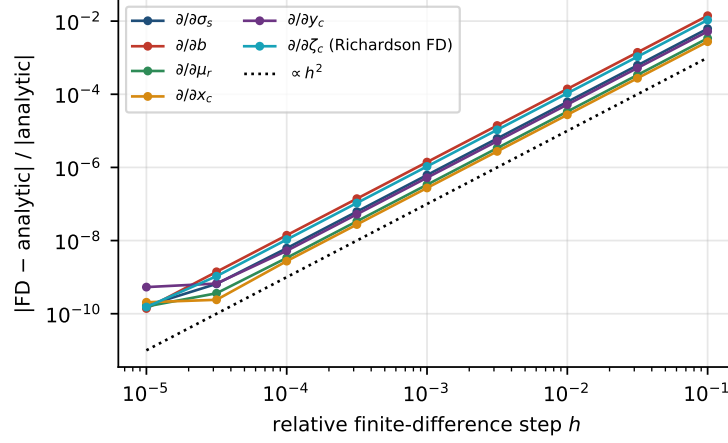


Figure 18: Analytic derivatives of the sphere response against central finite differences as a function of the step: the deviation falls as h^2 to the round-off floor; the depth derivative is itself a Richardson-extrapolated difference.

and 10^{-10} (x_c , y_c) for electric, dB/dt and loop receivers; the finite-difference error decreases as h^2 until round-off.

Reciprocity. In a two-layer earth with the sphere in the basement, $\hat{\mathbf{d}}_B \cdot \mathbf{E}_B(p_A) = \hat{\mathbf{d}}_A \cdot \mathbf{E}_A(p_B)$ holds to $2 \cdot 10^{-15}$ for electric dipoles at arbitrary surface positions, and $\mathbf{E}(\text{VMD at } B) = i\omega\mu_0\mathbf{H}(\text{HED at } A)$ exactly.

A.1.6 Coaxial loop against independent single-order references

For the 1:100 design (loop radius 200 m, sphere $b = 180$ m at 250 m depth, $\sigma_{\text{sph}} = 1$ S/m, $\sigma_h = 0.01$ S/m) the full assembly reproduces (Fig. 3) two independent single-order references (tests/reference/) in the frequency domain to 10^{-10} (half-space, Lee 1981 physics with all degrees) and 10^{-16} (full space, Singh 1973 physics). In the time domain it agrees with the full-space reference, whose transform also evaluates the filter directly, to 10^{-12} from 0.05 to $12\tau_1$, and with the half-space reference to 10^{-5} up to $2.3\tau_1$; beyond that the cubic-spline transform of the latter limits the agreement (10^{-4} at $7\tau_1$, $4 \cdot 10^{-3}$ at $12\tau_1$, Section 2.17). With the host conductivity set to 10^{-9} S/m it reproduces the free-space multipole response of Nabighian (1970) to $3 \cdot 10^{-5}$ – $3 \cdot 10^{-4}$; the degree-one formula of Lee (1983) is shown for comparison (Section 5.1).

A.2 Terrain over a half-space

The terrain expansion is verified with the wire and the Gaussian of Section 6 (Fig. 5) over a half-space of $\sigma = 0.01$ S/m ($100 \Omega\text{m}$), with heights up to 150 m (31°) for the convergence study. Because the half-space is homogeneous, the responses scale with σ : time with σ , the electric field with $1/\sigma$.

A.2.1 Flat-earth surface field

For this half-space the closed-form surface fields agree with **empymod** (Werthmüller, 2017) (quasi-static, receivers $1 \mu\text{m}$ below the surface) to 10^{-6} for E_x, E_y and 10^{-8} – 10^{-6} for H_z between 10^{-3} Hz and 100 kHz. The continuation of the surface data to $z = \pm 8$ m by the recursions (30) reproduces the **empymod** field at that depth or height to 10^{-4} up to 30 kHz. The filter transform of the closed-form spectra reproduces the closed-form transients $e_x(t)$ and $\partial_t b_z(t)$ of the flat half-space to 10^{-9} between 10^{-5} and 10 s, and $e_y(t)$ vanishes identically.

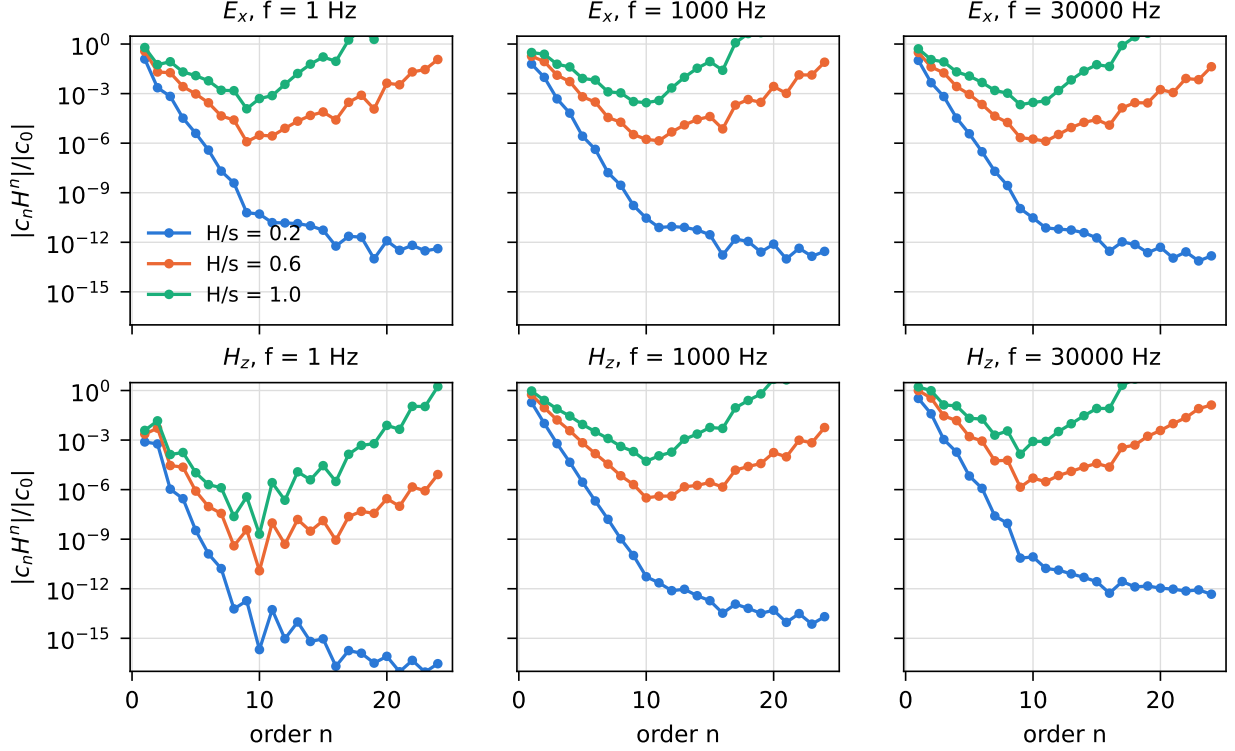


Figure 19: Terms of the height series on the summit, relative to the flat-earth value, for three heights and three frequencies; top E_x , bottom H_z .

A.2.2 Convergence of the series

Figure 19 shows the terms $|c_n H^n|$ on the summit. For $H/s = 0.2$ they fall by about one decade per order and reach 10^{-12} of the flat field after twelve orders, at all frequencies shown. For $H/s = 0.6$ (20°) the smallest term is 10^{-5} – 10^{-7} near order ten; beyond it the terms grow again. For $H/s = 1$ (31°) the smallest term is 10^{-3} – 10^{-4} . The growth beyond the optimal order is the ill-conditioning of the field expansion known from the wave problem (Nicholls and Reitich, 2001, 2004): the factors u_1^j and λ^j amplify the residual high-wavenumber content. Optimal truncation therefore leaves smallest terms of about 10^{-12} , 10^{-6} and 10^{-4} of the flat field for the three slopes; these indicate the truncation error but are not independently established error bounds.

A.2.3 Independent formulation at direct current

At DC the problem reduces to Laplace’s equation for the potential with a homogeneous Neumann condition on the terrain and point electrodes on the flat ground. We expanded this scalar problem independently, with the same grid but without any reference to the magnetic field. Figure 20 compares both formulations over hill and depression of ± 90 m: they agree to 10^{-7} of the profile maximum, while the terrain changes E_x by up to 37%. The hill lowers $|E_x|$ on the summit and raises it on the flanks; the depression does the opposite, as known from resistivity surveys (Fox et al., 1980).

A.2.4 Numerical parameters

Doubling the box to 10.24 km changes the terrain effect for $H = 30$ m by at most 1.2×10^{-3} of its size (periodic images; largest for H_z at low frequency); halving the grid spacing, moving the low-pass edge between 0.15 and 0.3 rad/m or widening the window changes it by less than 3×10^{-5} . For $H = 90$ m the result depends more on these choices. Moving the low-pass edge down to 0.15 rad/m

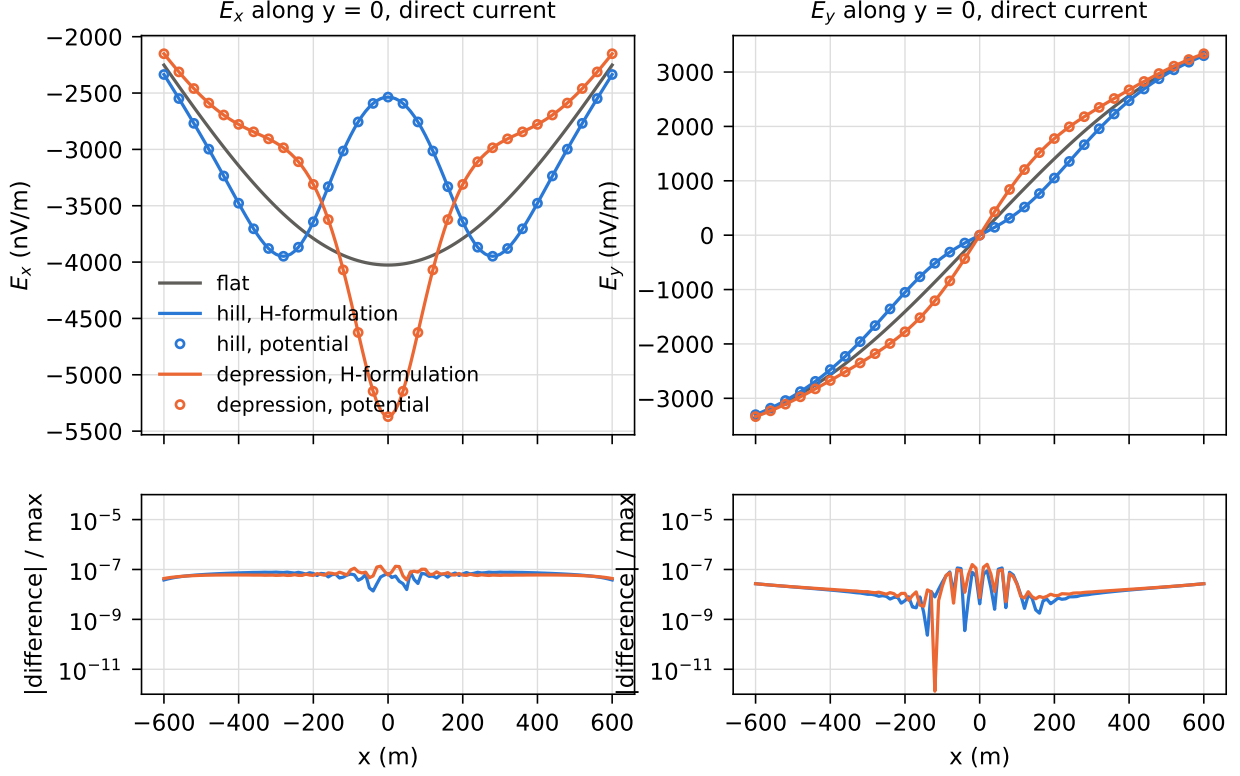


Figure 20: Direct-current check along $y = 0$. Lines: magnetic-field formulation; circles: independent scalar-potential expansion; grey: flat earth. Bottom: difference of the two formulations.

changes the terrain effect by less than 6×10^{-4} , but a finer grid changes the magnetic terrain effect by up to 2 %, and a wider low-pass (0.3 rad/m) by up to 40 %, because more high-wavenumber content reaches the higher orders. The electric field is less sensitive (at most 4×10^{-3} of its maximum). We therefore use $\lambda_c = 0.2$ rad/m and estimate, without a dedicated stability study, an uncertainty of a few per cent of the terrain effect for the $H = 90$ m magnetic results.

A.3 Unified code and coupling

A.3.1 Grounded dipole and loops

The closed-form surface field of a rotated HED ($\psi = 0.7$) agrees with the layered 1-D code of Section 2.2 at four receivers between 0.6 and 1.2 km to 1.3×10^{-5} ($\mathbf{E}$) and 10^{-8} ($\mathbf{H}$) from 0.1 Hz to 1 kHz; the layered code is evaluated 1 mm below the surface for $\mathbf{E}$, because its digital filter loses the galvanic field to 5×10^{-4} when source and receiver lie exactly in one plane (the 1 mm offset itself changes the fields by $O(\gamma \cdot 1 \text{ mm})$). A rectangular loop (800×600 m, 12 Gauss points per side) and a circular loop (radius 350 m, 256-gon against a 1024-gon) agree with the layered code inside and outside the loop to 10^{-7} – 2×10^{-9} ($\mathbf{H}$ at the surface) and 10^{-4} at 1 Hz to 10^{-6} at 100 Hz ($\mathbf{E}$, 1 mm deep). A square loop of 2.6 km side centred on the hill reproduces the 90° symmetry of the configuration on the two summit profiles to 8×10^{-14} in all orders; its electric field vanishes at DC to 4×10^{-16} of its value at 10 Hz, and the hill of 30 m changes H_z at 10 Hz by 0.4 %.

For a grounded point dipole (100 m, at $(0, -1500)$ m), the DC limit of the magnetic-field series ($f = 10^{-6}$ Hz, 12 orders) agrees with the independent potential expansion of Appendix A.2.3 to 3×10^{-8} – 3×10^{-7} for hill and depression of ± 30 m, where the terrain changes E_x and E_y by up to 12 %. Rotating the dipole and its position by 90° about the hill axis, from $(0, -1500)$ m in x to $(1500, 0)$ m in y , rotates the fields of all orders on the two summit profiles into each other to 2.7×10^{-15} .

A.3.2 Layers below the terrain

The layer correction of the surface field (Section 2.3) is tested against the layered 1-D code for a three-layer earth (0.01, 0.1, 0.002 S/m, interfaces at 80 and 300 m) and a rotated HED: the correction alone, layered minus half-space in both codes, agrees to 10^{-11} – 10^{-15} ($\mathbf{H}$) and 2×10^{-7} – 5×10^{-6} ($\mathbf{E}$, 1 mm deep) of the total field from 0.1 Hz to 30 kHz. The DC potential expansion was extended to layers below the terrain with the potential reflection $\rho_V = (\sigma_1 - \sigma_2)/(\sigma_1 + \sigma_2)e^{-2\lambda d_2}$ (and its recursion for more layers) and the layered surface potential; it shares with the magnetic-field series only the grid. For a conductive basement (0.1 S/m below 100 m) and a resistive intermediate layer (0.001 S/m between 60 and 200 m over 0.05 S/m), hill and depression of ± 30 m, the two agree to 3×10^{-8} – 10^{-6} of the profile maximum, while the terrain changes the field by 2–20 % and the layering by a factor of up to nine.

A.3.3 Sphere below the terrain

Order zero. The sphere's response to the sources on surface points from the radial tables of Section 3 agrees with the receiver map of the sphere model (Section 2.7) evaluated point by point to 10^{-10} – 6×10^{-8} in all five components from 1 Hz to 3 kHz (sphere $b = 60$ m, 1 S/m, centre 140 m deep and off-axis, two layers).

Higher orders. The grid route of Section 2.14 is tested with a smooth incident field: a vertical magnetic dipole 200 m above the ground, whose down-going field at $z = 0$ is given on the grid as the input of an order. The sphere's field at surface points returned on the grid agrees with the sphere model to 2.5×10^{-3} (3 Hz) and 2×10^{-4} (300 Hz) in the 5.12 km box and to 4.6×10^{-4} and 2.6×10^{-5} in the 10.24 km box: the residual is the periodicity of the box (images of source and sphere), which falls by a factor 5–9 per doubling. At null contrast ($\sigma_{\text{sph}} = \sigma_\ell$) the coupled series equals the layered series to 9×10^{-17} . With the sphere on the hill axis and the source rotated by 90° , the coupled coefficients of all orders rotate into each other to 3.9×10^{-15} , while the sphere changes the terrain orders by 32 %; this checks the azimuthal bookkeeping of the coupling for all orders m .

Coupled series and box. For the configuration of Section 6 the smallest term of the coupled series at optimal truncation is 2×10^{-8} ($H = 30$ m), 10^{-4} (60 m) and 10^{-2} (90 m) of the flat field at 3–3000 Hz, against 2×10^{-13} to 9×10^{-7} without the sphere; for 60 and 90 m the terms grow again beyond order 9–12, a consequence of the distance between the expansion plane and the sphere (Section 6). Doubling the box from 5.12 to 10.24 km at 3 Hz (12 orders, $N_g = 512$ against 1024) changes the terrain effect by 3×10^{-5} – 4×10^{-3} of its maximum with the sphere and by the same amounts without it (3×10^{-5} – 3.9×10^{-3}); the sphere coupling adds no periodicity error of its own beyond that of the terrain orders, whose leading low-frequency part the box extrapolation of Section 3 reduces.

B Sources about the sphere axis

The source scalars (7) are referred to the vertical axis through the *sphere* centre by the addition theorems of Neumann and Graf (Abramowitz and Stegun, 1964): with $(\rho_{\text{tx}}, \varphi_{\text{tx}})$ the source position relative to that axis,

$$\begin{aligned} J_0(\lambda\rho') &= \sum_m J_m(\lambda\rho)e^{im\varphi} C_m^0, & C_m^0 &= J_m(\lambda\rho_{\text{tx}})e^{-im\varphi_{\text{tx}}}, \\ J_1(\lambda\rho') \cos \varphi' &= \sum_m J_m(\lambda\rho)e^{im\varphi} C_m^{\cos}, & C_m^{\cos} &= \frac{1}{2}[J_{m-1}(\lambda\rho_{\text{tx}})e^{-i(m-1)\varphi_{\text{tx}}} - J_{m+1}(\lambda\rho_{\text{tx}})e^{-i(m+1)\varphi_{\text{tx}}}], \\ J_1(\lambda\rho') \sin \varphi' &= \sum_m J_m(\lambda\rho)e^{im\varphi} C_m^{\sin}, & C_m^{\sin} &= -\frac{i}{2}[J_{m-1}(\lambda\rho_{\text{tx}})e^{-i(m-1)\varphi_{\text{tx}}} + J_{m+1}(\lambda\rho_{\text{tx}})e^{-i(m+1)\varphi_{\text{tx}}}], \end{aligned} \tag{38}$$

where $\partial_x J_0(\lambda\rho') = -\lambda J_1(\lambda\rho') \cos \varphi'$ and $\partial_y J_0 = -\lambda J_1 \sin \varphi'$ with φ' the azimuth of the field point about the source. The resulting mode amplitudes $A_q^\pm(\lambda, m)$ of the wave $J_m(\lambda\rho)e^{im\varphi}e^{-u|z-z_{\text{tx}}|}$ are

Table 3: Mode amplitudes of the point sources about the sphere axis, $c_\psi = \cos \psi$, $s_\psi = \sin \psi$, k^2 and u of the source layer. A^+ : $z > z_{\text{tx}}$; A^- : $z < z_{\text{tx}}$.

source	TE (H_z): A^+ , A^-	TM (J_z): A^+ , A^-
VMD m_z	$m_z \frac{\lambda^3}{4\pi u} C_m^0$ (both)	0
loop I , radius a	$\frac{Ia}{2} \frac{\lambda^2}{u} J_1(\lambda a) C_m^0$ (both)	0
HED p , ψ	$p \frac{\lambda^2}{4\pi u} [c_\psi C_m^{\text{sin}} - s_\psi C_m^{\text{cos}}]$ (both)	$\pm p \frac{\lambda^2}{4\pi} [c_\psi C_m^{\text{cos}} + s_\psi C_m^{\text{sin}}]$
HMD m_h , ψ	$\pm m_h \frac{\lambda^2}{4\pi} [c_\psi C_m^{\text{cos}} + s_\psi C_m^{\text{sin}}]$	$k^2 m_h \frac{\lambda^2}{4\pi u} [c_\psi C_m^{\text{sin}} - s_\psi C_m^{\text{cos}}]$ (both)

listed in Table 3; the sign of ∂_z distinguishes A^+ (above the source) from A^- . The decomposition reduces the spectral integral over both horizontal wavenumbers, which Wang and Chew (2004) keep (their eqs. 22 and 24–26), to one λ -integral per azimuthal order; Zhao and Hong (2026) use the same addition theorem (their eq. 11).

C Identity (III) in the notation of Han et al.

Identity (III), eq. (17), is written in our normalisation. With their cylindrical and spherical vector wavefunctions $\mathbf{m}_{m\lambda}$, $\mathbf{n}_{m\lambda}$ and $\mathbf{m}_{mn}$, $\mathbf{n}_{mn}$, ours are $\mathbf{W}_m^{\text{TE}} = (i/\lambda) \mathbf{m}_{m\lambda}$, $\mathbf{W}_m^{\text{TM}} = -(1/\lambda) \mathbf{n}_{m\lambda}$, $\mathbf{M}_{nm}^{(1)} = N_{nm} \mathbf{m}_{mn}$ and $\mathbf{N}_{nm}^{(1)} = N_{nm} \mathbf{n}_{mn}$, so that $A_{nm}^{\text{TE}} = iA_{mn}/(\lambda N_{nm})$, $B_{nm}^{\text{TE}} = B_{mn}/(\lambda N_{nm})$, $A_{nm}^{\text{TM}} = iB_{mn}/(\lambda N_{nm})$ and $B_{nm}^{\text{TM}} = -A_{mn}/(\lambda N_{nm})$ with their A_{mn} , B_{mn} .

D Surface field of a dipole on a half-space

For an x -directed HED of moment p at the origin and a receiver at (x, y) on the surface of the homogeneous half-space, $r = |\mathbf{x}|$, $\gamma = (-i\omega\mu_0\sigma_1)^{1/2}$ with $\text{Re } \gamma > 0$,

$$\begin{aligned} E_x &= \frac{p}{2\pi\sigma_1 r^3} \left[\frac{3x^2}{r^2} - 2 + (1 + \gamma r)e^{-\gamma r} \right], & E_y &= \frac{p}{2\pi\sigma_1 r^3} \frac{3xy}{r^2}, \\ H_z &= \frac{py}{2\pi\gamma^2 r^5} [3 - (3 + 3\gamma r + \gamma^2 r^2)e^{-\gamma r}] \end{aligned} \quad (39)$$

(Ward and Hohmann, 1988, written there for $e^{+i\omega t}$). Writing $\gamma_0 = \gamma$, the two-dimensional Fourier transform of H_z is $\tilde{H}_z = -ip \xi_y/(u + \lambda)$ with $u = (\lambda^2 + \gamma_0^2)^{1/2} = u_1$. On the air side $\mathbf{H} = -\nabla\Phi$ with $\tilde{\Phi} = \tilde{H}_z/\lambda$, so $\Phi = p \partial_y G$ with $\tilde{G} = -1/[\lambda(u + \lambda)]$. Hence

$$H_x = -p \frac{xy}{r^2} \left(G'' - \frac{G'}{r} \right), \quad H_y = -p \left(\frac{y^2}{r^2} G'' + \frac{x^2}{r^3} G' \right), \quad (40)$$

$$G' = \frac{1}{4\pi r} + \frac{1}{2\pi} \int_0^\infty \frac{\lambda - u}{2(u + \lambda)} J_1(\lambda r) d\lambda, \quad G'' = -\frac{G'}{r} + \frac{1}{2\pi} \int_0^\infty \frac{\lambda(\lambda - u)}{2(u + \lambda)} J_0(\lambda r) d\lambda. \quad (41)$$

Both kernels are smooth and decay like λ^{-2} and λ^{-1} ; the constant $1/(4\pi r)$ is the DC (Biot–Savart) part. Numerically, $\partial_x H_y - \partial_y H_x$ vanishes to 10^{-12} and $\nabla_h \cdot \mathbf{H}_h + \partial_z H_z$ to the accuracy of the finite differences used (2×10^{-6} – 5×10^{-5}).

E Derivatives of the sphere response

The derivatives of the Mie coefficients (21) are closed: with $z = mx$, the Riccati–Bessel equation gives $q'_\psi = n(n+1)/x^2 - 1 - q_\psi^2$, $q'_\xi = n(n+1)/x^2 - 1 - q_\xi^2$, $D'_n(z) = n(n+1)/z^2 - 1 - D_n^2$,

and $(\ln J_n)' = q_\psi - q_\xi$, i.e. $(\ln[J_n/x^{2n+1}])' = q_\psi - q_\xi - (2n+1)/x$ for the scaled form that is evaluated; the chain rule with $dx/db = k$, $dz/db = k_{\text{sph}}$, $dz/d\sigma_{\text{sph}} = z/2\sigma_{\text{sph}}$, $dm/d\sigma_{\text{sph}} = m/2\sigma_{\text{sph}}$, $dz/d\mu_r = z/2\mu_r$, $dm/d\mu_r = m/2\mu_r$ and the explicit μ_r in $G_n^{a,b}$ completes them. The horizontal position (x_c, y_c) of the sphere enters only the source and receiver maps; moving the centre by $+\delta$ equals moving all sources and receivers by $-\delta$, and for a mode $\psi_m = J_m(\lambda\rho)e^{im\varphi}$

$$\partial_x \psi_m = \frac{\lambda}{2}(\psi_{m-1} - \psi_{m+1}), \quad \partial_y \psi_m = \frac{i\lambda}{2}(\psi_{m-1} + \psi_{m+1}), \quad (42)$$

so that the derivative of the receiver map is the same map with orders $m \pm 1$, and that of the source map follows from the same identity applied to the Graf coefficients (38). The derivatives are checked against finite differences in Appendix A.1.5.

F Numerical details and cost

Sphere: scaling. Without it the rows of degrees above 25 spread over fifty orders of magnitude and their round-off pollutes the low degrees at the 10^{-5} level; with it the degree series of the 1:100 design ($b/h_\uparrow = 0.72$) converges like $(b/c)^{2n}$, c the distance from the centre to the loop wire: $N = 10$ gives 10^{-7} (low frequency) to 10^{-5} (skin-depth regime), $N = 15$ gives 10^{-9} to $3 \cdot 10^{-8}$, $N = 20$ gives 10^{-12} , and $N = 30$ or 40 reach machine precision. A skin-depth guard sets the sphere response to zero when $\text{Im } k(h-b) > 60$, where the Mie ratios j_n/h_n would overflow and the response is below 10^{-26} anyway.

Sphere: quadrature. The panels have 24 points each and cover $[0, (45 + 2.5N)/h_{\text{min}}]$, with the panel count proportional to $\lambda_{\text{max}}\rho_{\text{max}}$. Doubling the panels, raising the cut-off by half or increasing the points per panel to 32 changes the response of the 1:100 design by 10^{-9} or less. Sources on the sphere axis excite $m = 0$ (loops) or $m = 0, \pm 1$ (dipoles) only. All Bessel functions of the model (the Graf coefficients of the sources, the loop factors $J_1(\lambda a)$ and the receiver weights) depend on the geometry but not on the frequency; they are computed once per transient and reused. One frequency then costs 0.2s for the coaxial loop (one order), 0.6s for an off-axis loop with a dB_z/dt receiver (21 orders, $N = 10$) and 2.3s for a 500m grounded wire with seven Gauss points and two receivers (25 orders, $N = 12$) on one core; a transient at 50 lagged times (three decades, 250 frequencies) thus takes 1, 2.5 and 10 minutes. The frequencies are independent and can be distributed over cores.

Terrain. One frequency with 20 orders costs about 10s on 12 cores; the cost grows quadratically with the number of orders and like $N_g^2 \log N_g$ with the grid. The sphere's response to the sources on the grid is computed per azimuthal order from the receiver map of Section 2.7 as five radial functions (cylindrical components of $\mathbf{E}$ and $\mathbf{H}$) on a table in ρ , with Gauss–Legendre panels in λ up to $40/(d_c - b)$, and splined; the Bessel table depends on the geometry only and is reused for all frequencies. The coupling of a higher order uses about 8×10^4 wavenumbers for $N_g = 512$: the regular coefficients are sums of identity (III) over these wavenumbers, the outgoing spectra identity (IV) at the same wavenumbers, and the two-dimensional density of Section 2.14 is multiplied by $N_g^2 \Delta \xi^2 e^{-i\xi \cdot (\mathbf{x}_0 + \mathbf{x}_c)}$, $\Delta \xi = 2\pi/(N_g \Delta x)$, $\mathbf{x}_0$ the grid origin and $\mathbf{x}_c$ the sphere axis, to give FFT coefficients. For the production grid, 20 orders, a wire of 96 Gauss points and a sphere with degrees up to $N = 10$, one frequency costs about 70s on 12 cores, of which 50s are the series and 20s the reference field.

Degree of the sphere near two interfaces. For the sphere of Section 7.4 ($b = 180$ m, centre 400 m deep, interfaces at 200 and 600 m) and the in-line wire under the 90 m hill, the series was repeated with degrees $N = 12, 15$ and 20 (without the box correction, which does not depend on N). The change of the sphere anomaly from $N = 12$ to 15 and to 20 is 10^{-8} – 2×10^{-7} of the anomaly maximum per component at 0.1 and 10 Hz and at most 4×10^{-7} at 1 kHz on both profiles, except for E_z on the y -profile at 1 kHz (1.8×10^{-6}), where the anomaly is 0.3% of the field. $N = 15$ and 20 give the same change, so $N = 12$ is converged to this level.

Transients with terrain. The spectrum of a transient is the flat-earth field at every filter frequency plus the terrain part of the series up to a cut-off frequency f_c , the last frequency from 1 Hz upwards

at which the smallest term of the series stays below 10^{-3} of the largest terrain effect (the default **tolerance**); above f_c the terrain part is continued as a constant, and the box correction is faded out towards its upper frequency. f_c is written with every transient result. This continuation is a numerical choice; transients with terrain, and with terrain and sphere, are not validated separately in this paper, and early times governed by frequencies above f_c should not be used as a reference.