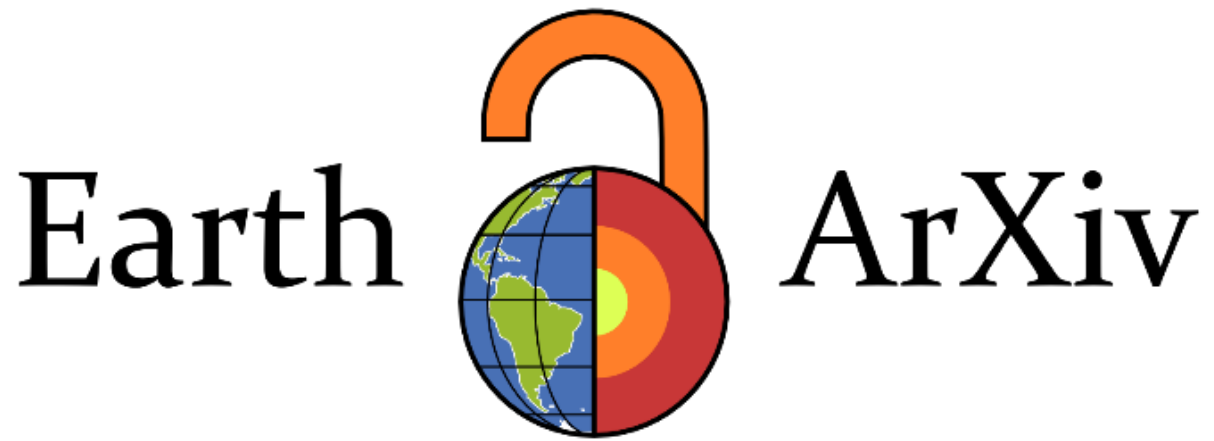




Peer review status:

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The Dynamics of Nonlinear Triads of Rossby-Haurwitz and Gravity Shallow Water Waves on the Sphere

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(Received xx; revised xx; accepted xx)

Understanding the interplay between nonlinearity and multiple scales is one of the cornerstones of geophysical fluids and climate modelling. Nonlinear interactions between atmospheric waves of disparate scales, from fast inertia-gravity waves to slow planetary-scale Rossby-Haurwitz waves, are central to energy cascades and to low-frequency phenomena. Yet, they remain poorly understood beyond simplified f-plane and beta-plane settings, and are often left only partially represented once gravity waves are filtered out of forecast models. Here, we study these couplings directly in the rotating shallow-water equations on the sphere with realistic atmospheric parameters for a single layer representing the troposphere. We derive reduced models of three, four and five nonlinearly interacting Hough harmonics to quantify the efficiency and timescale of energy exchange between wave families including both slow (Rossby-Haurwitz) and fast (inertia-gravity) modes. We show, in several configurations, that inertia-gravity modes can influence the slow Rossby-Haurwitz modes both through exchange of energy and by generating new low-frequency oscillations.

Key words: Atmospheric flows, Waves in rotating fluids, Low-Dimensional models, General fluid mechanics, Computational methods

1. Introduction

The complexity of Earth's atmospheric dynamics arises, in part, from the interplay between nonlinearity and multiple spatio-temporal scales. One instance is the interaction of atmospheric waves with disparate characteristic frequencies, such as acoustic waves (minutes to hours), inertia-gravity waves (days) and Rossby waves (days to weeks) (Vallis 2017; Salmon 1998), whose distinct dispersion branches have recently been linked to the topology of the underlying rotating-fluid symmetry (Delplace *et al.* 2017).

An important aspect of the theory of atmospheric dynamics and of numerical modeling of the atmosphere is deriving appropriate models to describe slow/large-scale phenomena by

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systematically discarding small-scale/fast oscillations (Salmon 1998). Several models have been proposed to filter out inertia-gravity waves and thereby isolate vortical/Rossby modes, such as the barotropic vorticity equation, the quasi-geostrophic equations and the semi-geostrophic equations. Besides isolating the physics, these models are cheaper to integrate, since the Courant-Friedrichs-Lewy (CFL) condition ties the time step to the fastest waves. While this approximation can yield useful physical insight, it neglects important phenomena. For instance, a realistic representation of energy and enstrophy cascades (Fjørtoft 1953) requires accounting for gravity waves (Rocha *et al.* 2018).

On the impact of gravity waves on slow Rossby-wave dynamics, A. Majda and collaborators (Embid & Majda 1996; Majda & Embid 1998; Majda 2003) showed that in the limit $\epsilon \rightarrow 0$, where $\epsilon \sim Ro$, $\epsilon \sim F$, the slow vortical modes evolve independently of the fast gravity waves. Although initially formulated for the shallow-water equations on an f-plane, the results were extended to the stratified Boussinesq equations (Majda 2003). The limitations of Majda’s theory of fast-wave averaging for large-scale atmospheric oscillations are that (1) the result is asymptotic, corresponding to the case of infinitely fast rotation/thin layer, and (2) on the f-plane Rossby waves degenerate to zero-frequency vortical modes. More recent work has sought to relax the asymptotic limit itself, for instance via stochastic corrections to the slow-manifold closure (Chekroun *et al.* 2021). Departing from the same f-plane shallow-water model, Eden *et al.* (2019) derived a kinetic equation in the weakly nonlinear limit, showing that the leading-order generation of gravity-wave energy scales as the square of the Rossby number.

The interaction of gravity waves and geostrophic flows plays an important role in oceanic dynamics as well. Vanneste (2013) reviews balance and spontaneous wave generation in geophysical flows more broadly. For instance, Xie & Vanneste (2015) used a generalized-Lagrangian-mean approach to study interactions between near-inertial waves and a geostrophically balanced mean flow. Both spontaneous and stimulated generation of near-inertial waves have been proposed as important mechanisms for extracting energy from balanced flows (Nagai *et al.* 2015), subsequently cascading energy toward small scales where it can be dissipated. In a similar line, Chouksey *et al.* (2018) and Chouksey *et al.* (2022) examined the emission of internal gravity waves from balanced, sheared flow across different dynamical regimes, and characterized this fast-slow energy exchange in the ocean in detail.

All the aforementioned studies were based on Cartesian, f-plane models, in which Rossby modes degenerate to zero-frequency vortical modes. Resonant fast-slow triad interactions continue to be studied in such flat, periodic-domain settings even in recent work (Thomas 2016; Carbone & Dutykh 2024). When considering the latitudinal variation of the Coriolis parameter, either in a beta-plane or on the sphere (Majda 2003), Rossby-wave frequencies remain small compared with those of gravity modes, but they are non-zero. Understanding the interaction between Rossby and gravity waves therefore requires a theory beyond the f-plane.

The nonlinear dynamics of Rossby-Haurwitz waves have been studied in the context of the barotropic vorticity equation, which eliminates gravity modes. Lynch (2009) explored the dynamics of a quasi-resonant triad, showing that its behavior persists in the complete spectral model, with slow energy leakage to non-resonant modes. More recently, Raphaldini *et al.* (2022) explored the role of multiple connected triads and concluded that under the precession-resonance regime (Bustamante *et al.* 2014) stronger energy fluxes occur between slow waves, generating low-frequency oscillations and peaks in the efficiency of energy and enstrophy transfers when wave phases align. This complements the broader statistical characterization of three-wave interactions in geophysical flows (Harris 2013). A generalization of the barotropic nondivergent model is the quasi-geostrophic equation on the sphere (Schubert *et al.* 2009; Verkley 2009), whose modes, expressed in terms of spheroidal

harmonics, satisfactorily approximate the Rossby-Haurwitz waves present in the shallow-water model. An accurate representation of the RH-wave structure, here provided by the Hough harmonics, is a prerequisite for any analysis of their nonlinear interactions.

Earlier analytic treatments of nonlinear wave-wave interactions were carried out on the equatorial β -plane (Ripa 1982, 1983). For the rotating shallow-water (RSW) system on the sphere, Vanneste & Vial (1994) derived analytical expressions for wave interaction coefficients under the assumption of a slowly rotating sphere. The RSW model can be decomposed into rotational and divergent parts, enabling analytical consideration of coupling coefficients and interactions between scales. For the full RSW equations on the sphere, however, several questions about nonlinear wave interactions remain open. The main difficulty is that the eigenfunctions of the linear shallow water operator on the sphere, the Hough functions (Longuet-Higgins 1968), cannot be written explicitly and must be approximated numerically by combinations of vector spherical harmonics (Swarztrauber & Kasahara 1985).

In this article, we examine the influence of gravity waves on slow Rossby waves using realistic parameters for Earth's atmosphere on the sphere. We use the full RSW equations on the sphere to study how inertia-gravity waves alter the dynamics of Rossby-Haurwitz waves, through exchanged energy and frequency shifts, and how slow oscillations emerge from nonlinear interactions. The RSW model is also relevant beyond a single layer, since the primitive equations used in weather forecasting separate into a series of shallow-water problems with different equivalent depths (Majda 2003). Nonlinear coupling between Rossby, Kelvin and gravity waves has previously been examined in the equatorial β -plane waveguide (Raupp & Silva Dias 2006; Raupp *et al.* 2008, 2019). We extend this line of investigation to the full sphere, where both wave families are represented by the exact Hough harmonic eigenfunctions of the linear problem, without restriction to the equatorial region or to the assumption of a slowly rotating sphere. To our knowledge, this is the first study of nonlinear energy exchange between Rossby-Haurwitz and inertia-gravity waves in this global, unapproximated setting.

Section 2 presents the RSW equations on the sphere, their linearization and the Hough harmonics. Section 3 derives the nonlinear interaction equations and discusses triadic and resonant interactions. Section 4 analyses four-wave interactions through coupled triads, §5 extends this to five waves, and §6 concludes.

2. The shallow water model on the sphere

Previous studies on the interaction between slow and fast atmospheric waves have been mainly based on the shallow water equations on an f -plane (Majda & Embid 1998; Embid & Majda 1996; Owen *et al.* 2018). In this context, the slow mode degenerates into a geostrophically balanced vortical mode with zero linear frequency, while the fast modes consist of eastward and westward inertia-gravity modes. Actual Rossby waves require the meridional gradient of the Coriolis parameter, which we retain by using the full shallow-water equations on the sphere. The price is that the eigenfunctions of the linear shallow-water operator, the Hough functions, have no closed form and must be constructed numerically (Swarztrauber & Kasahara 1985).

2.1. The rotating shallow water equations on the sphere

The inviscid one-layer RSW model with a flat lower surface is written in spherical coordinates as (Vallis 2017; Zeitlin 2018)

$$\frac{Du}{Dt} - \left(2\Omega \sin \phi + \frac{u \tan \phi}{a} \right) v + \frac{g}{a \cos \phi} \frac{\partial h}{\partial \lambda} = 0, \quad (2.1)$$

$$\frac{Dv}{Dt} + \left(2\Omega \sin \phi + \frac{u \tan \phi}{a} \right) u + \frac{g}{a} \frac{\partial h}{\partial \phi} = 0, \quad (2.2)$$

$$\frac{Dh}{Dt} + \frac{h + h_0}{a \cos \phi} \left(\frac{\partial u}{\partial \lambda} + \frac{\partial(v \cos \phi)}{\partial \phi} \right) = 0, \quad (2.3)$$

where λ is the longitude, ϕ the latitude, a the Earth's radius, Ω is the Earth rotation rate, g the gravitational acceleration, h_0 the mean fluid height and h its perturbation, the Coriolis parameter is $f = 2\Omega \sin \phi$, u is the zonal velocity, v the meridional velocity, and the material derivative in spherical coordinates writes

$$\frac{D\psi}{Dt} = \frac{\partial \psi}{\partial t} + \frac{u}{a \cos \phi} \frac{\partial \psi}{\partial \lambda} + \frac{v}{a} \frac{\partial \psi}{\partial \phi}. \quad (2.4)$$

Where ψ is a differentiable scalar field. It is convenient to define dimensionless variables (Swarztrauber & Kasahara 1985)

$$u' = \frac{u}{\sqrt{gh_0}}, \quad v' = \frac{v}{\sqrt{gh_0}}, \quad h' = \frac{h}{h_0}, \quad t' = 2\Omega t, \quad (2.5)$$

where the mean height (h_0) is considered as a vertical scale, the pure gravity wave velocity ($\sqrt{gh_0}$) as a velocity scale, and the inverse of the rotation rate (2Ω) as a time scale. After dropping primes, the system (2.1)-(2.3) is rewritten as

$$\frac{\partial u}{\partial t} + \gamma \left(\frac{u}{\cos \phi} \frac{\partial u}{\partial \lambda} + v \frac{\partial u}{\partial \phi} \right) - (\sin \phi + \gamma u \tan \phi) v + \frac{\gamma}{\cos \phi} \frac{\partial h}{\partial \lambda} = 0, \quad (2.6)$$

$$\frac{\partial v}{\partial t} + \gamma \left(\frac{u}{\cos \phi} \frac{\partial v}{\partial \lambda} + v \frac{\partial v}{\partial \phi} \right) + (\sin \phi + \gamma u \tan \phi) u + \gamma \frac{\partial h}{\partial \phi} = 0, \quad (2.7)$$

$$\frac{\partial h}{\partial t} + \gamma \left(\frac{u}{\cos \phi} \frac{\partial h}{\partial \lambda} + v \frac{\partial h}{\partial \phi} \right) + \frac{(1+h)\gamma}{\cos \phi} \left(\frac{\partial u}{\partial \lambda} + \frac{\partial(v \cos \phi)}{\partial \phi} \right) = 0, \quad (2.8)$$

where $\gamma = \sqrt{gh_0}/2\Omega a$, which is related to the so-called *Lamb's number* ϵ by

$$\epsilon = \frac{1}{\gamma^2} = \frac{4a^2\Omega^2}{gh_0} = \left(\frac{a}{\sqrt{gh_0}/(2\Omega)} \right)^2. \quad (2.9)$$

The Lamb's number measures the relative importance of the Earth's rotation velocity ($2\Omega a$) and the gravity-wave speed ($\sqrt{gh_0}$). Rotation dominates the dynamics for large Lamb's numbers, while the gravity-wave speed grows as the Lamb's number approaches zero.

The linear part of the system (2.6)-(2.8) can be represented by introducing the linear operator

$$\mathcal{L} = \begin{pmatrix} 0 & -\sin \phi & \frac{\gamma}{\cos \phi} \partial_\lambda \\ \sin \phi & 0 & \gamma \partial_\phi \\ \frac{\gamma}{\cos \phi} \partial_\lambda & \frac{\gamma}{\cos \phi} \partial_\phi (\cos \phi (\cdot)) & 0 \end{pmatrix},$$

152 while the non-linear part by the *bilinear* operator

$$153 \quad \mathcal{N}(\vec{u}_1, \vec{u}_2) = \gamma \begin{pmatrix} -\frac{u_1}{\cos \phi} \frac{\partial u_2}{\partial \lambda} - v_1 \frac{\partial u_2}{\partial \phi} + u_1 v_2 \tan \phi \\ -\frac{u_1}{\cos \phi} \frac{\partial v_2}{\partial \lambda} - v_1 \frac{\partial v_2}{\partial \phi} - u_1 u_2 \tan \phi \\ -\frac{u_1}{\cos \phi} \frac{\partial h_2}{\partial \lambda} - v_1 \frac{\partial h_2}{\partial \phi} - \frac{h_2}{\cos \phi} \left(\frac{\partial u_1}{\partial \lambda} + \frac{\partial(v_1 \cos \phi)}{\partial \phi} \right) \end{pmatrix}. \quad (2.10)$$

154 The RSW equations then take the operator form

$$155 \quad \frac{\partial \vec{u}}{\partial t} + \mathcal{L} \vec{u} = \mathcal{N}(\vec{u}, \vec{u}), \quad (2.11)$$

156 where $\vec{u} = (u, v, h)^T$ is the state vector.

157 2.2. Properties of the RSW equations on the sphere

158 The shallow water model on the sphere conserves potential vorticity, defined as

$$159 \quad Q = \frac{\zeta + f}{h + h_0}.$$

160 Here, the *vorticity* ζ define by

$$161 \quad \zeta = \vec{n} \cdot \nabla \times \vec{u} = \frac{1}{a \cos \phi} \left(\frac{\partial v}{\partial \lambda} - \frac{\partial}{\partial \phi} (u \cos \phi) \right), \quad (2.12)$$

162 where $\vec{n}$ is the local (radial) vertical unit vector.

163 The total energy of the system, defined as

$$164 \quad E = \frac{a^2}{2} \int_0^{2\pi} \int_{-\pi/2}^{\pi/2} [(h + h_0)(u^2 + v^2) + g(h + h_0)^2] \cos \phi \, d\phi d\lambda \quad (2.13)$$

165 is also conserved. Substituting the non-dimensional variables (2.5) and dropping primes, the
166 total energy (2.13) becomes

$$167 \quad E = \frac{gh_0^2 a^2}{2} \int_0^{2\pi} \int_{-\pi/2}^{\pi/2} [(1 + h)(u^2 + v^2) + (1 + h)^2] \cos \phi \, d\phi d\lambda. \quad (2.14)$$

168 Since the perturbation height is defined to have zero average,

$$169 \quad \langle h \rangle \equiv \frac{1}{4\pi} \int_0^{2\pi} \int_{-\pi/2}^{\pi/2} h \cos \phi \, d\phi d\lambda = 0,$$

170 and the Hamiltonian is defined up to additive constants, (2.14) reduces, apart from the
171 multiplicative factor $gh_0^2 a^2/2$, to the dimensionless quantity below. The RSW energy is
172 *cubic*, i.e. it contains third-order terms. Expanding (2.14), the quadratic part, comprising the
173 *kinetic* and *available potential energy*, is

$$174 \quad E^{(2)} = \frac{1}{4\pi} \int_0^{2\pi} \int_{-\pi/2}^{\pi/2} (u^2 + v^2 + h^2) \cos \phi \, d\phi d\lambda = \frac{1}{2} \|\vec{u}\|^2, \quad (2.15)$$

175 while the cubic term is the integral

$$176 \quad E^{(3)} = \frac{1}{4\pi} \int_0^{2\pi} \int_{-\pi/2}^{\pi/2} h(u^2 + v^2) \cos \phi \, d\phi d\lambda. \quad (2.16)$$

177 The next section uses the total energy to quantify energy transfer between Rossby-Haurwitz
178 and gravity waves, in both directions.

2.3. Normal modes: Hough harmonics

Assuming that the variables are perturbations of a hydrostatic and steady basic state with equilibrium height h_0 , that is

$$\begin{aligned}\vec{u}'(\lambda, \phi, t) &= \vec{u}_0 + \vec{u}(\lambda, \phi, t), \\ h'(\lambda, \phi, t) &= h_0 + h(\lambda, \phi, t),\end{aligned}$$

where $\vec{u}_0 = (u_0, v_0) = 0$. Neglecting terms quadratic in the perturbation amplitude, and proceeding with the change of variables (2.5) and dropping primes, the linearized RSW equations on the sphere, also known as the Laplace's tidal equations (Swarztrauber & Kasahara 1985), are written as:

$$\frac{\partial u}{\partial t} - \sin \phi v + \frac{\gamma}{\cos \phi} \frac{\partial h}{\partial \lambda} = 0, \quad (2.17)$$

$$\frac{\partial v}{\partial t} + \sin \phi u + \gamma \frac{\partial h}{\partial \phi} = 0, \quad (2.18)$$

$$\frac{\partial h}{\partial t} + \frac{\gamma}{\cos \phi} \left(\frac{\partial u}{\partial \lambda} + \frac{\partial (v \cos \phi)}{\partial \phi} \right) = 0. \quad (2.19)$$

For further considerations, the Laplace's tidal equations (2.17) - (2.19) can be written in the operator form as

$$\frac{\partial \vec{u}}{\partial t} + \mathcal{L} \vec{u} = 0. \quad (2.20)$$

Following the work of Swarztrauber & Kasahara (1985) and noting that the Laplace's tidal model (2.20) is linear with respect to time (t), their solutions can be expressed as a linear combination of complex exponentials

$$\vec{u}(\lambda, \phi, t) = H(\lambda, \phi) e^{-i\omega t}, \quad (2.21)$$

where ω is the dimensionless frequency scaled by the Earth's rotation rate 2Ω .

Substitution of (2.21) into (2.20) yields the eigenvalue problem

$$(\mathcal{L} - i\omega)H(\lambda, \phi) = 0. \quad (2.22)$$

that is, H is an eigenfunction of the Laplace's tidal operator $\mathcal{L}$, and constitutes a *normal mode* for the RSW equations on the sphere. Since the coefficients of $\mathcal{L}$ do not depend on the longitude (λ), for each *zonal wavenumber* $m \in \mathbb{Z}$, $H(\lambda, \phi)$ may be expressed as

$$H(\lambda, \phi) = \Theta(\phi) e^{im\lambda}, \quad (2.23)$$

where $\Theta(\phi)$ defines the *meridional* structure of the function.

Since for a fixed $m \in \mathbb{Z}$ there exist different meridional mode functions, we use the indices n and α to represent the n^{th} meridional mode of type α , where $\alpha \in \{1, 2, 3\}$ describes eastward gravity (EG) waves ($\alpha = 1$), westward gravity (WG) waves ($\alpha = 2$) and Rossby-Haurwitz (RH) waves ($\alpha = 3$). With this convention,

$$H_{n,\alpha}^m(\lambda, \phi) = \Theta_{n,\alpha}^m(\phi) e^{im\lambda}.$$

$H_{n,\alpha}^m$ is defined as the *Hough harmonic* of zonal wavenumber m , meridional wavenumber n and type α , while $\Theta_{n,\alpha}^m$ is the *Hough function*, which is related to the velocity and height fields by

$$\Theta_{n,\alpha}^m(\phi) = \begin{pmatrix} u_{n,\alpha}^m(\phi) \\ i v_{n,\alpha}^m(\phi) \\ h_{n,\alpha}^m(\phi) \end{pmatrix}. \quad (2.24)$$

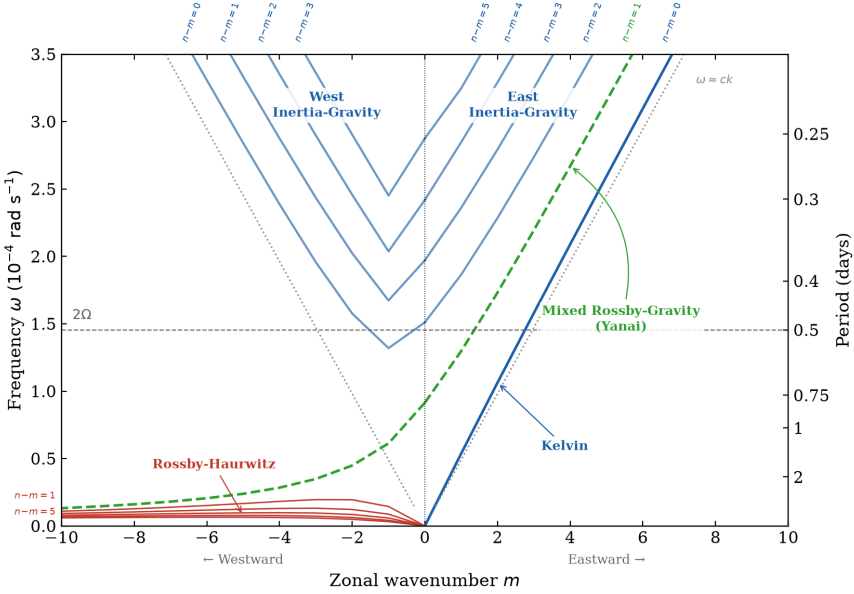


Figure 1: Dispersion relation of the RSW equations on the sphere at $h_e = 10\,000\text{m}$ (the equivalent depth, identified with h_0 throughout this paper): dimensionless frequency ω (left axis) and period (right axis) versus zonal wavenumber m , for the three wave families: eastward and westward gravity waves (EG/WG, also denoted EIG/WIG in the wave literature; blue) and Rossby-Haurwitz (RH, red), with the Kelvin and mixed Rossby-gravity (Yanai) modes highlighted. Dotted lines show the $\omega \approx ck$ reference and the dashed line marks $\omega = 2\Omega$.

215 By representing $H(\lambda, \phi)$ in terms of series of *spherical vector harmonics* with zonal index
 216 m , the substitution of (2.23) into (2.22) yields an infinite system, whose eigenvalues define
 217 the frequencies (ω), while its eigenvectors define the factors used in the linear combination of
 218 spherical vector harmonics, i.e., the eigenvectors of the associated system define the Hough
 219 harmonics. For a fixed m there are thus infinitely many eigenvalues, which are divided into
 220 three groups, defining the three wave types α .

221 Following the method proposed by Swartrauber & Kasahara (1985), the normal modes
 222 have either a *symmetric structure*, where both zonal velocity (u) and height (h) are symmetric,
 223 while the meridional velocity (v) is asymmetric around the equatorial plane, or an *anti-*
 224 *symmetric structure* where u and h are asymmetric while v is symmetric around the Equator.
 225 Considering the zonal wavenumber (m) and the meridional wavenumber (n), the symmetric
 226 Rossby-Haurwitz modes are the ones where $m + n$ is *odd*, while the symmetric gravity modes
 227 have $m + n$ *even*.

228 The case with *zero zonal wavenumber* has its peculiarities: the frequencies of the gravity
 229 waves appear as a pair of positive and negative values with the same magnitude, and the
 230 frequencies of Rossby-Haurwitz (rotational) waves are all zero (Kasahara 1977). Rossby-
 231 Haurwitz modes are absent in the case $m = 0$. A complete basis of eigenfunctions must
 232 include these missing modes, which correspond to zonal geostrophic modes. We follow the
 233 method of Swartrauber & Kasahara (1985), which computes the orthonormal basis without
 234 Gram-Schmidt orthogonalization.

235 The dispersion relation for the three kinds of waves, plotted in figure 1, follows directly
 236 from the eigenvalues of (2.22), truncated at $N = 3$ vector spherical harmonics per wave
 237 family.

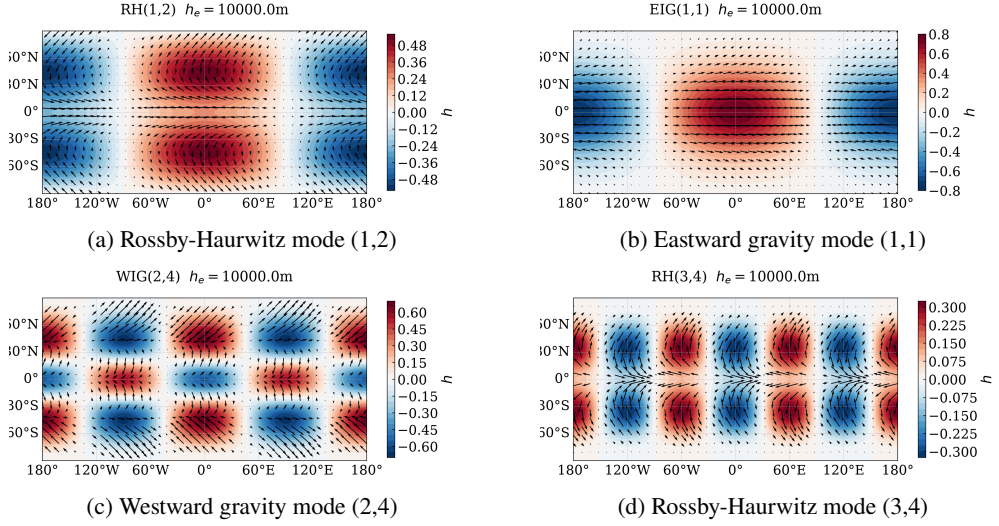


Figure 2: Spatial structure of four Hough harmonics at $h_e = 10\,000\text{m}$: height field h (shaded contours) and horizontal velocity (u, v) (arrows), reconstructed via $\text{Re}[(U, V, Z)(\phi) e^{im\lambda}]$ from the latitudinal Hough functions.

The three wave types occupy distinct regions of the dispersion diagram (figure 1): eastward gravity waves the positive quadrant, westward gravity waves the unbounded fourth quadrant, and Rossby-Haurwitz waves a bounded region that converges asymptotically to zero.

Figure 2 illustrates the spatial structure of four representative Hough harmonics used throughout this work, one from each wave family plus a second Rossby-Haurwitz mode, all reconstructed on the sphere from the height field (h , shaded contours) and the horizontal velocity field (u, v , arrows). Each mode's eigenfunction is built from a series of spherical vector harmonics truncated at $N = 30$ terms and normalized by a Gauss-Legendre quadrature with 60 points. The RH mode (1,2) exhibits the large-scale, low-frequency meridional cell structure characteristic of Rossby-Haurwitz waves, with velocities predominantly zonal, while the Kelvin-like EG mode (1,1) is equatorially trapped, with height and velocity perturbations concentrated within a narrow latitudinal band around the Equator, reflecting its much higher frequency and different dynamical balance. The WG mode (2,4) shows the finer-scale, westward-tilted structure typical of higher-order gravity modes, while the RH mode (3,4), which appears in several triads in §4, illustrates how the RH meridional cell structure narrows and multiplies as the zonal and meridional wavenumbers increase.

3. Wave interactions

3.1. General wave interaction

Solutions of the full RSWE can be analysed in terms of their linear eigenmodes. Expanding a solution of (2.11) in terms of Hough harmonics, with $A_{n,\alpha}^m(t)$ the complex amplitude of

258 mode (m, n, α) ,

$$\begin{aligned}
 259 \quad \vec{u}(\lambda, \phi, t) &= \sum_{n=0}^{\infty} \sum_{m=0}^n \sum_{\alpha=1}^3 A_{n,\alpha}^m(t) H_{n,\alpha}^m(\lambda, \phi) e^{-i\omega_{n,\alpha}^m t} \\
 260 &= \sum_{n=0}^{\infty} \sum_{m=0}^n \sum_{\alpha=1}^3 A_{n,\alpha}^m(t) \Theta_{n,\alpha}^m(\phi) e^{i(\lambda m - \omega_{n,\alpha}^m t)} + c.c., \quad (3.1)
 \end{aligned}$$

261 where 'c.c.' indicates the complex conjugate, ensuring that $\vec{u}$ is a real field. The non-linear
 262 term (2.10) writes

$$\begin{aligned}
 263 \quad \mathcal{N}(\vec{u}, \vec{u}) &= \sum_b A_b(t) e^{-i\omega_b t} \mathcal{N}(\Theta_b(\phi) e^{i\lambda m_b}, \vec{u}) \\
 264 &= \sum_b \sum_c A_b(t) A_c(t) e^{-i(\omega_b + \omega_c)t} \mathcal{N}(\Theta_b(\phi) e^{i\lambda m_b}, \Theta_c(\phi) e^{i\lambda m_c}), \\
 265 &\quad (3.2)
 \end{aligned}$$

266 where a compact notation has been used, in which $a = (m_a, n_a, \alpha_a)$, $b = (m_b, n_b, \alpha_b)$ and
 267 $c = (m_c, n_c, \alpha_c)$ represent arbitrary normal modes.

268 $\vec{u}(\lambda, \phi, t)$ is a solution of (2.11) provided that

$$\begin{aligned}
 269 \quad \mathcal{N}(\vec{u}, \vec{u}) &= \frac{\partial \vec{u}}{\partial t} + \mathcal{L}\vec{u} = \sum_a \frac{dA_a}{dt} H_a e^{-i\omega_a t} - i\omega_a A_a H_a e^{-i\omega_a t} + \sum_a A_a e^{-i\omega_a t} \mathcal{L}(H_a) \\
 270 &= \sum_a \frac{dA_a}{dt} H_a e^{-i\omega_a t} - i\omega_a A_a H_a e^{-i\omega_a t} + \sum_a i\omega_a A_a H_a e^{-i\omega_a t} \\
 271 &= \sum_a \frac{dA_a}{dt} H_a e^{-i\omega_a t}, \quad (3.3)
 \end{aligned}$$

272 where the eigenfunction property of Hough harmonics (2.22) was used.

273 For a given mode with subscript a , the equation (3.3) translates as

$$\begin{aligned}
 274 \quad \frac{dA_a}{dt} &= \langle \sum_a \frac{dA_a}{dt} H_a e^{-i\omega_a t}, H_a e^{-i\omega_a t} \rangle = \langle \mathcal{N}(\vec{u}, \vec{u}), H_a e^{-i\omega_a t} \rangle \\
 275 &= -i\gamma \sum_b \sum_c A_b(t) A_c(t) e^{-i(\omega_b + \omega_c - \omega_a)t} \langle \mathcal{B}, \Theta_a(\phi) e^{im_a \lambda} \rangle, \quad (3.4)
 \end{aligned}$$

276 where the orthogonality property of Hough harmonics was used and $\mathcal{B} \equiv (\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3)^T$ is
 277 defined below. Using (2.22), the bilinear term $\mathcal{N}(\Theta_b(\phi) e^{i\lambda m_b}, \Theta_c(\phi) e^{i\lambda m_c})$ can be rewritten
 278 as $-i\gamma (\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3)^T e^{i\lambda(m_b + m_c)}$, where

$$279 \quad \mathcal{B}_1 = \frac{m_c}{\cos \phi} u_b u_c + v_b \frac{\partial u_c}{\partial \phi} - u_b v_c \tan \phi, \quad (3.5)$$

$$280 \quad \mathcal{B}_2 = i \frac{m_c}{\cos \phi} u_b v_c + i v_b \frac{\partial v_c}{\partial \phi} - i u_b u_c \tan \phi, \quad (3.6)$$

$$281 \quad \mathcal{B}_3 = \frac{m_c}{\cos \phi} u_b h_c + v_b \frac{\partial h_c}{\partial \phi} + \frac{m_b}{\cos \phi} u_b h_c + \frac{1}{\cos \phi} h_c \frac{\partial (v_b \cos \phi)}{\partial \phi}, \quad (3.7)$$

282 with u_b, v_b, h_b and u_c, v_c, h_c the components of Θ_b and Θ_c as in (2.24), so that $\mathcal{B} \equiv$
 283 $(\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3)^T e^{i\lambda(m_b + m_c)}$.

Here, the projections are performed using the inner product

$$\begin{aligned}\langle X, Y \rangle &= \frac{1}{2\pi} \int_0^{2\pi} \int_{-\pi/2}^{\pi/2} X \cdot Y^* \cos \phi \, d\phi \, d\lambda \\ &= \frac{1}{2\pi} \int_0^{2\pi} \int_{-\pi/2}^{\pi/2} (X_1 Y_1^* + X_2 Y_2^* + X_3 Y_3^*) \cos \phi \, d\phi \, d\lambda.\end{aligned}\quad (3.8)$$

In order for the nonlinear coupling coefficient $\langle \mathcal{B}, \Theta_a e^{im_a \lambda} \rangle$ to be nonzero, the zonal wavenumbers must satisfy

$$m_a = m_b + m_c, \quad (3.9)$$

a constraint due to the longitudinal integral.

The integrand of the latitudinal integral also cannot be antisymmetric about the Equator, so only *three symmetric* modes, or *two antisymmetric* modes with *one symmetric* mode, can interact. These two conditions, the zonal-wavenumber resonance condition (3.9) and the meridional symmetry rule, are the selection rules for three-wave resonant interactions studied more broadly in the theory of weak (wave) turbulence (Craig 1988; Nazarenko 2011; Chouksey *et al.* 2026), where an ensemble of weakly nonlinear dispersive waves exchanges energy predominantly through such resonant triads. The triadic system derived below is the simplest realization of this mechanism for the Hough harmonics of the RSW equations on the sphere.

3.2. Triadic interaction

The three-wave equations are the minimal model of nonlinear interaction. Using the ansatz consisting of an expansion of the form (3.1) truncated for only three modes

$$\begin{aligned}\vec{u}(\lambda, \phi, t) &= A_a(t) \Theta_a(\phi) e^{i(\lambda m_a - \omega_a t)} + A_b(t) \Theta_b(\phi) e^{i(\lambda m_b - \omega_b t)} \\ &\quad + A_c(t) \Theta_c(\phi) e^{i(\lambda m_c - \omega_c t)} + c.c.,\end{aligned}\quad (3.10)$$

where 'c.c.' indicates complex conjugates.

To express the initial amplitudes of the triad ODEs in terms of the initial velocity and height fields, note that

$$A_a \begin{pmatrix} u_a \\ iv_a \\ h_a \end{pmatrix} + A_a^* \begin{pmatrix} u_a \\ -iv_a \\ h_a \end{pmatrix} = 2 \begin{pmatrix} \operatorname{Re}(A_a) u_a \\ -\operatorname{Im}(A_a) v_a \\ \operatorname{Re}(A_a) h_a \end{pmatrix}, \quad (3.11)$$

and consequently, an initial real amplitude characterizes a zonal jet at the Equator, while a pure imaginary amplitude characterizes a meridional jet. A zonal velocity of U m/s is then obtained by setting

$$A_a = \frac{U}{2\|u_a\|\sqrt{gh_0}},$$

where the velocity scaling factor $\sqrt{gh_0}$ was used and $\|u_a\|$ denotes the same latitudinal L^2 norm as (2.15), restricted to the scalar field $u_a(\phi)$. In particular, $\left(2\|u_a\|\sqrt{gh_0}\right)^{-1}$ corresponds to a zonal velocity 1 m/s.

Choosing c as the mode that plays the sum role in the selection rule (3.9), i.e. setting $m_c = m_a + m_b$, in equation (3.4), we obtain the *three-wave equations*

$$\frac{dA_a}{dt} = -i\gamma (\langle \mathcal{B}(H_b^*, H_c), H_a \rangle + \langle \mathcal{B}(H_c, H_b^*), H_a \rangle) A_b^* A_c e^{it\delta}, \quad (3.12)$$

$$\frac{dA_b}{dt} = -i\gamma (\langle \mathcal{B}(H_a^*, H_c), H_b \rangle + \langle \mathcal{B}(H_c, H_a^*), H_b \rangle) A_a^* A_c e^{it\delta}, \quad (3.13)$$

$$\frac{dA_c}{dt} = -i\gamma (\langle \mathcal{B}(H_a, H_b), H_c \rangle + \langle \mathcal{B}(H_b, H_a), H_c \rangle) A_a A_b e^{-it\delta}, \quad (3.14)$$

where $\delta = \omega_a + \omega_b - \omega_c$ is the *frequency mismatch* and the superscript (*) indicates the complex conjugate.

Defining new variables

$$A'_a = A_a e^{-it\omega_a}, A'_b = A_b e^{-it\omega_b}, A'_c = A_c e^{-it\omega_c},$$

and the *coupling coefficients*

$$\alpha_{b,c}^a = -\gamma (\langle \mathcal{B}(H_b^*, H_c), H_a \rangle + \langle \mathcal{B}(H_c, H_b^*), H_a \rangle), \quad (3.15)$$

$$\alpha_{a,c}^b = -\gamma (\langle \mathcal{B}(H_a^*, H_c), H_b \rangle + \langle \mathcal{B}(H_c, H_a^*), H_b \rangle), \quad (3.16)$$

$$\alpha_{a,b}^c = -\gamma (\langle \mathcal{B}(H_a, H_b), H_c \rangle + \langle \mathcal{B}(H_b, H_a), H_c \rangle), \quad (3.17)$$

equations (3.12)-(3.14), after dropping primes, can be rewritten in the simplified form

$$\frac{dA_a}{dt} = -i\omega_a A_a + i\alpha_{b,c}^a A_b^* A_c, \quad (3.18)$$

$$\frac{dA_b}{dt} = -i\omega_b A_b + i\alpha_{a,c}^b A_a^* A_c, \quad (3.19)$$

$$\frac{dA_c}{dt} = -i\omega_c A_c + i\alpha_{a,b}^c A_a A_b. \quad (3.20)$$

As noted in §3.1, the coupling coefficients can be non-zero only if the modes a, b, c are all symmetric, or two are anti-symmetric and one is symmetric. This is one of the so-called *Ellsaesser conditions* (Raphaldini *et al.* 2022).

Recalling from (2.15) and (2.16) that the quadratic energy term is $E^{(2)} = \frac{1}{2} \langle \vec{u}, \vec{u} \rangle$ in terms of the inner product (3.8), a straightforward calculation yields for the ansatz (3.10)

$$E^{(2)}(t) = |A_a(t)|^2 + |A_b(t)|^2 + |A_c(t)|^2, \quad (3.21)$$

while the cubic term is written as

$$E^{(3)}(t) = 2\Re e(A_a^* A_b^* A_c) \int_{-\pi/2}^{\pi/2} [h_a(u_b u_c + v_b v_c) + h_b(u_a u_c + v_a v_c) + h_c(u_a u_b - v_a v_b)] \cos \phi \, d\phi, \quad (3.22)$$

where u_a, v_a, h_a are as in (2.24).

The *conservation of total energy* now is written as

$$\frac{E^{(2)}(t) + E^{(3)}(t)}{E^{(2)}(0) + E^{(3)}(0)} = 1,$$

for all times $t \geq 0$. Noting that

$$\begin{aligned} \frac{d}{dt}|A_a(t)|^2 &= 2\Re\left(A_a^* \frac{d}{dt}A_a(t)\right) \\ &= 2\Re\left(-i\omega_a|A_a|^2 + i\alpha_{b,c}^a A_a^* A_b^* A_c\right) \\ &= 2\alpha_{b,c}^a \Im(A_a A_b A_c^*), \end{aligned} \quad (3.23)$$

and

$$\Re\left(A_b^* A_c \frac{d}{dt}A_a^*\right) = \omega_a \Im(A_a A_b A_c^*),$$

and similarly for b and c , one concludes

$$\frac{d}{dt}E^{(2)}(t) = 2\left(\alpha_{b,c}^a + \alpha_{a,c}^b - \alpha_{a,b}^c\right) \Im(A_a A_b A_c^*),$$

and

$$\frac{d}{dt}E^{(3)}(t) = 2\delta S_{abc} \Im(A_a A_b A_c^*), \quad (3.24)$$

where S_{abc} is the integral in (3.22), so that the total energy is preserved provided

$$\left(\alpha_{b,c}^a + \alpha_{a,c}^b - \alpha_{a,b}^c\right) + \delta S_{abc} = 0. \quad (3.25)$$

This identity holds for *any* three-wave truncation, regardless of the mismatch δ . Coefficients $\alpha_{b,c}^a$, $\alpha_{a,c}^b$, $\alpha_{a,b}^c$ and S_{abc} are all projections of the same bilinear operator $\mathcal{N}$ and the same cubic-energy integral onto a single, self-contained triad, so the truncated system (3.18)-(3.20) is exactly the Hamiltonian flow of $E^{(2)} + E^{(3)}$ restricted to that triad, which conserves its Hamiltonian by construction. This can be seen by noting that $\alpha_{b,c}^a$, $\alpha_{a,c}^b$, $\alpha_{a,b}^c$ and S_{abc} are all real, since the (U, iV, Z) phase structure of the Hough eigenfunctions (3.11) is matched by the same i in $\mathcal{B}_2$ (3.6), so that differentiating $E^{(2)}$ and $E^{(3)}$ along (3.18)-(3.20) leaves no term beyond (3.25). This holds exactly for the continuous Hough eigenfunctions. The finite spectral truncation N used to compute them numerically satisfies it only up to an N -dependent error, which grows with Lamb's parameter $\varepsilon = 1/\gamma^2$ as h_e decreases.

3.3. Resonant triads

When the mismatch $\delta = \omega_a + \omega_b - \omega_c$ equals zero in (3.12) - (3.14), the three wave system is called a *resonant triad*. As noted by Lynch (2009), the resonance condition lets the first two modes interact nonlinearly to produce a total phase that matches the third wave, which allows a *strong interaction* among the three. Resonant triads are also associated with the leading-order nonlinear effects (Raphaldini *et al.* 2022) and are likely one of the main sources of significant energy exchange among waves.

Energy exchange is not restricted to resonant processes. Longuet-Higgins & Gill (1967) showed for the β -plane approximation that when the wave with intermediate wavenumber is a zonal current, the coupling coefficients vanish, and there is no transfer of energy, concluding that the zonal currents cannot gain or lose any energy by the resonance mechanism. Relaxing the resonance condition, Reznik *et al.* (1993) considered the role of *quasi-resonant* triads in generating an intense energy redistribution among Rossby modes. For the stability of the wave field, Raphaldini *et al.* (2022) associate nearly resonant triads with a lower amplitude threshold for instability, while attributing the spectral broadening that characterizes the transition to a turbulent system specifically to non-resonant triad interactions.

On the sphere, this general theory of triadic resonance acquires three distinguishing

features. First, since the Hough functions have no closed form, the coupling coefficients $\alpha_{b,c}^a$ must be computed numerically for every candidate triad, rather than read off a dispersion relation in closed form as on the β -plane. Second, the equatorial-symmetry rule noted above (only three symmetric modes, or two anti-symmetric modes with one symmetric, may interact) has no direct β -plane analogue, and it alone can force $\alpha_{b,c}^a \equiv 0$ for a triad that would otherwise be a good resonance candidate. Third, because the RSW energy is cubic rather than quadratic, its conservation under (3.18)-(3.20) requires not only the coupling-coefficient identity familiar from quadratic-energy models such as the barotropic non-divergent equations, but also the additional mismatch-weighted term in (3.25), a three-wave, cubic-energy effect with no quadratic-energy analogue. The phenomenon noted by Longuet-Higgins & Gill (1967) also occurs in the RSW model, which has resonant triads that do not interact.

From now on, modes are labelled by type and (zonal, meridional) wavenumber, e.g. RH(m, n) for a Rossby-Haurwitz mode, EG(m, n)/WG(m, n) for an eastward/westward gravity mode. An example of the second point is the triad RH(1, 6)+EG(3, 7)+EG(4, 7) at $h_e = 10\,000\text{m}$ (a typical equivalent depth for barotropic flows, (Holton & Hakim 2013)), whose mismatch $\delta \approx -9.4 \times 10^{-5}$ is small, making it a good resonance candidate, but whose coupling coefficient vanishes to machine precision because the triad mixes two symmetric modes with one anti-symmetric mode. Further examples of the same kind exist with comparably small mismatches (e.g. $\delta \approx 10^{-5}$) and vanishing coupling coefficients (e.g. RH(3, 12)+EG(9, 15)+EG(12, 15), or RH(3, 16)+EG(10, 19)+EG(13, 19), or RH(2, 10)+EG(4, 11)+EG(6, 11)).

For a resonant triad, the exact solutions of (3.18)-(3.20) in terms of Jacobi elliptic functions (Bretherton 1964; Craik 1988; Kartashova & Bustamante 2011) identify a *pump mode*, which either supplies energy to the other two or receives it from them. Its kinetic energy has the same period as theirs but is out of phase, maximal when the others are minimal. As a direct consequence of the energy-conservation identity (3.25) at $\delta = 0$, the *pump mode is the mode with the largest coupling coefficient in absolute value*. A triad has no pump mode only when the three coupling-coefficient signs cannot satisfy this identity consistently. This happens for some quasi-resonant or non-resonant triads, never for a resonant one. In the special case where the triad's dynamical phase (see Appendix A) is identically zero, which is always attainable by an appropriate, generally complex, choice of initial amplitudes, the exact solutions further yield closed-form per-mode efficiency (3.26) expressions, showing that the efficiency of a resonant triad does not depend on the amplitude of its initial condition, only on how the initial energy is distributed among the three modes and on the ratios of their coupling coefficients. The full derivation and the exact pump-mode classification are also given in Appendix A.

Following the barotropic terminology of Raphaldini *et al.* (2022), we call $E_a \equiv |A_a(t)|^2$ the *kinetic energy* of mode a , both here and for general wave sets (§4.1.1, eq. 4.7). For the rotating shallow-water equations it is in fact the full quadratic energy $E^{(2)}$ (2.15) of the mode, kinetic plus available potential. Since the total energy of a triad is conserved, the efficiency of transfer is defined as (Raphaldini *et al.* 2022)

$$\mathcal{E}_a = \frac{\max_t E_a(t) - \min_t E_a(t)}{E_{\text{total}}(0)}, \quad (3.26)$$

mode a 's maximum-minus-minimum energy over the integration, normalized by the (conserved) initial total energy so that $\mathcal{E}_a \leq 1$. It measures the share of the total energy that cycles through mode a .

In the following examples, the system (3.18) - (3.20) is integrated with the classical 4-

Table 1: Table of triad properties at $h_e = 10\,000\text{m}$: each mode’s frequency ω , linear period (days), and coupling coefficient α ; each triad’s mismatch δ and pump mode.

Triad	Mode	Frequency (ω)	Period (days)	Coeff.	Pump
Triad A					
Rossby-only	RH(3,10)	−0.02629	19.02	−0.00328	pump
Near-resonant	RH(1,2)	−0.09944	5.028	−0.01229	
$\delta = -4.86 \times 10^{-4}$	RH(4,5)	−0.12524	3.992	−0.01558	
Triad B					
Rossby-only	RH(3,4)	−0.13435	3.722	0.44403	pump
Non-resonant	RH(1,2)	−0.09944	5.028	−0.02873	
$\delta = -0.1085$	RH(4,5)	−0.12524	3.992	0.36420	
Triad C					
Kelvin	EG(1,1)	0.36969	1.352	−0.13256	pump
Rossby flow	RH(3,4)	−0.13435	3.722	−0.59112	
$\delta = 0.3606$	RH(4,5)	−0.12524	3.992	−0.55801	
Triad D					
Gravity with	EG(6,9)	3.21205	0.1557	−0.55823	pump
Rossby catalyst	RH(1,7)	−0.01652	30.26	0.00287	
$\delta = 1.46 \times 10^{-5}$	EG(7,9)	3.19551	0.1565	−0.55536	

stage, fourth-order Runge-Kutta scheme (RK44) at a step of $\Delta t = 0.01$, at an equivalent height of $h_e = 10\,000\text{m}$ throughout, even though there exist analytic solutions for this system (Appendix A). Table 1 summarizes, for the four triads considered below, each mode’s frequency, linear period, and coupling coefficient, together with the triad’s mismatch δ and pump mode.

3.3.1. Rossby-only triads

The RH modes (1,2) and (4,5) are nearly resonant with RH(3,10), which minimizes the mismatch among all triads containing these two modes. This is Triad A in table 1. Replacing RH(3,10) by RH(3,4) gives Triad B, which is less resonant but couples the same two modes more strongly.

In each case, the *target mode*, i.e. the mode whose efficiency is measured, is held initially at rest, and the other two are swept over their initial zonal velocities, up to 100m/s for RH modes and 50m/s for gravity modes (realistic jet-stream and inertia-gravity wind-anomaly amplitudes, respectively). Figure 3 compares Triad A and Triad B. For Triad A (top row, target RH(3,10)), the efficiency map peaks at 21%. The time integration, with RH(1,2) and RH(4,5) initialized at 58 and 100m/s, shows a slow, roughly 80-day half-cycle exchange dominated by RH(1,2) and RH(4,5), with RH(3,10) only weakly excited. This is consistent with table 1, where RH(3,10) has the smallest coupling coefficient of the triad and is not its pump mode. For Triad B (bottom row, target RH(3,4)), the efficiency map peaks at 56% and the exchange period drops to a few days. RH(3,4) has the largest coupling coefficient in table 1 and is the triad’s pump mode. Its kinetic energy peaks exactly when RH(1,2) and RH(4,5) are both minimal.

The contrast between Triad A and Triad B shows that frequency mismatch alone does not predict the strength of energy transfer. Triad B’s mismatch ($\delta \approx -0.109$) is more than two orders of magnitude larger than Triad A’s ($\delta \approx -4.9 \times 10^{-4}$), yet it transfers energy several times more efficiently and several times faster. The coupling-coefficient magnitude and the

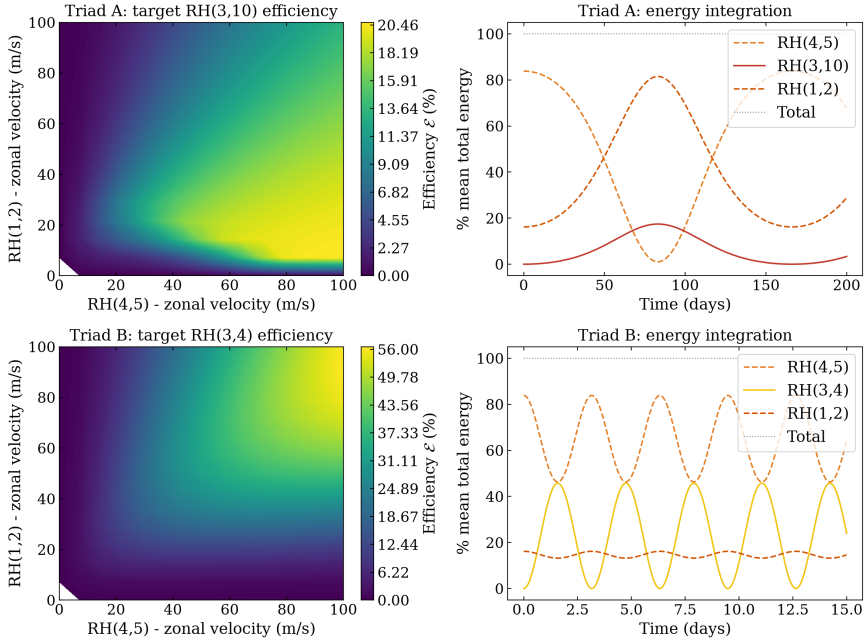


Figure 3: Rossby-only triads. Top row: Triad A (RH(3,10)+RH(1,2)+RH(4,5)); bottom row: Triad B (RH(3,4)+RH(1,2)+RH(4,5)). Left: efficiency of energy transfer (%) into the target mode (titled), swept over the initial zonal velocities of the other two modes with target mode held initially at rest. Right: kinetic-energy time integration with RH(1,2) and RH(4,5) initialized at 58 and 100m/s (target at rest), reaching 17% efficiency for Triad A and 46% for Triad B, on these specific initial conditions. The target mode is drawn solid, the other two dashed/dotted.

457 resulting pump-mode structure set the efficiency and timescale of exchange. RH(4,5) couples
 458 weakly to RH(1,2) when the third mode is RH(3,10), and strongly when it is RH(3,4).

459 3.3.2. Combined Rossby-gravity triads

460 Energy can also flow between length scales, from the long Rossby-Haurwitz waves to the
 461 shorter inertia-gravity waves, or in the opposite direction.

462 Triad C (EG(1,1)+RH(3,4)+RH(4,5)) is examined with two choices of target mode, which
 463 shows the direction of transfer directly. In both cases the target starts at rest and the other two
 464 modes are initialized at the velocities given in the caption of figure 4. With target EG(1,1)
 465 (figure 4, top row), the efficiency map peaks at 2%, and the Kelvin mode's trajectory stays
 466 within that range. Energy moves from the two RH modes toward the gravity mode, but weakly.
 467 With RH(3,4) as target instead, the efficiency map peaks at 14% (figure 4, middle row), and
 468 its trajectory grows from rest to a share of the total energy comparable to RH(4,5)'s, while
 469 EG(1,1)'s share stays near 3% throughout. RH(3,4) therefore most likely gains its energy
 470 from RH(4,5) rather than from the gravity mode, so this case shows transfer into the Rossby
 471 mode without isolating a gravity-to-Rossby pathway.

472 In the opposite scenario, the energy exchange can happen almost entirely between two
 473 gravity modes, with a Rossby-Haurwitz mode present but essentially inert. This is illustrated
 474 by Triad D (EG(6,9)+RH(1,7)+EG(7,9)). EG(6,9) and EG(7,9) have nearly equal coupling
 475 coefficients, the largest in the triad, while RH(1,7)'s is more than two orders of magnitude
 476 smaller (table 1). The efficiency map for EG(6,9) peaks near 98% (figure 4, bottom row).
 477 With EG(6,9) started at rest and the initial energy placed on EG(7,9) instead, the time

478 integration shows EG(6,9) growing to a comparably large share of the total energy while
 479 EG(7,9) falls by nearly as much and RH(1,7) stays close to its initial value throughout.
 480 Because the Rossby-Haurwitz mode's frequency is low, its coupling coefficient with the two
 481 gravity modes is small, so it exchanges little energy with either. It still sets the frequency
 482 mismatch, and therefore the period, of the exchange between the two gravity modes. We
 483 call such a mode a *catalyst*, since it paces an exchange between two other modes without
 484 absorbing an appreciable share of the energy. This is the same mechanism Bartello (1995)
 485 identifies for the barotropic/rotational mode in a catalytic two-wave-plus-rotational-mode
 486 interaction driving geostrophic adjustment in rotating stratified turbulence. RH(1,7)'s linear
 487 period exceeds 30 days, while EG(6,9) and EG(7,9) both oscillate with linear periods under
 488 four hours (table 1), a separation of roughly two orders of magnitude between the catalyst
 489 and the modes it paces and the largest fast/slow contrast among the triads considered here.

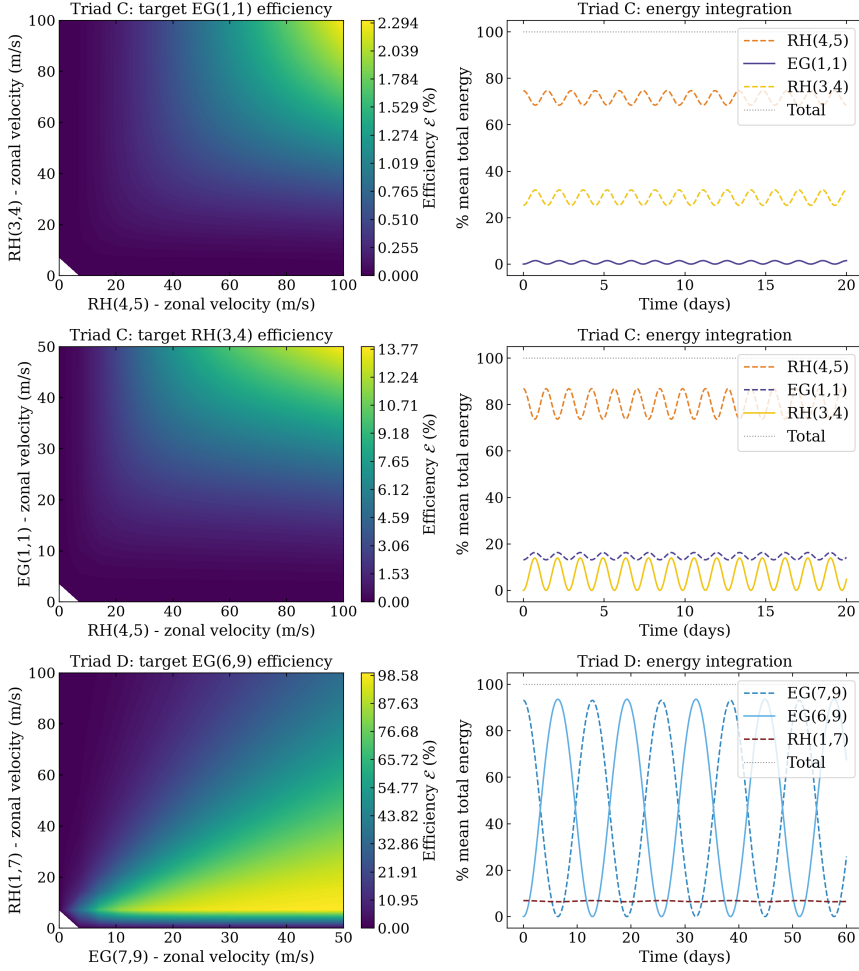


Figure 4: Combined Rossby-gravity triads. Top row: Triad C (EG(1,1)+RH(3,4)+RH(4,5)), target EG(1,1); time integration with RH(3,4) and RH(4,5) initialized at 63.89 and 100m/s (target at rest), reaching 1% efficiency on these specific initial conditions. Middle row: the same Triad C, retargeted to RH(3,4); time integration uses initial velocity for EG(1,1) and RH(4,5) at 50 and 100m/s, with target at rest, reaching 14% efficiency on these specific initial conditions. Bottom row: Triad D (EG(6,9)+RH(1,7)+EG(7,9)), target EG(6,9); time integration with RH(1,7) and EG(7,9) initialized at 20 and 50m/s (target at rest), reaching 94% efficiency on these specific initial conditions. The target mode is drawn solid, the other two dashed/dotted.

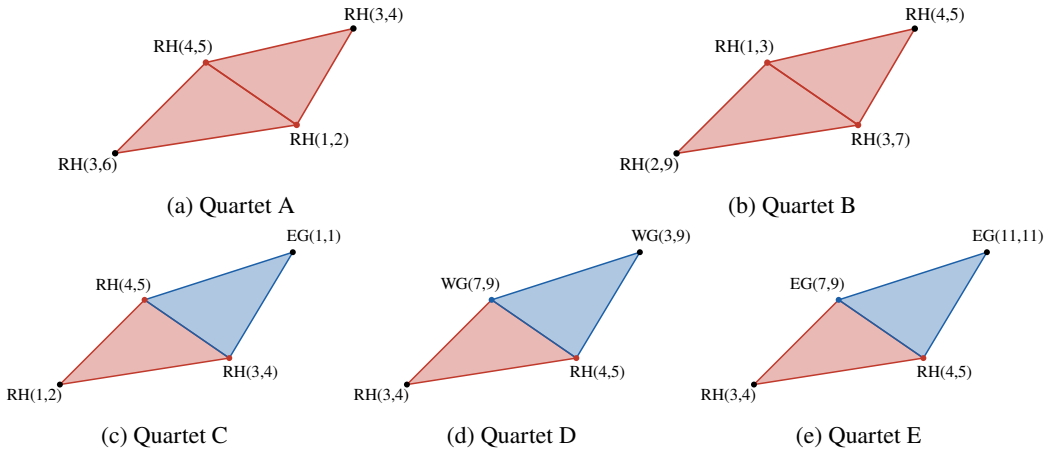


Figure 5: Topology overview of the quartets examined in this section: (a) all-Rossby quartet (table 2), (b) all-Rossby quartet reproducing Raphaldini *et al.* (2022)’s topology (table 4), (c) quartet with the Kelvin-like mode EG(1,1) sharing edge RH(4,5)+RH(3,4) with Quartet A’s triad (table 2), (d) quartet with WG(7,9) as the shared sum mode of both constituent triads and RH(4,5) as the other shared member, completed by RH(3,4) and WG(3,9) (private) (table 2), (e) quartet in which EG(7,9) is both the sum mode of the RH-only triad and a member of a second, higher triad closed by EG(11,11), completed by RH(3,4) and EG(11,11) (private) (table 2).

4. Coupled triads

Atmospheric and oceanic dynamics are characterized by a complex network of coupled triads (Kartashova & L’vov 2007), in which a single normal mode can participate in several triads and energy is redistributed through cascades. In the full partial differential equations, energy initially concentrated in a few modes leaks toward other wavenumbers (Lynch 2009).

The simplest couplings of triads involve four or five waves. We focus on topologies in which all constituent triads share the same two-mode edge (figure 5), so that a quartet adds one triad to the shared edge and a quintet adds two.

Although two coupled triads may seem distant from the full model, this simplification can shed light on the *low frequency variability* of atmospheric flows (Kartashova & L’vov 2007), such as the *intraseasonal variability*. Using four- and five-wave configurations, Raphaldini *et al.* (2022) related the efficiency of energy exchange to the precession resonance (Bustamante *et al.* 2014), which also enhances the low-frequency part of the spectrum. The same work suggests that resonant triads may control the instability of a Rossby mode excited by forcing mechanisms such as baroclinic instability.

Throughout this section, a **quartet** denotes four Hough modes coupled through two constituent resonant triads that share one common edge (a pair of modes), and a **quintet** denotes five modes coupled through three constituent triads sharing the same edge. A mode that belongs to only one constituent triad is marked (*private*) where the distinction from the shared-edge modes matters. See figure 5 for the topology of every quartet examined here: the all-Rossby Quartet A (edge RH(4,5)+RH(1,2), panel a), the all-Rossby Quartet B reproducing Raphaldini *et al.* (2022)’s topology (edge RH(1,3)+RH(3,7), panel b), the Rossby-gravity Quartet C sharing edge RH(4,5)+RH(3,4) with Quartet A’s triad, completed by the Kelvin-like mode EG(1,1) (panel c), and Quartet D, in which the gravity mode WG(7,9) belongs to the shared edge together with RH(4,5), and each triad is completed by one further mode, RH(3,4) or WG(3,9) (panel d).

Mode	Frequency (ω)	Period (days)	Coeff. 1	Coeff. 2	Pump
Quartet A: $\delta_1 = -0.108548$, $\delta_2 = -0.0400376$					
RH(4,5)	-0.12524	3.99232	0.364202	0.552536	2
RH(1,2)	-0.0994375	5.02829	-0.028726	0.208514	
RH(3,4)	-0.134351	3.7216	0.444031		1
RH(3,6)	-0.0658405	7.59411		0.343659	
Quartet B: $\delta_1 = 0.0149092$, $\delta_2 = -0.0312087$					
RH(4,5)	-0.12524	3.99232	0.558519		1
RH(1,3)	-0.0601672	8.31017	0.361554	-0.385223	
RH(3,7)	-0.050164	9.96731	0.197138	-0.848058	2
RH(2,9)	-0.0212055	23.5788		-0.466506	
Quartet C: $\delta_1 = -0.108548$, $\delta_2 = 0.360579$					
RH(4,5)	-0.12524	3.99232	0.364202	-0.55801	2
RH(3,4)	-0.134351	3.7216	0.444031	-0.591116	
RH(1,2)	-0.0994375	5.02829	-0.028726		1
EG(1,1)	0.36969	1.35249		-0.132558	
Quartet D: $\delta_1 = 3.01536$, $\delta_2 = -0.137364$					
WG(7,9)	-3.27495	0.152674	-0.143769	-0.925608	1,2
RH(4,5)	-0.12524	3.99232	-0.680596	-0.0355634	
RH(3,4)	-0.134351	3.7216	-0.776361		
WG(3,9)	-3.28707	0.152111		-0.935921	
Quartet E: $\delta_1 = -3.45511$, $\delta_2 = -0.769247$					
EG(7,9)	3.19551	0.156469	-0.338506	0.942072	1,2
RH(4,5)	-0.12524	3.99232	-0.766192	-0.0366433	
RH(3,4)	-0.134351	3.7216	-0.899854		
EG(11,11)	3.83952	0.130225		1.18603	

Table 2: Properties of every quartet examined in §4 (figure 5): each mode's frequency ω , linear period (days), and coupling coefficient within whichever constituent triad(s) contain it (Coeff.₁/Coeff.₂, – where a mode is not in that triad), the triad's mismatch δ , and each triad's pump mode (from the signs of its coupling coefficients, table 6). Quartet B's precession-comparison table (table 4) is separate.

4.1. Four-wave systems: quartets

The expansion (3.1) truncated at the four mentioned RH modes yields in equation (3.4) the four-wave system

$$\frac{dA_a}{dt} = -i\omega_a A_a + i\alpha_{b,c}^a A_b A_c + i\alpha_{b,d}^a A_b A_d, \quad (4.1)$$

$$\frac{dA_b}{dt} = -i\omega_b A_b + i\alpha_{a,c}^b A_a A_c^* + i\alpha_{a,d}^b A_a A_d^*, \quad (4.2)$$

$$\frac{dA_c}{dt} = -i\omega_c A_c + i\alpha_{a,b}^c A_a A_b^*, \quad (4.3)$$

$$\frac{dA_d}{dt} = -i\omega_d A_d + i\alpha_{a,b}^d A_a A_b^*, \quad (4.4)$$

whose *Manley-Rowe* relations, which are preserved during integration, are

$$J_{a,c,d} = |A_a|^2 + \frac{\alpha_{b,c}^a}{\alpha_{a,b}^c} |A_c|^2 + \frac{\alpha_{b,d}^a}{\alpha_{a,b}^d} |A_d|^2,$$

$$J_{b,c,d} = |A_b|^2 - \frac{\alpha_{a,c}^b}{\alpha_{a,b}^c} |A_c|^2 - \frac{\alpha_{a,d}^b}{\alpha_{a,b}^d} |A_d|^2.$$

Although the system (4.1) - (4.4) is derived from the RSW equations, whose total energy is given by the Hamiltonian (2.13), the four-wave system *does not necessarily conserve* the Hamiltonian evaluated on the truncated solution (Appendix B). For models with quadratic total energy, such as the barotropic nondivergent model, this quantity is conserved for *any* truncation (Majda 2003). In the RSW equations, some Galerkin truncations (3.1) are therefore inconsistent with the full system, in the sense that they do not conserve its total energy.

4.1.1. Diagnostics

In the rest of this study we evaluate how efficiently a pair of triads exchanges energy, and how a primary triad changes when a fourth mode interacts with two of its modes. The measures used to quantify these effects are defined below. The time average of a quantity f over a window $[0, T_f]$ is

$$\langle f \rangle_t \equiv \frac{1}{T_f} \int_0^{T_f} f(t) dt. \quad (4.5)$$

This average is applied to the energy contained in a truncated wave system, which is not conserved for arbitrary truncations of the shallow water equations (Appendix B). Every diagnostic below is therefore gated on the relative variation of the total energy,

$$\max_{t \in [0, T_f]} \frac{|E_{total}(t) - \langle E_{total} \rangle_t|}{\langle E_{total} \rangle_t} \leq \epsilon = 10\%, \quad (4.6)$$

and left undefined otherwise, since beyond this threshold the diagnostics would measure truncation error rather than physics. Following atmospheric normal-mode decompositions (Kasahara & Puri 1981; Marques *et al.* 2020), the physical energy of mode a is $EK_a = (gh_0^2 a^2 \pi) \rho \|A_a\|^2$, with $\rho = p_s / (gh_0) \approx 1.03 \text{ kg/m}^3$. All ratios below cancel this prefactor, so we work with the dimensionless per-mode energy

$$E_a \equiv \|A_a\|^2, \quad (4.7)$$

consistent with $E^{(2)} = \sum_a E_a$ (2.15).

For a mode a , we compare the fluctuation of E_a in the triad $\{a, b, c\}$ with its fluctuation in the quartet $\{a, b, c, d\}$, which quantifies how adding mode d affects mode a . In a system with modes $\Gamma = \{a, b, c, \dots\}$, the quantity

$$\Delta E_{\{a,b,c,\dots\}}^a = \max_{t \in [0, T_f]} E_a(t) - \min_{t \in [0, T_f]} E_a(t) \quad (4.8)$$

is the energy variation of the target mode a over $[0, T_f]$, computed within that wave set alone. The superscript denotes the mode whose energy is measured, and the subscript denotes the set Γ of all modes in the system. For example, $\Delta E_{\{a,b,c\}}^a$ is the energy variation of a within the triad $a + b + c$, and $\Delta E_{\{a,b,c,d\}}^a$ within the quartet.

Efficiency variation ($\Delta \mathcal{E}_a$). The triad efficiency (3.26) extends directly to a quartet or quintet once its instantaneous total energy is replaced by the time average (4.5), gated by

559 (4.6):

$$560 \quad \mathcal{E}_{\{a,b,c,\dots\}}^a = \frac{\Delta E_{\{a,b,c,\dots\}}^a}{\langle E_{total} \rangle_t}, \quad (4.9)$$

561 the share of the wave set's total (time-averaged) energy that cycles through mode a , which
 562 characterizes a single configuration. Comparing the dynamics of a across two configurations,
 563 such as a triad and the quartet that contains it, is a different question. The percent change
 564 between the two values of $\mathcal{E}^a$ (4.9) is misleading here, because the total-energy budgets
 565 $\langle E_{total} \rangle_t$ of the two configurations differ for reasons unrelated to the dynamics of a . A
 566 reference triad has fewer active modes, and hence typically a smaller budget, than the full
 567 quartet, and screening candidate modes for a slot (§4.3) changes the budget further through
 568 each candidate's driving velocity and modal structure. Normalizing both configurations by
 569 the *same* budget, that of the smaller (reference) configuration, which a lone triad conserves
 570 exactly (3.25), cancels it from the ratio and leaves

$$571 \quad \Delta \mathcal{E}_a = 100 \frac{\Delta E_{\{a,b,c,d\}}^a - \Delta E^{a*}}{\Delta E^{a*}}, \quad \Delta E^{a*} = \max \left(\Delta E_{\{a,b,c\}}^a, \Delta E_{\{a,b,d\}}^a \right) \quad (4.10)$$

572 (and analogously for b, c, d). This is the percent change in the *raw* energy exchange of a
 573 between the full wave set and its reference triad, and it does not depend on the total-energy
 574 budget of either configuration. For a mode shared by both constituent triads (a or b above),
 575 ΔE^{a*} uses the triad in which the mode has the larger ΔE , since a mode can be a weak
 576 or off-resonance participant in one triad while the other drives it strongly. For a mode that
 577 belongs to one triad only (private), the maximum is trivial.

578 **Dominant period and novel frequency content** ($\tau_{\text{novel}}^a, \rho_{\text{novel}}^a$). Given a mode's energy
 579 time-series $E_a(t)$, its dominant period is the period of the largest peak of the power spectrum
 580 $|\widehat{E_a}(f)|$, obtained via a real Fast Fourier Transform of the mean-subtracted series (periods
 581 within about 10% of T_f are unresolved and excluded, and every period reported below uses
 582 an integration window at least an order of magnitude longer than the period itself). For a
 583 target mode belonging to one or more constituent sub-triads, a *novel* period is a peak present
 584 in the full wave set's spectrum but absent from every containing sub-triad's spectrum. Novel
 585 peaks are identified by eye against the sub-triad spectra plotted alongside the full spectrum
 586 in each figure below (§4), rather than by an automated rule, since a small shift in an existing
 587 peak can otherwise register as spuriously novel. Each novel peak, at period τ_{novel}^a , is reported
 588 with a relevance ρ_{novel}^a , defined as its share of the total spectral power of a in the full wave
 589 set. The diagnostic is silent whenever the full wave set's spectrum is already explained by its
 590 constituent triads, and positive only when a new timescale appears.

591 A mode's linear frequency ω_a never appears as a peak in the spectrum of $E_a(t)$, at any
 592 coupling strength. The term $-i\omega_a A_a$ contributes nothing to the time derivative of $E_a = \|A_a\|^2$
 593 in (3.18)–(3.20) (or, for a quartet, (4.1)–(4.4)), since a pure phase rotation is orthogonal to a
 594 change in modulus,

$$595 \quad \frac{d}{dt} |A_a|^2 = 2 \operatorname{Re} \left(A_a^* \frac{dA_a}{dt} \right) = 2 \operatorname{Re} (-i\omega_a |A_a|^2) + (\text{nonlinear terms}) = 0 + (\text{nonlinear terms})$$

596 as derived for the triad in (3.23). The same cancellation holds mode by mode in the quartet
 597 system, however many constituent triads a mode belongs to. The periods in the spectrum
 598 of $E_a(t)$ are set only by the nonlinear coupling terms, together with the mismatch δ acting
 599 through the dynamical phase Φ defined below.

600 **Dynamical phase.** Consider one constituent triad with sum mode a (satisfying the zonal-
 601 wavenumber selection rule of §4.1, e.g. $m_a = m_b + m_c$ in eq. 4.1) and two member modes,
 602 labelled j, k because the pair differs between triads (b, c for one constituent triad of a quartet,

603 b, d for the other). The phase combination

$$604 \quad \Phi_{j,k}^a = \arg(A_a) - \arg(A_j) - \arg(A_k) \quad (4.11)$$

605 is the dynamical phase of an isolated triad (Appendix A). The coupling term in eq. 3.14
 606 depends on the three phases only through Φ (Appendix A derives this for the resonant case).
 607 Without coupling, Φ advances at the constant rate δ . With coupling, the nonlinear term can
 608 slow or arrest this drift. If the coupling is weak, Φ keeps circulating through 2π at a rate
 609 close to δ . Sufficiently strong coupling instead traps Φ in a bounded oscillation about a
 610 fixed value, a *synchronized* state that signals a resonant lock (Bustamante *et al.* 2014). We
 611 call the magnitude of the linear drift rate of Φ the *precession frequency*. It is large while Φ
 612 is unsynchronized and falls toward zero at a lock. We use it below (§4.2, §4.3) to test for
 613 precession resonance, and report it for every constituent triad both within the full wave set
 614 and for the triad integrated alone.

615 4.2. Rossby-only quartet

616 We now place the RH modes (1,2) and (4,5) in a network of coupled triads. As noted in §3.3,
 617 these two modes interact mainly with symmetric waves of zonal wavenumber $m = 3$ and
 618 meridional wavenumber $n = 4-10$, and the transfer to this third wave weakens as n increases.
 619 Letting them interact simultaneously with RH(3,4) and RH(3,6), we ask whether they prefer
 620 one of these two partners, using the kinetic energy of all four modes.

621 Figure 5 shows in panel (a) the all-Rossby quartet, hereafter **Quartet A**, composed of
 622 the triads RH(4,5)+RH(1,2)+RH(3,4) and RH(4,5)+RH(1,2)+RH(3,6), sharing the edge
 623 RH(4,5)+RH(1,2). Its coupling coefficients are given in table 2.

624 4.2.1. Partner preference

625 In the first example (table 2), RH(3,4) and RH(3,6) start at rest, so the energy initially in
 626 RH(4,5) and RH(1,2) is redistributed between them and any preference for one partner shows
 627 directly.

628 RH(4,5) and RH(1,2) interact preferentially with RH(3,4), whose kinetic energy reaches
 629 higher peaks than that of RH(3,6) (figure 6). The three RH modes with the smallest zonal
 630 wavenumbers are quasi-periodic, and the period of RH(3,4) is almost half that of the other
 631 two, which may explain its larger energy gain. RH(4,5) behaves less regularly. It receives and
 632 returns energy without a well-defined period, so the four-wave system already shows irregular
 633 dynamics, in contrast to the periodicity of a single triad. When the two triads are integrated
 634 separately (figure 6, left and middle panels), the triad containing RH(3,4) has the shorter
 635 period. In the quartet, the energy variation of RH(3,4) is 35.4% larger than in its triad alone
 636 ($\Delta\mathcal{E}_{RH(3,4)} = +35.4\%$), while that of RH(3,6) is 26.1% smaller ($\Delta\mathcal{E}_{RH(3,6)} = -26.1\%$). The
 637 difference in periods partly explains this, since the faster exchange favours RH(3,4). RH(1,2)
 638 is subject to two opposite influences, a stronger flux due to RH(3,6) and an almost stationary
 639 pattern due to RH(3,4), which may explain why its energy tends to the average of the values
 640 obtained in each triad.

641 The exchange periods of the triads affect how energy is redistributed, but they do not set
 642 how much energy a given partner captures. To test this, the partner RH(3,4) of the shared
 643 edge RH(4,5)+RH(1,2) (table 2) is replaced by each RH(3, n), $n = 4, \dots, 16$, in turn. Each
 644 row of table 3 is an isolated triad RH(4,5)+RH(1,2)+RH(3, n), with RH(4,5) and RH(1,2)
 645 driven at 40m/s and the target RH(3, n) initially at rest. The table also reports, through the
 646 efficiency variation $\Delta\mathcal{E}_a$ (eq. 4.10), whether this ranking survives when RH(3,4) is restored
 647 as a competitor for the same driving pair. Odd n are omitted because, by the equatorial-
 648 symmetry rule, RH(3, n) couples to this symmetric driving pair only when $3 + n$ is odd, so
 649 odd- n members have zero coupling coefficient and efficiency.

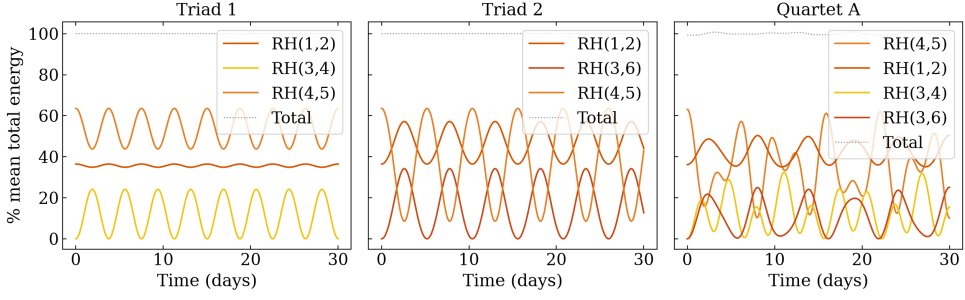


Figure 6: Integration of (left) the triad completed with RH mode (3,6), (middle) the triad completed with RH mode (3,4), and (right) the full four-wave configuration presented in table 2. The y-axis is each mode’s energy E_a (4.7) as a percentage share of the wave set’s mean total energy $\langle E_{total} \rangle_t$ (4.5). The grey dotted line is the wave set’s total energy on that same normalized scale; its small wander from 100% reflects the truncation’s non-conservation (Appendix B).

Table 3: Driving pair and RH(3, n) target family at $h_e = 10\,000\text{m}$. The driving pair RH(1,2)/RH(4,5) is fixed across the whole family with initial zonal velocity 40m/s and amplitude A_0 . Frequency ω and period are the mode’s linear (normal-mode) values; mismatch δ , computed from these linear frequencies, is per-triad. Efficiency $\mathcal{E}$ (eq. 3.26) is the fraction of the triad’s initial energy that ends up cycling through the target, which starts at rest ($A_0 = 0$). $\Delta\mathcal{E}_a$ (eq. 4.10) is the target’s percent change in raw energy exchange, four-wave quartet RH(4,5)+RH(1,2)+RH(3,4)+RH(3, n) versus its isolated triad RH(4,5)+RH(1,2)+RH(3, n) alone. *: RH(3,4) is fixed in the quartet. **: for $n = 12$ efficiency is negligible, leaving the ratio numerically undefined.

		Frequency (ω)	Period (days)	A_0	Coeff. ($ \alpha $)	$\mathcal{E}$ (%)	$\Delta\mathcal{E}_a$ (%)
Driving pair (both 40m/s)	RH(1,2)	-0.09944	5.028	0.0927	–	–	–
	RH(4,5)	-0.12524	3.992	0.1224	–	–	–
Targets RH(3,n)	δ						
$n = 4$ *	-0.1085	-0.1344	3.722	0	0.444	24.11	–
$n = 6$	-0.0400	-0.0658	7.594	0	0.344	34.18	-25.95
$n = 8$	-0.0137	-0.0395	12.659	0	0.046	14.49	-66.27
$n = 10$	-0.0005	-0.0263	19.020	0	0.0033	12.64	-99.08
$n = 12$	+0.0071	-0.0187	26.701	0	0.0002	0.001	**

Two results deserve comment. First, $n = 6$ captures more energy than $n = 4$ (34.2% vs. 24.1%) in isolated triads, although $n = 4$ has the stronger coupling. Second, $n = 10$ ’s mismatch is the smallest of the family ($\delta \approx -0.0005$, two orders of magnitude below $n = 4$ ’s), yet its converged isolated triad efficiency (12.64%) is comparable to $n = 8$ ’s. Mismatch therefore sets how much of the family’s resonant capacity a partner can reach, and the coupling strength sets how long it takes to get there. This is consistent with the pump-mode exact solutions of Appendix A, in which the same coupling coefficients set both the period (A 2) and the achievable efficiency. Beyond $n = 10$, both effects work against the target simultaneously (δ grows again in the opposite sign, $|\alpha|$ keeps shrinking).

The final column of table 3 gives $\Delta\mathcal{E}_a$ for the quartet RH(4,5)+RH(1,2)+RH(3,4)+RH(3, n), with RH(3,4) also initially at rest, relative to each target’s isolated triad.

The competition is one-sided and grows more severe as $n \rightarrow 10$, with $\Delta\mathcal{E}_a$ falling from -25.95% at $n = 6$ to -99.08% at $n = 10$. Despite its 12.64% isolated efficiency, RH(3,10)

receives almost no energy once RH(3,4) competes for it. Beyond $n = 10$, both the isolated and quartet efficiency collapse to negligible values.

4.2.2. Precession resonance

Raphaldini *et al.* (2022) tie the efficiency of Rossby-triad energy exchange to *precession resonance* (Bustamante *et al.* 2014). Two triads sharing an edge can, at a specific driving amplitude, generate a resonance between the nonlinear frequency of the first triad (i.e. the frequency of its amplitude oscillations) and the mismatch frequency of the second triad, locking their dynamical phases (eq. 4.11) together, producing a sharp, non-monotonic efficiency signature. At finite amplitude, the relevant frequency of the second triad is its precession frequency, the drift rate of Φ defined in §4.1.1, which reduces to the mismatch δ in the small-amplitude limit. This is the quantity we measure below, and a lock appears as its fall toward zero. Their demonstration uses the barotropic vorticity equation. We test whether the same mechanism operates in the RSW quartets, whose cubic energy is not conserved by the truncation, by building the topology of Raphaldini *et al.* (2022) in the RSW system and by examining Quartet A.

Raphaldini *et al.* (2022) report an efficiency peak of $\sim 9\text{--}10\%$ for the four-wave configuration that maps here as RH(1,3), RH(3,7), RH(4,5), RH(2,9) (figure 5b, hereafter **Quartet B**), at “about $\alpha = 15$ ” in their figure 2. In the RSW version of this topology, RH(1,3), RH(3,7) and RH(4,5), which form one constituent triad, start with comparable amplitudes, while RH(2,9) starts near zero and is the target whose growth the efficiency measures. A single overall amplitude scale drives all four modes. The phase Φ of the triad RH(4,5)+RH(1,3)+RH(3,7) is the one that reproduces the signature in figure 3 of Raphaldini *et al.* (2022), and its mismatch is δ_1 in table 4 below. The dynamical-phase diagnostic (eq. 4.11), applied to that same triad in both systems at a low-amplitude *control* scale and at each system’s efficiency peak, shows the phase becoming synchronized in the barotropic system but remaining unsynchronized throughout in RSW. The control scale is chosen, separately for each system, about one order of magnitude below that system’s efficiency peak in log-amplitude (barotropic: control = 2×10^{-4} , peak = 1.8×10^{-3} ; RSW: control = 3×10^{-3} , peak = 5×10^{-2}). At the barotropic control scale, Φ (eq. 4.11) advances by about 0.041 radians every day with no preferred value it settles toward, with full 2π revolutions roughly every 150 days. At the peak this drift nearly stops, with the precession frequency falling to +0.0001 rad/day, and Φ oscillates around a fixed angle, as reported by Raphaldini *et al.* (2022). For the RSW equations, the precession frequency *grows* from control (+0.192) to peak (+0.372), without locking. Figure 7b shows both curves over the full amplitude range.

In both systems the amplitude scale is varied along the initial-condition direction scale $\cdot (1, 1, 1, 10^{-3})$ (in each system’s mode order), on a log-spaced grid of 20 points from the control scale up to the largest value at which the precession-frequency diagnostic remains reliable (0.05 barotropic, 0.25 RSW, since at larger RSW scales, e.g. 1.09, the sign of the precession frequency changes with the length of the integration window). The RSW efficiency curve is not monotonic. It rises to a first local maximum at scale = 5×10^{-2} (21.8%, the “peak” value above), dips by about 5 percentage points by scale ≈ 0.1 , and rises again toward the upper end of the sweep without exceeding the first maximum (20.6% at scale = 0.25).

The two systems can share a qualitative efficiency signature (§4.2) without sharing the phase-locking behaviour. This is consistent with the barotropic model being an approximation of the RSW system valid in the small-Lamb’s-number limit, where gravity-wave coupling is discarded (Schubert *et al.* 2009; Verkley 2009). At any nonzero Lamb’s number, RSW does not reduce to it exactly.

We apply the same diagnostics to the all-Rossby Quartet A, which fixes the shared edge RH(4,5)+RH(1,2) at 40m/s and sweeps RH(3,6)’s initial velocity u from 10 to 150m/s. The

Table 4: Efficiency and dynamical-phase precession frequency (of the RH(4,5)+RH(1,3)+RH(3,7) triad’s Φ , per §4.2.2) for Quartet B (Raphaldini *et al.* (2022)’s topology, figure 5b) in the barotropic vorticity equation (their system) vs. this paper’s RSW system, at each system’s control point (amplitude scale 2×10^{-4} barotropic, 3×10^{-3} RSW) and efficiency peak (1.8×10^{-3} barotropic, 5×10^{-2} RSW).

	Barotropic (Raphaldini <i>et al.</i> 2022)	RSW (same topology)
δ_1	−0.003	+0.015
Efficiency peak (%)	10.0	21.8
Precession freq., control (rad/day)	−0.041	+0.192
Precession freq., peak (rad/day)	+0.0001	+0.372

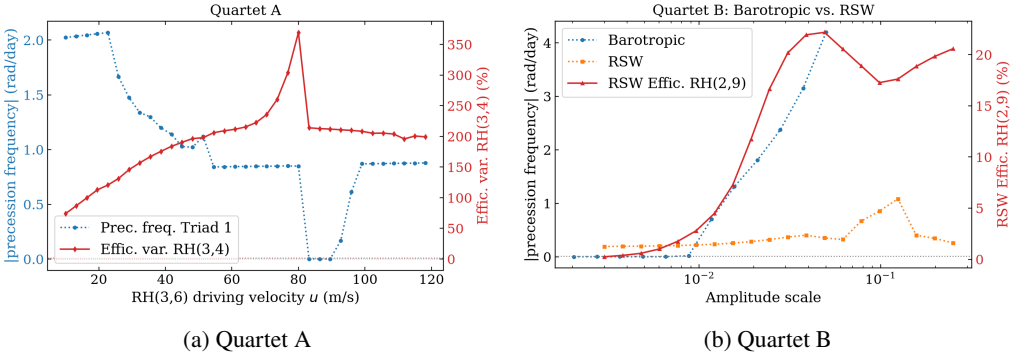


Figure 7: Precession frequency (left axis, $|\Phi|$ ’s linear drift rate) and a target mode’s efficiency (right axis) vs. driving amplitude. (a) Quartet A: precession frequency of Triad 1 (RH(4,5)+RH(1,2)+RH(3,4)) and efficiency variation of the target RH(3,4) (eq. 4.10), sweeping the driving velocity of RH(3,6) with the shared edge fixed at 40m/s. The triad of RH(3,6), which locks onto its forcing, is not shown. (b) Quartet B: barotropic and RSW precession frequency of the triad RH(4,5)+RH(1,3)+RH(3,7), which excludes the target RH(2,9), and time-averaged efficiency of the RSW target RH(2,9) (eq. 4.9).

712 triad of RH(3,6) locks onto its forcing from $u \approx 54$ m/s onward, as a strongly driven triad
713 eventually does. The more informative case is the undriven mode RH(3,4) (private), which
714 receives energy only through the shared edge. Its triad locks for $u \approx 83$ –92m/s, with sharp
715 boundaries (figure 7a). The lock coincides with a discontinuity in the efficiency variation
716 of RH(3,4). $\Delta\mathcal{E}_a$ rises from $\approx 236\%$ at $u \approx 70$ m/s to a peak of $\approx 369\%$ at $u \approx 80$ m/s,
717 drops abruptly to $\approx 214\%$ at lock onset, and then declines slowly to $\approx 199\%$ at $u \approx 118$ m/s.
718 Throughout the sweep RH(3,4) captures more energy than in its isolated triad, and the lock
719 marks a sharp drop in this excess rather than a reversal.

720 Locking, when it occurs, leaves a mark on efficiency. Quartet A’s target-triad lock coincides
721 with the sharp drop described above. But a smooth efficiency curve does not rule locking
722 out, and a peak does not require it. In RSW, Quartet B never locks within the range tested,
723 yet its efficiency still peaks, as in the barotropic system, where the peak does coincide with a
724 lock (table 4). A shared efficiency signature between two systems is therefore not, by itself,
725 evidence that the same phase-locking mechanism produced it.

726 Raphaldini *et al.* (2022) also report an individual-mode phase reversal at high driving
727 amplitude (their Section III.A), in which the raw phase of one mode bends enough for its
728 propagation direction to change sign. We checked for this in both quartets and in a direct

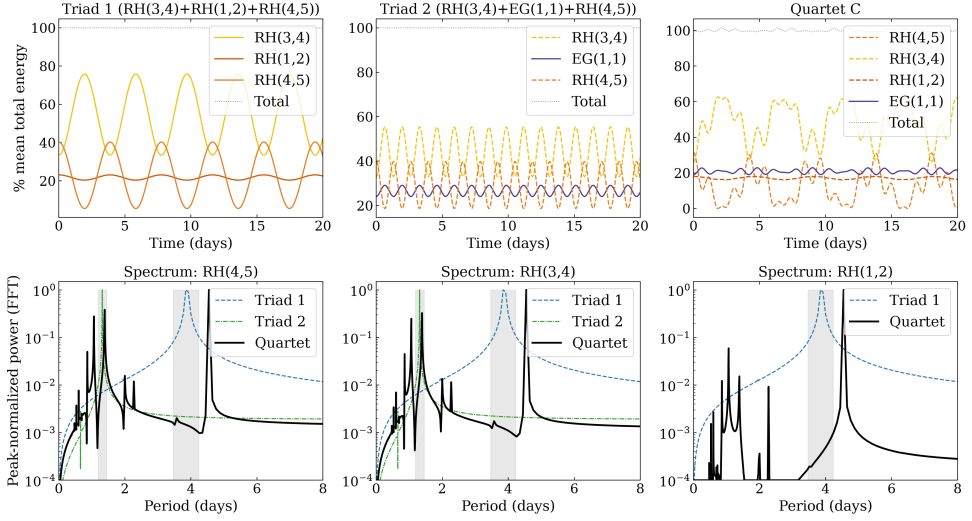


Figure 8: Quartet C (table 2), all four modes at 40m/s: (top row) energy evolution of Triad 1 (RH-only), Triad 2 (with EG(1,1)), and the full quartet; (bottom row) each Rossby mode’s novelty-frequency spectrum (§4.1.1) against whichever sub-triad(s) contain it, grey shading marking the union of each containing sub-triad’s dominant spectral peak, the reference window excluded when identifying a novel period. The y-axis of the top row follows figure 6: each mode’s energy E_a (4.7) as a percentage share of the wave set’s mean total energy $\langle E_{total} \rangle_t$ (4.5); the grey dotted line is the total energy on that same normalized scale.

reproduction of their barotropic system, but no such reversal appears at any scale tested, including through Quartet A’s lock. In both quartets, the efficiency peaks also coincide with elevated low-frequency spectral power in the kinetic energy of the target mode, as in Raphaldini *et al.* (2022).

4.3. Gravity-Rossby quartets

Although gravity waves are fast, gravity modes, including Kelvin waves, take part in several low-frequency meteorological phenomena, such as the aforementioned Madden-Julian Oscillation (Madden & Julian 1972; Zhang 2005; Teruya *et al.* 2024) and the quasi-biennial oscillation (Baldwin *et al.* 2001; Raphaldini *et al.* 2020).

Gravity waves also take part in energy cascades, for instance as near-inertial waves that extract energy from the inverse cascade of the balanced flow (Rocha *et al.* 2018). In coupled triads, a gravity mode can either attenuate the exchange among RH modes or route energy into them, and filtering gravity waves out of a model removes these effects.

The two main quartets of this section (figure 5c-d) couple the Rossby modes RH(4,5) and RH(3,4) to gravity modes in different ways. Quartet C adds the Kelvin-like mode EG(1,1) to Quartet A’s RH-only triad RH(4,5)+RH(3,4)+RH(1,2), as a fourth mode on the edge RH(4,5)+RH(3,4). In Quartet D the gravity mode WG(7,9) is instead the shared sum mode of both triads (§4.3.2).

4.3.1. Coupling a Rossby triad with a gravity Kelvin mode

We first couple a full Rossby triad with a Kelvin mode (Quartet C, table 2). All four modes, including EG(1,1), start at 40m/s, so any difference between the isolated RH-only triad and the full quartet comes from the coupling, not from an asymmetric initial condition.

The bottom row of figure 8 shows a sharp new spectral peak at 4.55d in all three Rossby

modes of the full quartet, outside both excluded windows (the RH-only triad's 3.85d, Triad 2's 1.32d). Coupling with EG(1,1) therefore generates a new, slower period through nonlinear interactions, rather than blending the two triads' periods. This new period accounts for 53–69% of each Rossby mode's total spectral power (RH(4,5) 53.3%, RH(3,4) 55.2%, RH(1,2) 69.3%), making it the dominant spectral feature in every case. The effect on the energy exchange differs between modes. The efficiency variation is positive for RH(4,5) and RH(3,4) ($\Delta\mathcal{E}_a = +15.85\%/+11.21\%$, eq. 4.10), which swing harder in the full quartet than in the RH-only triad, but negative for RH(1,2) ($\Delta\mathcal{E}_a = -10.77\%$), even though RH(1,2) shows the new period with the highest relevance of the three. EG(1,1) carries only a weak trace of the new period (8.95% relevance), and its spectrum stays dominated by its fast oscillation of about 1.3d. The slow period is thus an RH-mode phenomenon generated by the coupling, not imported from the timescale of the gravity mode.

When the three Rossby modes are held at 40m/s and the driving velocity of EG(1,1) is swept from 10 to 100m/s, the shift of the new period from the RH-only triad's 3.85d grows monotonically above 30m/s, reaching about 3.6d at 100m/s, where the new period is about 7.4d.

4.3.2. Coupling with faster gravity modes

Before changing the topology, we ask whether replacing EG(1,1) by a higher-wavenumber gravity mode could already weaken this coupling. In figure 9, EG(1,1) is replaced by each EG(1, n) and WG(1, n), $n = 1, 3, \dots, 15$. Panel (a) shows the candidate's coupling coefficient to the shared edge RH(4,5)+RH(3,4) (its member coefficient within Triad 2), and panel (b) the efficiency variation of RH(1,2) (eq. 4.10) relative to Triad 1, the RH-only triad, which does not contain the candidate.

The coupling coefficient is not monotonic in n . Both families rise from $n = 1$ ($|\alpha| = 0.133$ EG, 0.236 WG) to a peak near $n = 5$ ($|\alpha| = 0.682$ EG, 0.666 WG), then decay by roughly four orders of magnitude by $n = 15$ ($|\alpha| = 2.0 \times 10^{-5}$ EG, 1.9×10^{-5} WG). The shared edge therefore couples most strongly to an intermediate wavenumber, not to the lowest one. RH(1,2)'s efficiency variation broadly tracks the coefficient's decay past that peak, shrinking from a few percent to effectively zero by $n \gtrsim 11$ ($|\Delta\mathcal{E}| \lesssim 0.1\%$), but it is non-monotonic over $n = 1$ –5, where the coefficient is still rising, swinging between -10.8% ($n = 1$, EG) and $+5.2\%$ ($n = 5$, EG). A higher-wavenumber gravity mode therefore does not uniformly weaken the coupling. Only past the intermediate peak does increasing n steadily decouple it from the rest of the quartet.

Quartet D (figure 5d) has a different topology from Quartets A–C. Its two constituent triads share a member, RH(4,5), and their sum mode, the westward gravity mode WG(7,9). Triad 1 is RH(3,4)+RH(4,5)+WG(7,9) and Triad 2 is WG(3,9)+RH(4,5)+WG(7,9), so RH(3,4) and WG(3,9) each belong to one triad only (private).

The linear periods of WG(7,9) and WG(3,9) are about 0.15d, against 3.99d and 3.72d for RH(4,5) and RH(3,4) (table 2), a timescale separation of roughly 25. The two triads lie on opposite sides of resonance, with Triad 1 far from it ($\delta_1 = 3.015$) and Triad 2 an order of magnitude closer ($\delta_2 = -0.137$).

Figure 10 shows the energy evolution of each constituent triad alone (top row), the full quartet (bottom left), and RH(3,4)'s novelty-frequency spectrum against Triad 1, its only containing triad (bottom right). In each triad alone, RH(4,5) holds a large but weakly modulated share of the mean total energy, peaking at 63.9% in Triad 1 and 54.6% in Triad 2, and its energy variation $\Delta E_{\text{RH}(4,5)}$ is more than three times larger in Triad 1 (4.32×10^{-3}) than in Triad 2 (1.19×10^{-3}). In the full quartet its raw energy swing grows to 7.52×10^{-3} , larger than in either triad alone, giving $\Delta\mathcal{E}_{\text{RH}(4,5)} = +74.0\%$ against Triad 1 (its larger-efficiency reference). The quartet dynamics of this mode is therefore not a mild perturbation

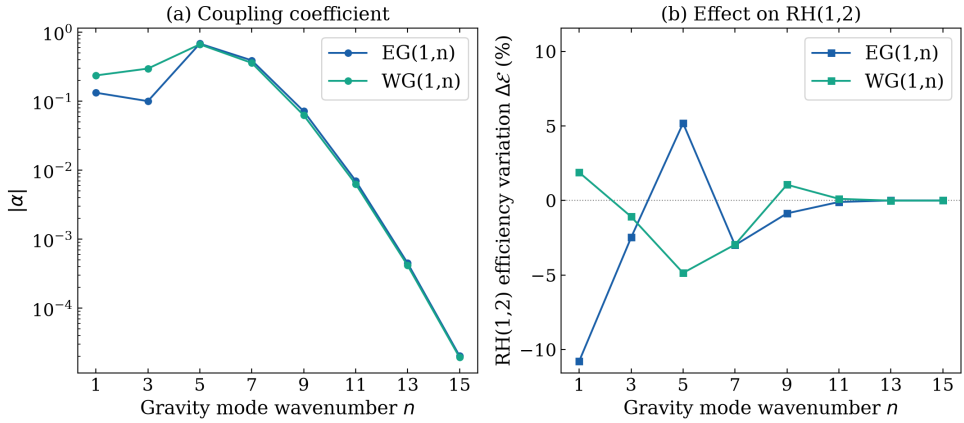


Figure 9: Quartet C with the fourth mode EG(1,1) replaced by EG(1, n)/WG(1, n), each point an independent four-wave integration with $T_f = 200d$. (a) The candidate's coupling coefficient $|\alpha|$ to the shared edge RH(4,5)+RH(3,4) (its member coefficient within Triad 2, computed from the linear eigenproblem without time integration; log scale). (b) RH(1,2)'s efficiency variation $\Delta\epsilon$ (eq. 4.10) against Triad 1 (RH-only, excludes the candidate).

of either triad. The dynamical phases also shift when the triads are coupled. The precession frequency of Triad 1 drops by 5.8% (from 6.031 to 5.678 cycles/day), while Triad 2 stays locked (0.00048 cycles/day alone, 0.00037 in the quartet).

The spectrum panel shows a more direct signature of the coupling. RH(3,4) belongs to Triad 1 only (private), so its spectrum is compared with that triad alone. Its dominant period in the full quartet, 0.175d, is close to that of Triad 1 (0.165d), but a second, much weaker peak appears at 1.053d (3% of the dominant peak's power) with no counterpart in the spectrum of Triad 1. This is the characteristic period of Triad 2, to which RH(3,4) does not belong. The peak is weak, but it is distinct from anything Triad 1 produces and is identified as novel, with $\tau_{\text{novel}}^a = 1.053d$ at 10.1% relevance. The shared sum mode WG(7,9) therefore transmits the periodicity of Triad 2 into a mode outside that triad. RH(4,5) shows no comparable new peak, since both of its peaks are already accounted for by the triads it belongs to.

Figure 11 shows how the efficiency variation $\Delta\epsilon_a$ (4.10) of the two Rossby modes responds when the driving velocity of WG(3,9) is varied from 0 to 60m/s, with all other initial conditions fixed. Panel (a) omits the two gravity modes, whose $\Delta\epsilon_a$ spans a much wider range and would obscure the smoother Rossby-mode curves. Both Rossby modes rise smoothly and almost identically, from $\approx 0.6\%$ at $u = 0$ to $\approx 102\%/101\%$ (RH(4,5)/RH(3,4)) at $u = 60\text{m/s}$. The growth is roughly linear at $\approx 0.23\%$ per m/s below $u \approx 24\text{m/s}$ and steepens to $\approx 2.8\%$ per m/s above $u \approx 32\text{m/s}$, with the transition between $u \approx 26$ and 30m/s . Panel (b) extends this to a two-dimensional sweep of the driving velocities of WG(3,9) and RH(3,4), for target RH(4,5). The effect is positive everywhere sampled, from $\approx 0.6\%$ at the origin to 149% at (WG(3,9), RH(3,4)) = (60, 20)m/s, and reaches 98% at (54, 10)m/s.

In Quartet E (figure 5e, table 2), EG(7,9) plays a double role. It closes the RH-only triad as its sum mode (Triad 1, with RH(4,5) and RH(3,4)) and is also a member of a second, higher triad closed by the higher-frequency gravity mode EG(11,11) (Triad 2, with RH(4,5)).

With EG(7,9), RH(4,5), RH(3,4) and EG(11,11) initialized at 50, 80, 50 and 50m/s, both triads oscillate fast when integrated alone, with dominant periods of 0.145d (Triad 1) and 0.54d (Triad 2). In the full quartet the two Rossby modes keep the period of Triad 1 (0.148d) and the two gravity modes that of Triad 2 (0.56d) (figure 12). The second triad increases the energy exchange of the Rossby modes by about a quarter ($\Delta\epsilon_a = +25.0\%/+24.7\%$ for

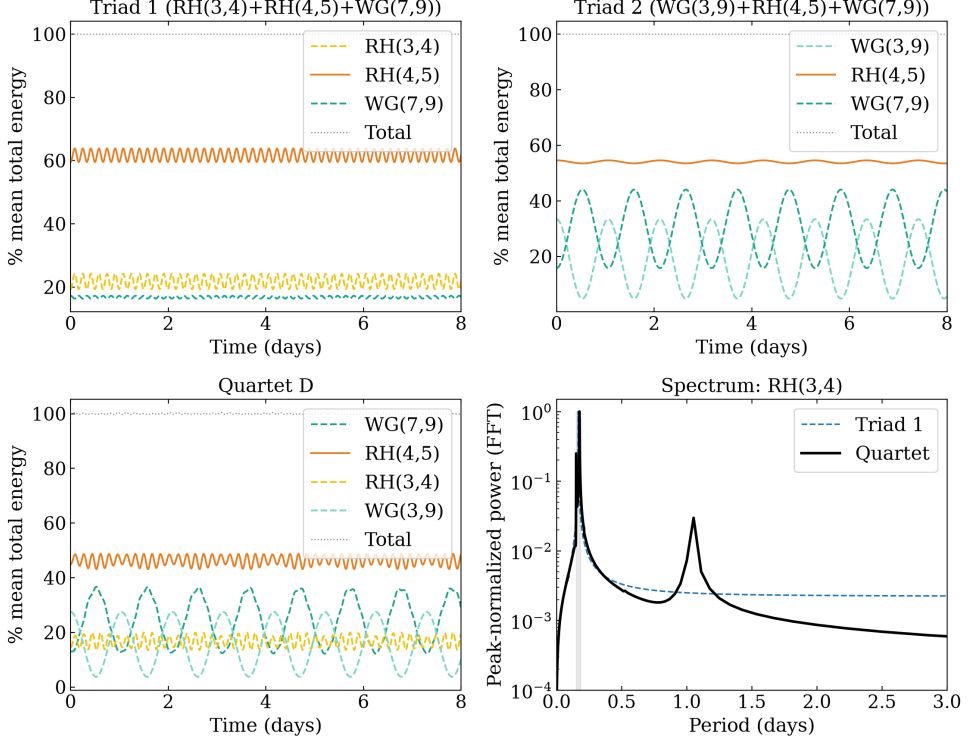


Figure 10: Quartet D (table 2): (top left) Triad 1 (RH(3,4)+RH(4,5)+WG(7,9)) integrated alone; (top right) Triad 2 (WG(3,9)+RH(4,5)+WG(7,9)) integrated alone; (bottom left) the full four-wave configuration; (bottom right) RH(3,4)'s novelty-frequency spectrum (§4.1.1) against Triad 1, its only containing triad, grey shading marking Triad 1's dominant spectral peak (as figure 8). The y-axis of the three evolution panels follows figure 6: each mode's energy E_a (4.7) as a percentage share of the wave set's mean total energy $\langle E_{total} \rangle_t$ (4.5); the grey dotted line is the total energy on that same normalized scale.

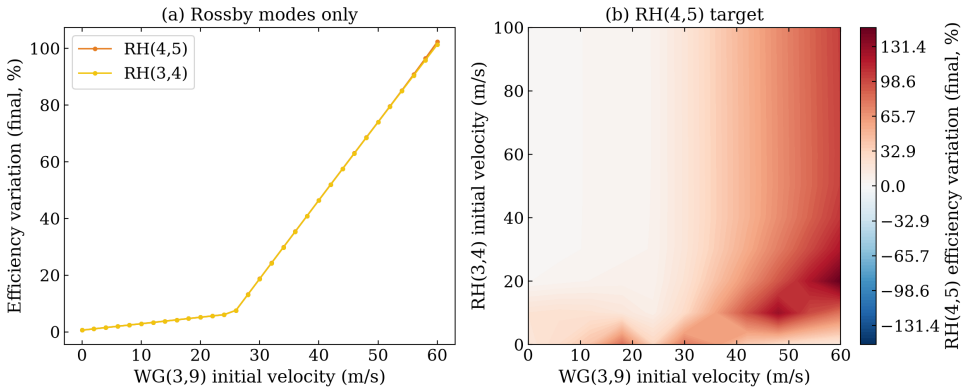


Figure 11: Quartet D, efficiency variation $\Delta \mathcal{E}_a$ (%) vs. WG(3,9)'s driving velocity: (a) RH(4,5)/RH(3,4) only, 0-60m/s; (b) 2D heatmap for target RH(4,5), sweeping WG(3,9) (0-60m/s) and RH(3,4) (0-100m/s) together.

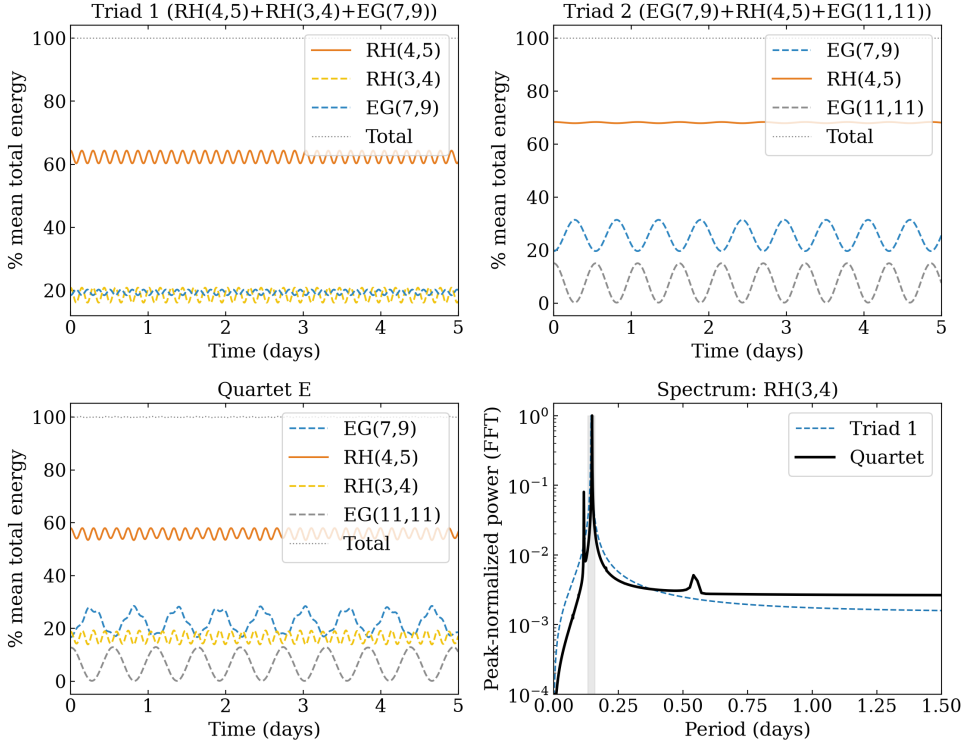


Figure 12: Quartet E (table 2): (top left) Triad 1 (RH(4,5)+RH(3,4)+EG(7,9)) integrated alone; (top right) Triad 2 (EG(7,9)+RH(4,5)+EG(11,11)) integrated alone; (bottom left) the full four-wave configuration; (bottom right) RH(3,4)'s frequency spectrum against Triad 1, its only containing triad. Axes as in figure 10.

831 RH(4,5)/RH(3,4) against Triad 1), a third of the increase found for RH(4,5) in Quartet D.
 832 As in Quartet D, the period of Triad 2 leaks into RH(3,4), which does not belong to that
 833 triad, but here only as a weak bump near 0.54d. The only novel peak, at 0.116d, is faster
 834 than both triads and carries less than 0.4% of the spectral power of RH(3,4). The precession
 835 frequencies of the two triads change by 2.3% and 1.4%, and neither triad locks. Adding a
 836 higher-frequency gravity triad to a gravity-closed edge thus strengthens the energy exchange
 837 of the Rossby modes but does not create a new low frequency.

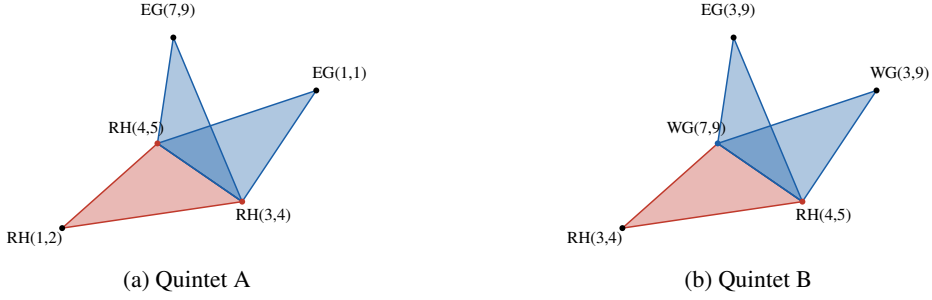


Figure 13: Topology of the two quintets examined in this section, each with three constituent triads sharing one common edge (as in figure 5): (a) Quintet A, Quartet C’s edge RH(4,5)+RH(3,4) closed a second time by EG(7,9) (table 5); (b) Quintet B, Quartet D’s shared sum mode WG(7,9) (paired with member RH(4,5)) closed a third time by EG(3,9) (private) (table 5).

Mode	Frequency (ω)	Period (days)	Coeff. 1	Coeff. 2	Coeff. 3	Pump
Quintet A: $\delta_1 = -0.108548$, $\delta_2 = 0.360579$, $\delta_3 = -3.45511$						
RH(4,5)	-0.12524	3.99232	0.364202	-0.55801	-0.766192	2
RH(3,4)	-0.134351	3.7216	0.444031	-0.591116	-0.899854	1
RH(1,2)	-0.0994375	5.02829	-0.028726			
EG(1,1)	0.36969	1.35249		-0.132558		
EG(7,9)	3.19551	0.156469			-0.338506	3
Quintet B: $\delta_1 = 3.01536$, $\delta_2 = -0.137364$, $\delta_3 = 6.40193$						
WG(7,9)	-3.27495	0.152674	-0.143769	-0.925608	0.203942	1,2
RH(4,5)	-0.12524	3.99232	-0.680596	-0.0355634	-0.00187474	
RH(3,4)	-0.134351	3.7216	-0.776361			
WG(3,9)	-3.28707	0.152111		-0.935921		
EG(3,9)	3.25223	0.153741			0.1555	3

Table 5: Properties of both quintets examined in §5 (figure 13): each mode’s frequency ω , linear period (days), and coupling coefficient within whichever constituent triad(s) contain it (Coeff.1/Coeff.2/Coeff.3, – where a mode is not in that triad), each triad’s mismatch δ , and its pump mode (from the signs of its coupling coefficients, table 6).

5. Five-Wave systems: Quintets

Combining two quartets on the same shared edge extends the star topology of §4 to three constituent triads and five total modes. We consider two such quintets, each built by adding a third triad to the shared edge of a quartet examined above. **Quintet A** extends Quartet C (§4.3.1) with a second gravity mode, EG(7,9), which closes the shared edge RH(4,5)+RH(3,4) a second time. **Quintet B** extends Quartet D (§4.3.2) with a third triad member, EG(3,9) (private), on the same shared sum mode WG(7,9). Both topologies are shown in figure 13, and their combined properties in table 5. Further quintet topologies (in particular the “hourglass” case, two independent triads sharing a single mode rather than an edge) are left for future work.

5.1. Quintet A: a second gravity mode on the Kelvin-mode edge

Quintet A is obtained from Quartet C (§4.3.1) by adding a third triad on its shared edge. The five modes form three triads (figure 13a), all of which contain RH(3,4) and RH(4,5).

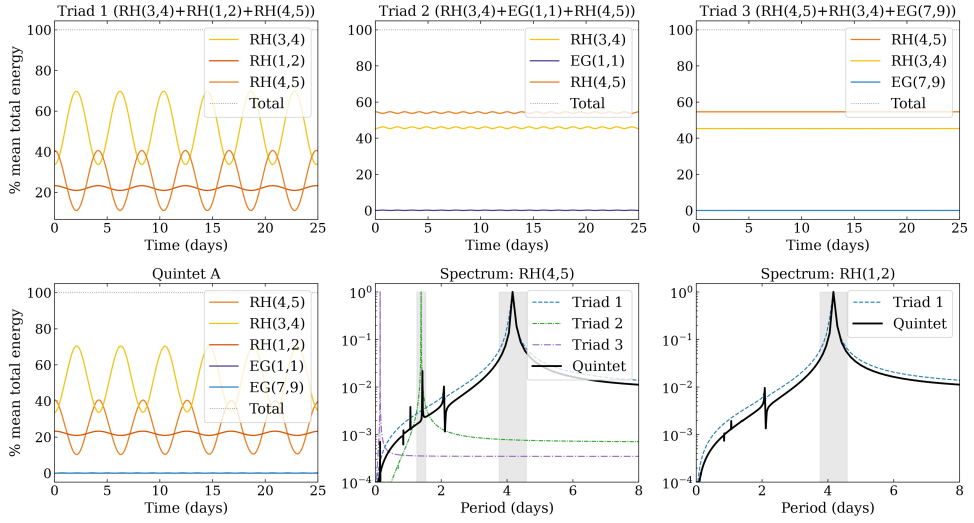


Figure 14: Quintet A (table 5), all three Rossby modes driven at 30m/s (EG(1,1)/EG(7,9) both at rest): (top row) energy evolution of Triad 1 (RH-only), Triad 2 (with EG(1,1)), and Triad 3 (with EG(7,9)); (bottom row) the full quintet, and RH(4,5)’s/RH(1,2)’s novelty-frequency spectra (§4.1.1) against whichever sub-triad(s) contain them, grey shading marking each containing sub-triad’s dominant spectral peak (as figure 8). The y-axis of the top row and the full-quintet panel follows figure 6: each mode’s energy E_a (4.7) as a percentage share of the wave set’s mean total energy $\langle E_{total} \rangle_t$ (4.5); the grey dotted line is the total energy on that same normalized scale.

851 When the Galerkin expansion (3.1) is truncated into three RH modes (4,5), (3,4) and (1,2),
 852 along with the eastward gravity waves (1,1) and (7,9), which satisfy the zonal wavenumber
 853 conditions

$$854 \quad m_a = m_b + m_c,$$

$$855 \quad m_a = m_b + m_d,$$

$$856 \quad m_e = m_a + m_b,$$

857 the nonlinear interactions are expressed as the five-equation system

$$858 \quad \frac{dA_a}{dt} = -i\omega_a A_a + i\alpha_{b,c}^a A_b A_c + i\alpha_{b,d}^a A_b A_d + i\alpha_{b,e}^a A_b^* A_e, \quad (5.1)$$

$$859 \quad \frac{dA_b}{dt} = -i\omega_b A_b + i\alpha_{a,c}^b A_a A_c^* + i\alpha_{a,d}^b A_a A_d^* + i\alpha_{a,e}^b A_a^* A_e, \quad (5.2)$$

$$860 \quad \frac{dA_c}{dt} = -i\omega_c A_c + i\alpha_{a,b}^c A_a A_b^*, \quad (5.3)$$

$$861 \quad \frac{dA_d}{dt} = -i\omega_d A_d + i\alpha_{a,b}^d A_a A_b^*, \quad (5.4)$$

$$862 \quad \frac{dA_e}{dt} = -i\omega_e A_e + i\alpha_{a,b}^e A_a A_b, \quad (5.5)$$

863 where the coupling coefficients are as in (3.15) - (3.17).

864 Quintet A’s Triad 1 and Triad 2 are, respectively, Quartet C’s RH-only triad and its Kelvin-
 865 mode triad ($\delta_1 = -0.1085$, $\delta_2 = 0.3606$, identical to Quartet C’s values, table 2). Triad
 866 3 is Quartet E’s Triad 1, RH(4,5)+RH(3,4)+EG(7,9) ($\delta = -3.455$). With all three Rossby
 867 modes at 30m/s and EG(1,1) and EG(7,9) at rest, adding the second gravity mode leaves the

dominant period unchanged. Every Rossby mode's dominant period in the full quintet stays at 4.17d, as in Triad 1 alone, and no novel period is detected (figure 14, bottom row). This differs from Quartet C's 40m/s-driven configuration (§4.3.1), which generates a new 4.55d period absent from either constituent triad. The efficiency variations are correspondingly small but positive ($\Delta\mathcal{E}_a = +2.43\%/+2.19\%/+0.71\%$, $\text{RH}(4,5)/\text{RH}(3,4)/\text{RH}(1,2)$, eq. 4.10), compared with Quartet C's $+15.85\%/-10.77\%$.

For the two gravity modes the effect runs the other way. Both start at rest, and their raw energy swing in the reference triads is too small relative to numerical noise to define a meaningful efficiency variation (§4.1.1). Yet each picks up a distinct spectral peak at the 4.17d period of the RH-only triad (14.2%/31.8% relevance, $\text{EG}(1,1)/\text{EG}(7,9)$). The slow Rossby period thus leaks into the spectrum of an otherwise fast, weakly excited gravity mode, the reverse direction of the coupling documented for Quartet C.

5.2. Quintet B: a third triad on Quartet D's shared sum mode

Quintet B is obtained from Quartet D (§4.3.2) by adding a third triad on its shared sum mode $\text{WG}(7,9)$ (figure 13b). The new member is the eastward gravity mode $\text{EG}(3,9)$ (private), which satisfies the same selection rule as $\text{RH}(3,4)$ and $\text{WG}(3,9)$ in Quartet D ($m_e = m_a - m_b = 3$). Unlike in Quintet A, all three constituent triads share the sum mode a , as both triads of Quartet D already do, so the system is the four-wave system (4.1)-(4.4) extended by one more member:

$$\frac{dA_a}{dt} = -i\omega_a A_a + i\alpha_{b,c}^a A_b A_c + i\alpha_{b,d}^a A_b A_d + i\alpha_{b,e}^a A_b A_e, \quad (5.6)$$

$$\frac{dA_b}{dt} = -i\omega_b A_b + i\alpha_{a,c}^b A_a A_c^* + i\alpha_{a,d}^b A_a A_d^* + i\alpha_{a,e}^b A_a A_e^*, \quad (5.7)$$

$$\frac{dA_c}{dt} = -i\omega_c A_c + i\alpha_{a,b}^c A_a A_b^*, \quad (5.8)$$

$$\frac{dA_d}{dt} = -i\omega_d A_d + i\alpha_{a,b}^d A_a A_b^*, \quad (5.9)$$

$$\frac{dA_e}{dt} = -i\omega_e A_e + i\alpha_{a,b}^e A_a A_b^*, \quad (5.10)$$

with $a = \text{WG}(7,9)$, $b = \text{RH}(4,5)$, $c = \text{RH}(3,4)$, $d = \text{WG}(3,9)$, $e = \text{EG}(3,9)$. The coupling coefficient of $\text{EG}(3,9)$ to the shared edge is not negligible ($|\alpha| = 0.156$), but the coefficient of $\text{RH}(4,5)$ within this third triad ($|\alpha| = 1.9 \times 10^{-3}$) is smaller than in either triad of Quartet D (0.681, 0.036, table 5). $\text{RH}(4,5)$ is therefore an even more passive catalyst in Triad 3 than in Quartet D's Triad 2, and $\text{WG}(7,9)$ and $\text{EG}(3,9)$ exchange energy directly with each other. The small-coupling bound on $\mathcal{E}_b$ (A 4) documented for Triad D applies here as well (Appendix A).

The efficiency variations of $\text{RH}(4,5)$ and $\text{RH}(3,4)$ against Triad 1 are almost unchanged from Quartet D ($+74.0\%/+73.9\%$ here, against $+74.0\%$ for $\text{RH}(4,5)$ in Quartet D, §4.3.2). Consistent with the weak coupling of Triad 3 to $\text{RH}(4,5)$, adding $\text{EG}(3,9)$ barely perturbs the Rossby modes. $\text{WG}(7,9)$ and $\text{WG}(3,9)$ themselves change little against Triad 2 ($\Delta\mathcal{E}_a = +5.3\%/+1.2\%$). A new period, 1.03d, absent from every triad taken alone, appears in $\text{RH}(3,4)$'s spectrum (7.4% relevance) and, far more prominently, in $\text{EG}(3,9)$'s spectrum (25.3% relevance), even though the raw energy variation of $\text{EG}(3,9)$ is almost zero. As for $\text{RH}(3,4)$ in Quartet D (§4.3.2), the characteristic period of Triad 2 (1.07d alone) is transmitted through the shared sum mode $\text{WG}(7,9)$ into modes outside that triad. For $\text{RH}(4,5)$, a member of all three triads, the same peak is not counted as novel, because it falls inside Triad 2's excluded window once all containing triads are considered. In the pairwise comparison with

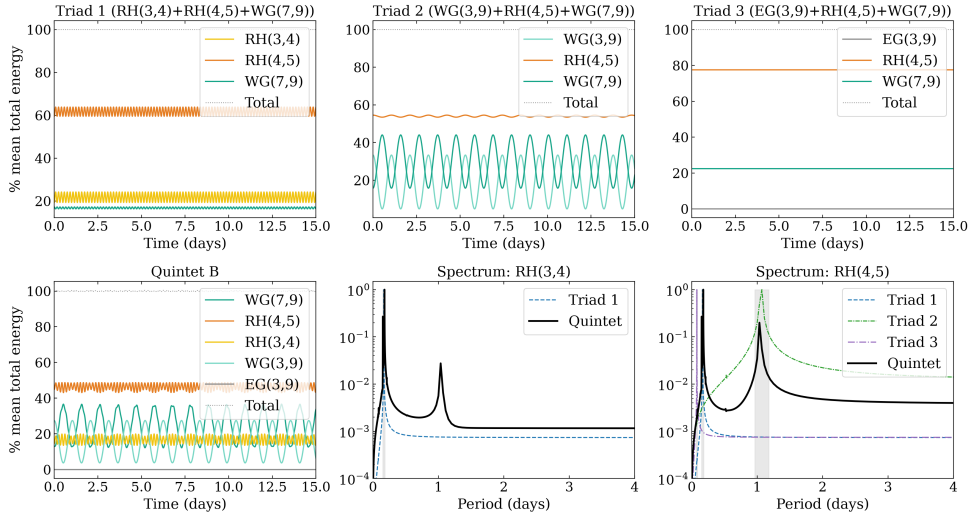


Figure 15: Quintet B (table 5), initial velocities (WG(7,9)/RH(4,5)/RH(3,4)/WG(3,9) at 50/80/50/50m/s, EG(3,9) at rest: (top row) energy evolution of Triad 1 (RH(3,4)+RH(4,5)+WG(7,9)), Triad 2 (WG(3,9)+RH(4,5)+WG(7,9)), and Triad 3 (EG(3,9)+RH(4,5)+WG(7,9)); (bottom row) the full quintet, and RH(3,4)’s/RH(4,5)’s novelty-frequency spectra against whichever sub-triad(s) contain them. Axes as in figure 14.

Triad 1 alone it would count as novel. The two readings, “novel relative to any one triad” and “novel relative to every containing triad”, disagree here, and we adopt the second (§4.1.1).

Triad 3 alone precesses at 12.80 cycles/day, essentially unsynchronized (consistent with its large mismatch $\delta = 6.40$). Within the full quintet its precession frequency falls to 3×10^{-5} cycles/day, but this apparent lock depends on EG(3,9) starting exactly at rest, after which it remains weakly excited. An initial velocity of only 1m/s for EG(3,9) restores a precession frequency of 12.46 cycles/day, close to that of Triad 3 alone, for each initial phase tested. Triad 1 is unaffected by the third triad (5.679 cycles/day in the full quintet against 5.678 in Quartet D). Triad 2, already locked in Quartet D, stays locked (over the same 30-day window, 1.6×10^{-4} cycles/day in Quartet D and 1.5×10^{-4} in the full quintet).

6. Conclusions

The interaction between slow Rossby-Haurwitz and fast inertia-gravity waves is central to the multiscale dynamics of the atmosphere, yet it has mostly been studied in simplified settings, such as the f-plane, where Rossby modes degenerate into zero-frequency vortical modes (Majda & Embid 1998), and the equatorial beta-plane (Raupp *et al.* 2008). Here, we studied it in the full rotating shallow-water equations on the sphere, with the numerically computed Hough harmonics as a basis and low-order models of three to five interacting modes, the building blocks of the complete PDE dynamics.

In single triads, the efficiency of energy transfer depends on the coupling coefficients as much as on the frequency mismatch, so a strongly coupled triad far from resonance can exchange energy faster and more efficiently than a nearly resonant one, in line with results for the barotropic vorticity equation (Lynch 2009; Raphaldini *et al.* 2022). In a triad that mixes the two wave families, a Rossby-Haurwitz mode coupled with two gravity modes acts as a catalyst, setting the period of their exchange while its own energy barely changes.

Connected triads open further exchange mechanisms, among them the precession resonance (Bustamante *et al.* 2014; Raphaldini *et al.* 2022, 2019).

First we investigate a class of all-Rossby quartets, fixing a primary triad RH(4,5)+RH(1,2)+RH(3,4) and letting it interact, via modes RH(4,5) and RH(1,2), with an extra Rossby-Haurwitz mode RH(3,*n*). Two partner modes (i.e. members of the second triad) compete for the energy of the same driving pair, and the most resonant candidate, RH(3,10), receives almost none because its coupling coefficient is small. The RSW version of the quartet of Raphaldini *et al.* (2022) has an efficiency peak without the phase locking found in the barotropic model, so a shared efficiency signature is not sufficient evidence of precession resonance.

Coupling a Rossby triad with the Kelvin mode EG(1,1) (Quartet C, with Rossby modes RH(4,5)+RH(1,2)+RH(3,4)) generates a new, slower period of 4.55d that dominates the spectra of all three Rossby modes and lengthens as the Kelvin mode is driven harder. When a gravity mode is the shared sum mode of both triads (Quartet D), the energy exchange of the Rossby modes grows by 74–149%, while the imprint on their frequencies is weak. Adding the higher-frequency gravity mode EG(11,11) on a gravity-closed edge (Quartet E) raises this exchange by about 25% but creates no new low frequency. In the quintets, a third triad changes the energy of the Rossby modes little, but in Quintet B a new 1.03d period, close to that of Triad 2, appears in two modes that do not belong to that triad.

At realistic atmospheric parameters, fast and slow waves on the sphere do not evolve independently, as they would in the asymptotic limit of Majda & Embid (1998). Inertia-gravity modes change the Rossby-Haurwitz modes both by exchanging energy with them and by generating new low-frequency oscillations, and the topology of the coupled triads decides which effect dominates. A model that filters out gravity waves represents neither effect.

Examples of quartets containing both Rossby-Haurwitz and westward inertia-gravity waves were also investigated. By coupling triads RH(3,4)+RH(4,5)+WG(7,9) and WG(3,9)+RH(4,5)+WG(7,9) (Quartet D), we observe the appearance of a new, weak, low-frequency oscillation (period 1.05d, against the 0.175d dominant exchange) in the Rossby mode RH(3,4) that is absent when the same triads are isolated.

Small-scale inertia-gravity waves remain hard to represent, and even the ERA5 reanalysis of the European Centre for Medium-Range Weather Forecasts underestimates their energy (Ratynski *et al.* 2025). Extending the present analysis to the spectrally truncated shallow-water equations and to the primitive equations is a natural next step. The results also provide a theoretical basis for the statistical evidence of interaction between Rossby and inertia-gravity modes found in normal-mode projections of reanalysis data (Raphaldini *et al.* 2021a,b).

Appendix A. Exact solutions of the resonant three-wave system

The three-wave equations (3.18)-(3.20), written in the resonant-triad short notation

$$\frac{dA_a}{dt} = i\alpha_{b,c}^a A_b^* A_c e^{it\delta}, \quad \frac{dA_b}{dt} = i\alpha_{a,c}^b A_a^* A_c e^{it\delta}, \quad \frac{dA_c}{dt} = i\alpha_{a,b}^c A_a A_b e^{-it\delta},$$

admit exact solutions in terms of *Jacobi elliptic functions* (Craig 1988), whose periodicity characterizes the energy exchange among coupled triads (Bustamante *et al.* 2014; Raphaldini *et al.* 2022) and identifies the *pump mode*. Assuming, without loss of generality, $\alpha_{b,c}^a \leq \alpha_{a,c}^b$, four sign combinations of the coupling coefficients arise (table 6). Case IV, in which no pump mode exists and all three kinetic energies rise and fall together, cannot occur for a resonant triad, since the energy-conservation identity at $\delta = 0$,

$$\alpha_{b,c}^a + \alpha_{a,c}^b - \alpha_{a,b}^c = 0, \quad (\text{A } 1)$$

Table 6: Pump mode of the three-wave system by sign combination of the coupling coefficients.

	Case I	Case II	Case III	Case IV
$\alpha_{b,c}^a$	+ -	-	-	+ -
$\alpha_{a,c}^b$	+ -	+	+	+ -
$\alpha_{a,b}^c$	+ -	-	+	- +
Pump mode	mode c	mode a	mode b	none

forces $\alpha_{a,b}^c$ to share the sign of at least one of the other two coefficients. Equivalently, the pump mode of a resonant triad is the mode with the largest $|\alpha|$.

For Case I, the change of variables $A'_a = -i\sqrt{\alpha_{a,c}^b \alpha_{a,b}^c} A_a$, $A'_b = -i\sqrt{\alpha_{a,b}^c \alpha_{b,c}^a} A_b$, $A'_c = -i\sqrt{\alpha_{b,c}^a \alpha_{a,c}^b} A_c$ reduces the system, in the resonant case, to the complex-triad dynamical system of Kartashova & Bustamante (2011), admitting two Manley-Rowe invariants $I_{a,c} = |A'_a|^2 + |A'_c|^2$ and $I_{b,c} = |A'_b|^2 + |A'_c|^2$. Writing the amplitudes in polar form $A'_j = B'_j e^{i\varphi_j}$ yields a Hamiltonian system in $(B'_a, B'_b, B'_c, \Phi_{a,b}^c)$, with dynamical phase $\Phi_{a,b}^c = \varphi_c - \varphi_a - \varphi_b + \delta t$ and Hamiltonian $\mathcal{H} = B'_a B'_b B'_c \sin \Phi_{a,b}^c$. Defining $\rho = I_{b,c}/I_{a,c}$ and $\nu \in [0, \pi]$ via

$$\cos \nu = \frac{(-2 + 3\rho + 3\rho^2 - 2\rho^3)I_{a,c}^3 - 27\mathcal{H}}{2(1 - \rho + \rho^2)^{3/2}I_{a,c}^3},$$

the moduli B'_j are translations of a common Jacobi elliptic-sine function of modulus $\mu = \cos(\nu/3 + \pi/6)/\cos(\nu/3 - \pi/6)$, with period

$$T = \frac{3^{1/4}\sqrt{2}K(\mu)}{(1 - \rho + \rho^2)^{1/4}\sqrt{\cos(\nu/3 - \pi/6)}\sqrt{I_{a,c}}}, \quad (\text{A } 2)$$

$K(\mu)$ the complete elliptic integral of the first kind. Because mode c alone carries the opposite sign of this common function, mode c is the pump mode of Case I. By an analogous change of variables mapping each case onto Case I, or by a direct comparison of energy derivatives, modes a and b are recovered as the pump modes of Cases II and III respectively (table 6). For non-resonant triads, Harris (2013) provides an analogous solution and period formula, T_{Harris} , which agrees closely with (A 2) near resonance but can differ by more than a day from it for the more strongly mismatched triads considered in §3.3.1 (Dourado 2025).

Choosing the (always attainable) initial phases for which $\mathcal{H} = 0$, the primed system's maximum attainable efficiency reduces, after a lengthy but elementary trigonometric simplification of (A 2) (Dourado 2025), to the simple result

$$\max_{t \geq 0} |B'_a(t)|^2 - \min_{t \geq 0} |B'_a(t)|^2 = \min(I_{a,c}, I_{b,c}),$$

identical for all three (primed) modes. Because the resonant three-wave equations are invariant under the rescaling $A_j \rightarrow \lambda A_j$, $t \rightarrow t/\lambda$, this efficiency does not depend on the amplitude of the initial condition, only on how the initial energy is distributed among the three modes and on the ratios of their coupling coefficients. Undoing the change of variables, for $I_{a,c} \leq I_{b,c}$ (otherwise modes a and b exchange roles) the per-mode efficiency (3.26) of

1007 the original system is

$$1008 \quad \mathcal{E}_a = \frac{1}{E_{total}(0)} \left(|A_a|^2 + \frac{|\alpha_{b,c}^a|}{|\alpha_{a,b}^c|} |A_c|^2 \right), \quad (\text{A } 3)$$

$$1009 \quad \mathcal{E}_b = \frac{1}{E_{total}(0)} \left(\frac{|\alpha_{a,c}^b|}{|\alpha_{b,c}^a|} |A_a|^2 + \frac{|\alpha_{a,c}^b|}{|\alpha_{a,b}^c|} |A_c|^2 \right), \quad (\text{A } 4)$$

$$1010 \quad \mathcal{E}_c = \frac{1}{E_{total}(0)} \left(\frac{|\alpha_{a,b}^c|}{|\alpha_{b,c}^a|} |A_a|^2 + |A_c|^2 \right). \quad (\text{A } 5)$$

1011 For Triad D (§3.3.2), where $|\alpha_{a,c}^b|$ (the Rossby-Haurwitz mode's coupling coefficient) is
 1012 orders of magnitude smaller than $|\alpha_{b,c}^a|$ and $|\alpha_{a,b}^c|$, (A 4) is bounded by a correspondingly
 1013 small factor regardless of the initial amplitudes. This is the analytic counterpart of the
 1014 near-zero RH efficiency observed numerically in figure 4.

1015 Appendix B. Energy conservation of the four-wave truncation

1016 Section 4 truncates the Galerkin expansion (3.1) to a four-wave ansatz satisfying $m_a =$
 1017 $m_b + m_c$ and $m_a = m_b + m_d$, giving the system (4.1)-(4.4). For this truncation the quadratic
 1018 and cubic energies are

$$1019 \quad E^{(2)}(t) = |A_a|^2 + |A_b|^2 + |A_c|^2 + |A_d|^2,$$

$$1020 \quad E^{(3)}(t) = 2\Re e(A_a^* A_b A_c) S_{a,b,c} + 2\Re e(A_a^* A_b A_d) S_{a,b,d},$$

1021 where $S_{a,b,c}$ is the integral in (3.22). Differentiating both, substituting (4.1)-(4.4), and using
 1022 the two constituent triads' energy-conservation identities (3.25) to cancel the terms already
 1023 accounted for by each isolated triad, the derivative of the total energy reduces to

$$1024 \quad \frac{d}{dt} \left(E^{(2)} + E^{(3)} \right) = 2 \left(\alpha_{a,c}^b S_{a,b,d} - \alpha_{a,d}^b S_{a,b,c} \right) |A_a|^2 \Im m(A_c A_d^*)$$

$$1025 \quad + 2 \left(\alpha_{b,d}^a S_{a,b,c} - \alpha_{b,c}^a S_{a,b,d} \right) |A_b|^2 \Im m(A_c A_d^*). \quad (\text{B } 1)$$

1026 This vanishes for every initial condition if, and only if,

$$1027 \quad \frac{\alpha_{a,c}^b}{S_{a,b,c}} = \frac{\alpha_{a,d}^b}{S_{a,b,d}} \quad \text{and} \quad \frac{\alpha_{b,c}^a}{S_{a,b,c}} = \frac{\alpha_{b,d}^a}{S_{a,b,d}}, \quad (\text{B } 2)$$

1028 relations between the coupling coefficients and cubic-energy integrals of the two constituent
 1029 triads that do not hold in general. The four-wave (and, by the same argument, five-wave)
 1030 Galerkin truncation of the RSW equations therefore does not exactly conserve the total
 1031 energy inherited from the full system, in contrast with quadratic-energy models such as
 1032 the barotropic non-divergent equations, for which any truncation is automatically energy-
 1033 conserving (Majda 2003). By (B 1), the imbalance is fourth-order in the modal amplitudes,
 1034 so it remains numerically small for the moderate initial velocities used in §4-§4.3 and only
 1035 becomes significant for amplitudes well beyond realistic atmospheric values (see Dourado
 1036 2025 for a worked numerical example).

1037 Acknowledgements.

1038 **Funding.** M.A.D. was funded by Fundação de Amparo à Pesquisa do Estado de São Paulo (FAPESP process
 1039 2024/01000-9). B.R. was funded by FAPESP (processes 2023/16513-9 and 2024/20546-2). P.S.P. was funded

1040 by FAPESP (processes 2021/06176-0 and 2025/02678-1) and by Conselho Nacional de Desenvolvimento
1041 Científico e Tecnológico (CNPq process 301553/2026-2).

1042 **Declaration of interests.** The authors report no conflict of interest.

1043 **Data availability statement.** No external data were used in this study. The code that computes the Hough
1044 harmonics, integrates the reduced wave systems and generates every figure and table of this paper is openly
1045 available at <https://github.com/CGFD-USP/SWE-nonlinear-interactions>.

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1050 editing), based on his MSc dissertation (Dourado 2025). B.R.: conceptualization, methodology and writing
1051 (original draft, review and editing). A.S.W.T.: methodology (Hough harmonics), software and writing
1052 (review and editing). P.S.P.: conceptualization, methodology, software, formal analysis, investigation, valida-
1053 tion, visualization, writing (original draft), writing (review and editing), supervision, project administration
1054 and funding acquisition.

1055 **Declaration of generative AI use.** During the preparation of this work the authors used Claude (Anthropic)
1056 to review and edit manuscript text, to verify mathematical statements and cross-references against the paper’s
1057 own equations and tables, and to validate descriptions of the numerical implementation against the source
1058 code. After using this tool, the authors reviewed and edited the content as needed and take full responsibility
1059 for the content of the publication.

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