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Interior-fold and endpoint transitions in finite-depth capillary–gravity ship wakes

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Abstract

We study how finite water depth changes the principal stern caustics and wave patterns of linear capillary–gravity waves generated by a steadily moving pressure source on an inviscid fluid of uniform depth. Using the dispersion relation and ray directions relative to the source, we determine the conditions for caustic formation. Let H denote the water depth divided by the capillary length. No regular stern caustics exist for $0 < H \leq \sqrt{3}$. For $H > \sqrt{3}$, analytical results and numerical scans support a sequence of zero, two and one regular principal caustics in each stern half-plane as the source speed increases, under a weak-contribution approximation. The pair emerges at a finite wavenumber. At the long-wave speed, the lower branch reaches the zero-wavenumber endpoint, removing the transverse–divergent boundary while leaving the divergent–fan boundary regular. Local expansions quantify the wavenumber and angular separation near onset and the approach to the long-wave endpoint. Near the critical depth, the squared-speed width of the two-caustic regime decreases quadratically in $H^2 - 3$. Gaussian pressure-source calculations show that the transverse contribution stretches and weakens before its interval closes, while the surviving divergent and fan contributions continue to interfere. Source weighting and relative phases determine which caustics are prominent and where the largest response occurs. These results connect the creation and loss of principal caustics to changes in the visible wake and distinguish caustic directions from maximum-amplitude directions.

1 Introduction

The geometry of a ship wake depends on the wave dispersion relation and the source speed. Classical ship-wave theory describes the pattern through waves that are stationary relative to the moving source and rays determined by their relative group velocity (Thomson, 1887; Havelock, 1908; Keller, 1979). Caustics occur where neighbouring stationary wave contributions merge in the same ray direction and can produce locally enhanced wave amplitudes. The direction of the largest amplitude also depends on how strongly the source excites different wavelengths. Observations and calculations show that this apparent wake angle can differ from a caustic angle and vary with source size and speed (Rabaud and Moisy, 2013; Darmon et al., 2014; Pethiyagoda et al., 2014). We therefore distinguish the caustic directions from the direction of maximum wave amplitude.

In deep water, the combined effects of gravity and surface tension produce a minimum phase speed, which sets the threshold for stationary wave generation by a steadily moving disturbance (Raphaël and de Gennes, 1996). Liang and Chen (2017, 2018) described the transverse, divergent and fan-wave sectors and analysed the critical speed at which the divergent sector disappears. In the ray description used here, this transition corresponds to two caustics meeting at a finite wavenumber. Related studies have examined gravity and capillary cusp directions and the influence of the source spectrum on the visible wake (Moisy and Rabaud, 2014; Gnevyshev and

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Badulin, 2020). These deep-water results provide a reference for following the caustics as the water depth changes.

Finite depth introduces a long-wave limit that also affects the stationary wave pattern. For water depth h , the long-wave speed is $\sqrt{gh}$, where g is gravitational acceleration. The depth Froude number $Fr_h = U/\sqrt{gh}$ compares the source speed U with this speed. In linear pure-gravity wakes, the subcritical pattern contains transverse and divergent waves, whereas the supercritical pattern contains only divergent waves (Pethiyagoda et al., 2015). When surface tension is retained, this long-wave transition and the finite-wavenumber caustic transition can occur at different source speeds. We examine how they jointly determine the number of principal caustics and the wave sectors that these caustics separate.

We consider linear capillary-gravity waves generated by a steadily moving pressure source over an inviscid fluid of uniform depth. From the stationary-wave condition and the relative group velocity, we obtain the source speed required for a caustic at each wavenumber. We retain the conditions for radiation and sternward propagation when determining the allowed solutions. Analytical arguments exclude regular stern caustics for $0 < H \leq \sqrt{3}$, where H is the water depth divided by the capillary length. For $H > \sqrt{3}$, analytical results and numerical scans support a dominant minimum of the required-speed curve. Section 3.2 gives the numerical checks and the weak-contribution approximation used to classify the principal branches.

This minimum determines the speed at which two principal caustics first appear. As the source speed increases further to the long-wave speed, the lower-wavenumber caustic reaches zero wavenumber and is lost, while the other remains. The principal branches therefore follow a sequence of zero, two and one regular caustics in each stern half-plane. Continuation from deep water identifies the lost branch as the transverse-divergent boundary and the surviving branch as the divergent-fan boundary. Local expansions describe the separation of the caustics near onset, the loss of the lower branch, and the narrowing of the two-caustic speed interval near $H = \sqrt{3}$.

Finally, we use the retarded Green response in Eq. (2.7) and Fourier reconstruction to compute finite-time wavefields after a Gaussian pressure source accelerates to constant speed. Selected steady pole contributions isolate long-wave stretching and interference. The calculations show that the transverse contribution weakens before its interval closes, while the surviving divergent and fan contributions continue to produce an oblique wake and interference. Source weights and relative phases affect which caustics are visible and where the largest response occurs. These wave-pattern calculations complement the required-speed analysis used to establish the principal caustic count.

2 Formulation and caustic condition

2.1 Quadratic surface action

We consider small-amplitude waves on an incompressible, inviscid fluid of density ρ and uniform depth h . The flow is irrotational, and the bottom is horizontal and impermeable. Let $\eta(r, t)$ denote the upward free-surface displacement in the laboratory frame, with $r = (X, Y)$. The gravitational acceleration and surface tension are denoted by g and T , respectively. A prescribed pressure distribution moves in the positive X direction at speed U . We write the corresponding load as $J(X - Ut, Y)$, with $J = -p$ for a positive downward pressure p .

The standard variational formulation uses the surface displacement and the velocity potential at the free surface (Luke, 1967; Zakharov, 1968). Keeping terms to quadratic order and eliminating the surface potential gives

$$S_{\text{eff}}^{(2)}[\eta; J] = \int dt \, d^2\mathbf{r} \left[\frac{\rho}{2} (\partial_t \eta) \Lambda^{-1} (\partial_t \eta) - \frac{\rho g}{2} \eta^2 - \frac{T}{2} |\nabla_{\perp} \eta|^2 + J\eta \right]. \quad (2.1)$$

Here $\nabla_{\perp} = (\partial_X, \partial_Y)$ is the horizontal gradient, and $\Delta_{\perp} = \partial_X^2 + \partial_Y^2$ is the horizontal Laplacian. The linear Dirichlet-to-Neumann operator for finite depth is

$$\Lambda = |\nabla_{\perp}| \tanh(h |\nabla_{\perp}|), \quad |\nabla_{\perp}| = (-\Delta_{\perp})^{1/2}. \quad (2.2)$$

This operator maps the surface velocity potential to the vertical velocity at the undisturbed surface (Craig and Sulem, 1993; Lannes, 2013). Its Fourier symbol is $k \tanh(kh)$, where $k = |\mathbf{k}|$ is the horizontal wavenumber magnitude.

Since this symbol vanishes at $k = 0$, we define Λ^{-1} through its Fourier multiplier, $1/[k \tanh(kh)]$ for $k > 0$, and set its multiplier at $k = 0$ to zero. The mean surface level is fixed separately, and the following wave equations apply only to the nonzero Fourier modes. The long-wave endpoint used in the caustic analysis is obtained by taking $k \rightarrow 0^+$.

With vanishing spatial and temporal boundary terms, integration by parts gives

$$S_{\text{eff}}^{(2)}[\eta; J] = \int dt \, d^2\mathbf{r} \left[\frac{1}{2} \eta \Pi \eta + J \eta \right], \quad (2.3)$$

where

$$\Pi = -\rho \partial_t \Lambda^{-1} \partial_t + T \Delta_{\perp} - \rho g. \quad (2.4)$$

Under these boundary conditions, Π is formally self-adjoint. Varying the action with respect to η therefore gives, for the nonzero wave modes,

$$\Pi \eta = -J. \quad (2.5)$$

This equation combines the linear kinematic and dynamic free-surface conditions into a single equation for the surface displacement. In the next subsection, we use its retarded Green function to obtain the causal radiating response to the moving load.

2.2 Retarded response and stationary waves

We obtain the causal response using the retarded Green function G_R , defined by

$$\Pi G_R(\mathbf{r} - \mathbf{r}', t - t') = \delta(t - t') \delta^{(2)}(\mathbf{r} - \mathbf{r}'), \quad (2.6)$$

with $G_R = 0$ for $t < t'$. As in Section 2.1, these identities are understood on the nonzero wave modes. The surface displacement is then

$$\eta(\mathbf{r}, t) = - \int dt' d^2\mathbf{r}' G_R(\mathbf{r} - \mathbf{r}', t - t') J(\mathbf{r}', t'). \quad (2.7)$$

The minus sign follows from $\Pi \eta = -J$. We take the space-time Fourier transform, using the inverse-transform phase $e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)}$ and normalisation $(2\pi)^{-3}$. A related time-domain Fourier response for a moving pressure source in deep water is given by Closa et al. (2010).

For $k > 0$, define

$$\Lambda_k = k \tanh(kh), \quad \Omega_k^2 = \left(gk + \frac{T}{\rho} k^3 \right) \tanh(kh), \quad (2.8)$$

where Ω_k is the positive intrinsic wave frequency. In Fourier space, Π acts on each mode by multiplication with

$$\tilde{\Pi}(\mathbf{k}, \omega) = \frac{\rho}{\Lambda_k} (\omega^2 - \Omega_k^2). \quad (2.9)$$

Its retarded inverse is

$$\tilde{G}_R(k, \omega) = \lim_{\epsilon \rightarrow 0^+} \frac{\Lambda_k}{\rho [(\omega + i\epsilon)^2 - \Omega_k^2]}. \quad (2.10)$$

For $\epsilon > 0$, both frequency poles lie below the real axis, giving a Green function that vanishes before the forcing acts.

For the translating load $J(X - Ut, Y)$, each spatial Fourier component has frequency $\omega = Uk_X$, as in the moving-source wave description of [Lighthill \(1967\)](#). We denote the spatial transform of the load profile by $\tilde{J}(\mathbf{k})$. Evaluating the retarded response at this frequency gives the prescription $Uk_X + i\epsilon$. Equivalently, it follows by switching on the source with a factor $e^{\epsilon t}$ in the remote past and taking $\epsilon \rightarrow 0^+$ at fixed observation time.

Introduce coordinates relative to the source,

$$\mathbf{R} = (X - Ut, Y) = r(\cos \varphi, \sin \varphi). \quad (2.11)$$

The steady response is

$$\eta(\mathbf{R}) = - \lim_{\epsilon \rightarrow 0^+} \int \frac{d^2 \mathbf{k}}{(2\pi)^2} \frac{\Lambda_k \tilde{J}(\mathbf{k})}{\rho [(Uk_X + i\epsilon)^2 - \Omega_k^2]} e^{i\mathbf{k} \cdot \mathbf{R}}. \quad (2.12)$$

Define the steady dispersion denominator

$$\mathcal{D}(\mathbf{k}) = U^2 k_X^2 - \Omega_k^2. \quad (2.13)$$

The condition $\mathcal{D} = 0$ selects waves that are stationary relative to the source. On the positive-frequency branch, it becomes $Uk_X = \Omega_k > 0$. On regular portions of this dispersion curve, the Sokhotski-Plemelj identity gives

$$\lim_{\epsilon \rightarrow 0^+} \frac{1}{(Uk_X + i\epsilon)^2 - \Omega_k^2} = \text{PV} \frac{1}{\mathcal{D}} - i\pi \text{sgn}(k_X) \delta(\mathcal{D}). \quad (2.14)$$

This separates the response into a principal-value contribution and a delta-supported contribution on the dispersion curve. Following the far-field pole-term treatment of [Darmon et al. \(2014\)](#), we retain the delta-supported contribution for the following ray analysis and denote it by η .

To evaluate this contribution, write $\mathbf{k} = k(\cos \theta, \sin \theta)$. Let $\Theta_+ \subset (-\pi/2, \pi/2)$ contain the angles for which positive radial roots exist, and denote the roots of $\mathcal{D}(k, \theta) = 0$ by $k_j(\theta)$. Where these roots are simple, so that $\partial_k \mathcal{D}|_{k=k_j} \neq 0$, the radial integral can be evaluated directly. For a real load, pairing each positive-frequency mode with its negative-wavevector conjugate gives

$$\eta(r, \varphi) = \text{Re} \left\{ \frac{i}{2\pi\rho} \int_{\Theta_+} d\theta \sum_j \frac{k_j \Lambda_{k_j} \tilde{J}(k_j, \theta)}{|\partial_k \mathcal{D}(k, \theta)|_{k=k_j}} \exp[ik_j r \cos(\theta - \varphi)] \right\}, \quad (2.15)$$

where the sum includes the positive roots present at each angle. This on-shell representation retains the dispersion relation and phase structure used to locate caustics. The outgoing radiation condition is imposed when selecting the physical rays. Wave amplitudes and apparent wake angles are computed from the same linear model using the retarded response and Fourier reconstruction described in [Section 5.1](#).

2.3 Dispersion relation and nondimensionalisation

We define the capillary length and the associated velocity scale by

$$\ell_T = \sqrt{\frac{T}{\rho g}}, \quad c_T = \sqrt{g\ell_T}, \quad (2.16)$$

and introduce the dimensionless wavenumber, depth and source speed,

$$q = k\ell_T, \quad H = \frac{h}{\ell_T}, \quad \mathcal{U} = \frac{U}{c_T}. \quad (2.17)$$

For a representative clean-water example, take $T = 0.073 \text{ N m}^{-1}$, $\rho = 1000 \text{ kg m}^{-3}$ and $g = 9.81 \text{ m s}^{-2}$. Eq. (2.16) then gives $\ell_T \approx 2.7 \text{ mm}$ and $c_T \approx 0.16 \text{ m s}^{-1}$. The critical depth $H = \sqrt{3}$ corresponds to about 4.7 mm; for the $H = 3$ cases in Section 5, the depth is about 8.2 mm, and caustic onset and long-wave endpoint loss occur at approximately 0.26 and 0.28 m s^{-1} , respectively.

The ratio of capillary to gravitational restoring contributions is $q^2 = Tk^2/(\rho g)$. Thus, $q \ll 1$ and $q \gg 1$ correspond to gravity-dominated and capillary-dominated waves, respectively. The depth dependence is controlled by $qH = kh$, with $qH \ll 1$ and $qH \gg 1$ corresponding to shallow-water and deep-water behaviour.

The phase speed is $c_p = \Omega_k/k$. Using the dispersion relation in Eq. (2.8), we define the dimensionless squared phase speed as

$$\mathcal{C}(q; H) = \left(\frac{c_p}{c_T}\right)^2 = \left(q + \frac{1}{q}\right) \tanh(qH). \quad (2.18)$$

In the long-wave limit, $\mathcal{C}(q; H) \rightarrow H$ as $q \rightarrow 0^+$, corresponding to $c_p \rightarrow \sqrt{gh}$. For $0 < H \leq \sqrt{3}$, $\mathcal{C}$ increases strictly with q , and its infimum H is approached as $q \rightarrow 0^+$. For $H > \sqrt{3}$, it decreases to a unique global minimum at $q_p(H) \in (0, 1)$ and then increases. Here, q_p denotes the wavenumber of the phase-speed minimum.

To establish this classification, set $x = qH$. Differentiating at fixed H gives

$$\text{sgn}(\partial_q \mathcal{C}) = \text{sgn}[A(x) - H^2], \quad A(x) = x^2 \frac{\sinh(2x) + 2x}{\sinh(2x) - 2x}. \quad (2.19)$$

The function A increases strictly, since

$$\frac{dA}{dx} = \frac{8x \cosh^2 x}{[\sinh(2x) - 2x]^2} \left[(\sinh x - x)^2 + x(2 \sinh x + \tanh x - 3x) \right] > 0. \quad (2.20)$$

The final bracket is positive because $2 \sinh x + \tanh x - 3x$ vanishes at the origin and has derivative $(\cosh x - 1)^2(2 \cosh x + 1)/\cosh^2 x > 0$ for $x > 0$. With $A(0^+) = 3$ and $A(x) \sim x^2$ as $x \rightarrow \infty$, the derivative sign gives the two cases above. For $H > \sqrt{3}$, the unique root $A(q_p H) = H^2$ determines q_p ; equivalently,

$$(1 - q_p^2) \sinh(2q_p H) = 2q_p H(1 + q_p^2). \quad (2.21)$$

Since $A(x) > x^2$, this root satisfies $q_p < 1$.

The depth threshold $H = \sqrt{3}$ therefore separates the two cases: the lowest phase speed is approached in the long-wave limit or attained at a finite wavenumber. In terms of the depth-based Bond number $\text{Bo} = T/(\rho gh^2) = H^{-2}$, this is the classical threshold $\text{Bo} = 1/3$, with a finite-wavenumber phase-speed minimum occurring for $\text{Bo} < 1/3$ (Kim and Akylas, 2005).

Section 3 shows that regular stern caustics also require $H > \sqrt{3}$, together with an additional condition on the source speed.

2.4 Stationary modes and ray geometry

Using the wavevector angle θ introduced in Section 2.2, the steady Doppler condition $Uk \cos \theta = \Omega_k$ gives

$$\cos^2 \theta = P, \quad P = \frac{\mathcal{C}}{U^2}. \quad (2.22)$$

We restrict the following analysis to $q > 0$ and $0 < P < 1$, excluding the angular endpoints. Let $c_g = \partial_k \Omega_k$ be the group speed. We define the ratio of group speed to phase speed by

$$\kappa = \frac{c_g}{c_p} = 1 + \frac{q}{2\mathcal{C}} \partial_q \mathcal{C}, \quad (2.23)$$

where the derivative with respect to q is taken at fixed H .

To determine the ray directions, we apply the stationary-phase condition to the angular integral in Eq. (2.15), following the classical ship-wave treatment of Ursell (1960). On each smooth root branch, the exponential factor has phase

$$\Phi_j(\theta; r, \varphi) = k_j(\theta) r \cos(\theta - \varphi). \quad (2.24)$$

At a fixed observation position (r, φ) , the stationary phase condition is

$$\frac{\partial \Phi_j}{\partial \theta} = r \left[\frac{dk_j}{d\theta} \cos(\theta - \varphi) - k_j \sin(\theta - \varphi) \right] = 0. \quad (2.25)$$

Using $c_p = U \cos \theta$ and $c_g = \kappa c_p$, differentiating the steady Doppler relation along each branch at fixed U and h gives

$$(1 - \kappa) \cos \theta \frac{dk_j}{d\theta} = k_j \sin \theta. \quad (2.26)$$

Combining this relation with the stationary-phase condition eliminates $dk_j/d\theta$. Expanding the trigonometric terms then gives

$$\kappa \sin \theta \cos \theta \cos \varphi + (1 - \kappa \cos^2 \theta) \sin \varphi = 0. \quad (2.27)$$

The retarded prescription in Eq. (2.12) selects the outgoing direction. We first consider upper stern rays, for which $0 < \theta < \pi/2$. Let β be the non-negative angle measured from the stern centreline $-\mathbf{e}_X$, so that $\varphi = \pi - \beta$. Substitution into the preceding equation gives

$$(1 - \kappa \cos^2 \theta) \sin \beta = \kappa \sin \theta \cos \theta \cos \beta. \quad (2.28)$$

Hence, using $\cos^2 \theta = P$,

$$\tan \beta = \frac{\kappa \sin \theta \cos \theta}{1 - \kappa \cos^2 \theta} = \frac{\kappa \sqrt{P(1-P)}}{1 - \kappa P}. \quad (2.29)$$

For the present dispersion relation, $\kappa > 0$. With $0 < P < 1$, the upper stern sector $0 < \beta < \pi/2$ therefore requires

$$1 - \kappa P > 0. \quad (2.30)$$

The stationary dispersion relation is unchanged under $k_Y \mapsto -k_Y$. Reflection across $Y = 0$ therefore gives the lower rays, with the same positive β and $\varphi = \pi + \beta$ modulo 2π . Symmetry of the full wavefield additionally requires a source symmetric under $Y \mapsto -Y$. The boundaries $\beta = 0$ and $\beta = \pi/2$ correspond to the stern centreline and the lateral direction, respectively, and are excluded from the regular interior analysis.

We retain the conditions $q > 0$, $0 < P < 1$ and $1 - \kappa P > 0$ when solving the caustic condition below, since the reduced equation can also admit roots outside this domain. Figure 1 illustrates the ray geometry and the finite-depth dispersion curves. The resulting relation between q and β is used in Section 2.5 to determine the caustics.

2.5 Caustic condition

For a prescribed depth H and source speed $\mathcal{U}$, the caustic wavenumbers satisfy $\partial_q \beta = 0$ on the regular upper stern set defined in Section 2.4. All derivatives with respect to q below are taken at fixed H and $\mathcal{U}$. For a positive function $f(q)$, define its logarithmic derivative by

$$\chi_f = \frac{\partial_q f}{f}. \quad (2.31)$$

We also introduce

$$\Xi_0 = \chi_C + 2\chi_\kappa, \quad \Xi_1 = (2 - \kappa)\chi_C + 2\chi_\kappa. \quad (2.32)$$

Since $0 < \beta < \pi/2$, the condition $\partial_q \beta = 0$ is equivalent to $\partial_q \ln \tan^2 \beta = 0$. Starting from the expression for $\tan \beta$ in Eq. (2.29), we take the logarithm of $\tan^2 \beta$ and differentiate with respect to q , giving

$$\frac{\partial}{\partial q} \ln [\tan^2 \beta(q; H, \mathcal{U})] = \frac{\Xi_0 - P\Xi_1}{(1 - P)(1 - \kappa P)}. \quad (2.33)$$

Equivalently,

$$\partial_q \beta = \frac{\kappa \sqrt{P}}{2\sqrt{1 - P}[1 - \kappa(2 - \kappa)P]} (\Xi_0 - P\Xi_1). \quad (2.34)$$

Here,

$$1 - \kappa(2 - \kappa)P = (1 - \kappa P)^2 + \kappa^2 P(1 - P) > 0. \quad (2.35)$$

Together with $\kappa > 0$ and $0 < P < 1$, this shows that the prefactor is strictly positive. The caustic condition therefore reduces to

$$\mathcal{N}_\beta = \Xi_0 - P\Xi_1 = 0. \quad (2.36)$$

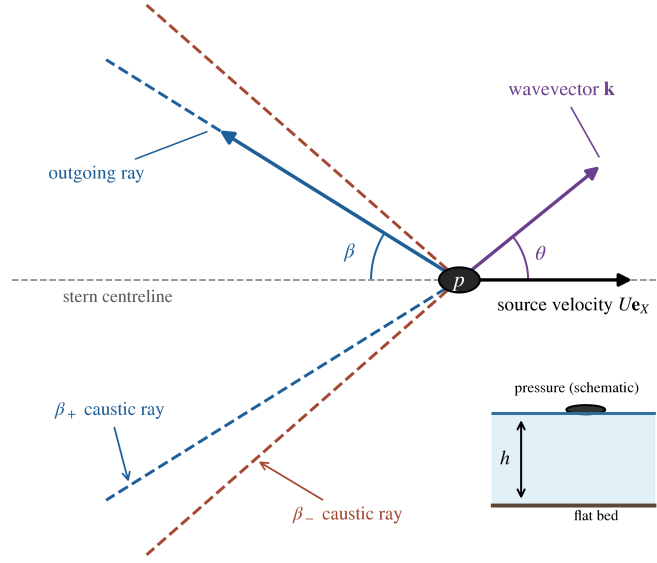
We now use this condition to find the source speed required for a caustic at a given wavenumber q and depth H . Solving for P , provided $\Xi_1 \neq 0$, gives

$$P_C = \frac{\Xi_0}{\Xi_1}. \quad (2.37)$$

Using $P = \mathcal{C}/\mathcal{U}^2$, the required dimensionless source speed is then determined by

$$\mathcal{U}_C^2 = \mathcal{V}(q, H) = \mathcal{C}(q; H) \frac{\Xi_1(q, H)}{\Xi_0(q, H)}, \quad (2.38)$$

(a) Stationary stern-ray geometry



(b) Finite-depth capillary-gravity dispersion

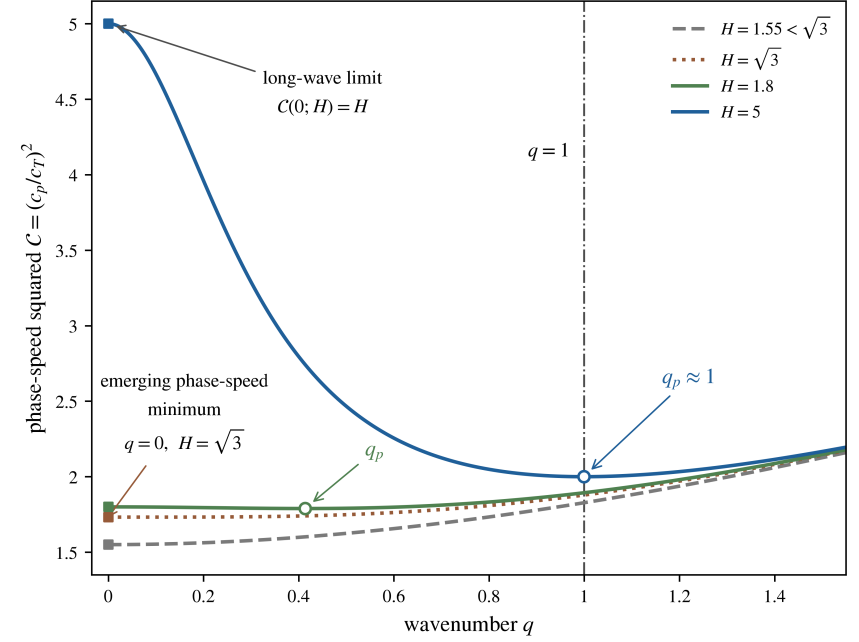


Figure 1: (a) illustrates the wavevector angle θ , measured from the source-motion direction, and the stern ray angle β , measured from the stern centreline. For $H = 5$ and $U \approx 2.07$, the solid blue arrow shows the outgoing ray for the q_+ mode. The dashed lines show the rays corresponding to the two caustic angles $\beta_{\pm}$ and their reflections across the stern centreline. Figure 1(b) shows the dimensionless squared phase speed $\mathcal{C}(q; H)$ for the indicated depths. Filled squares mark the long-wave limits $\mathcal{C}(0; H) = H$, and open circles mark the minima at positive wavenumbers for $H > \sqrt{3}$. The "emerging phase-speed minimum" annotation marks the limiting point $(q, \mathcal{C}) = (0, \sqrt{3})$ at $H = \sqrt{3}$. The vertical line $q = 1$ marks equal capillary and gravitational restoring contributions.

where this ratio is defined. For an imposed speed $\mathcal{U}$, the caustic wavenumbers are the admissible solutions of $\mathcal{V}(q, H) = \mathcal{U}^2$. For $q > 0$, $H > 0$ and $\mathcal{U} > 0$, we therefore define the admissible caustic-speed surface as

$$\mathcal{M}_C = \left\{ (q, H, \mathcal{U}) : \mathcal{U}^2 = \mathcal{V}(q, H) \right\}, \quad (2.39)$$

subject to

$$0 < P_C < 1, \quad 1 - \kappa P_C > 0, \quad \mathcal{V} > 0. \quad (2.40)$$

These conditions retain stationary modes in the stern sector with a real positive required speed. The existence and global structure of this set are analysed in Section 3.

3 Global structure of stern caustics

3.1 Absence of regular stern caustics for $0 < H \leq \sqrt{3}$

A regular stern caustic requires a stationary point of the ray-angle map, where $\partial_q \beta = 0$. For a simple caustic, this corresponds to two stationary wave contributions coalescing (Chester et al., 1957). At fixed H and $\mathcal{U}$, we examine the caustic condition in Eq. (2.36) on the regular stern set defined in Section 2.4.

The required group-speed sign follows from comparison with the critical depth. Set $x = qH$, $u = q^2/(1 + q^2)$ and $a(x) = x/\sinh(2x)$. The definition in Eq. (2.23) gives $\kappa = 1/2 + u + a$. Introduce

$$G = \frac{q}{\sqrt{\mathcal{C}}} \partial_q (\kappa \sqrt{\mathcal{C}}) = q \partial_q \kappa + \kappa (\kappa - 1). \quad (3.1)$$

At fixed x , direct differentiation gives $(\partial_u G)_x = 2(1 + a - u) > 0$. Since $0 < H \leq \sqrt{3}$ corresponds to $u \geq x^2/(x^2 + 3)$, the smallest G in this depth range occurs at $H = \sqrt{3}$. The elementary inequalities

$$0 < \xi \coth \xi - 1 < \frac{\xi^2}{3}, \quad \frac{\sinh \xi}{\xi} > \left(1 + \frac{\xi^2}{12} \right)^2 \quad (\xi > 0) \quad (3.2)$$

give, with $\xi = 2x$,

$$q \partial_q \kappa|_{H=\sqrt{3}} = \frac{6x^2}{(x^2 + 3)^2} + x \frac{da}{dx} > 0. \quad (3.3)$$

At this depth, $\kappa \rightarrow 1$ as $q \rightarrow 0^+$, so $\kappa > 1$ and $G > 0$. The comparison above extends this sign to every $0 < H \leq \sqrt{3}$. Hence, over the complete positive-wavenumber range,

$$\partial_q (\kappa \sqrt{\mathcal{C}}) > 0 \quad (q > 0). \quad (3.4)$$

To connect this property to the caustic condition, the definitions in Eqs. (2.31) and (2.32) give

$$\Xi_0 = 2 \partial_q \ln(\kappa \sqrt{\mathcal{C}}), \quad \Xi_0 - \Xi_1 = \frac{2}{q} (\kappa - 1)^2. \quad (3.5)$$

We can therefore write

$$\begin{aligned}
\mathcal{N}_\beta &= (1 - P)\Xi_0 + P(\Xi_0 - \Xi_1) \\
&= 2(1 - P)\frac{\partial_q(\kappa\sqrt{\mathcal{C}})}{\kappa\sqrt{\mathcal{C}}} + \frac{2P}{q}(\kappa - 1)^2.
\end{aligned} \tag{3.6}$$

The first term is strictly positive because $1 - P > 0$ and the group speed increases with q . The second term contains the squared relative difference between group and phase speeds, since $\kappa = c_g/c_p$, and is nonnegative. Thus $\mathcal{N}_\beta > 0$ throughout the regular radiating stern set.

Eqs. (2.34) and (2.35) show that $\partial_q\beta$ and $\mathcal{N}_\beta$ have the same sign on this set. Hence

$$\partial_q\beta > 0. \tag{3.7}$$

The ray angle therefore increases strictly with wavenumber and has no stationary point. Thus, no regular stern caustics exist for $0 < H \leq \sqrt{3}$.

3.2 Caustic onset for $H > \sqrt{3}$

For $H > \sqrt{3}$, the group speed decreases at small wavenumbers, allowing the two contributions to $\mathcal{N}_\beta$ in Eq. (3.6) to balance. At fixed H , caustic wavenumbers are the intersections of $\mathcal{V}(q, H)$ from Eq. (2.38) with the horizontal level $\mathcal{U}^2$.

On the regular stern set with $\kappa \neq 1$, the condition $\mathcal{N}_\beta = 0$ gives

$$\partial_q(\kappa\sqrt{\mathcal{C}}) = -\frac{\kappa\sqrt{\mathcal{C}}P}{q(1 - P)}(1 - \kappa)^2 < 0. \tag{3.8}$$

At this depth, the dimensionless group speed $\kappa\sqrt{\mathcal{C}}$ has a single minimum at $q_{\text{edge}}(H)$. Hence,

$$\partial_q(\kappa\sqrt{\mathcal{C}}) \begin{cases} < 0, & 0 < q < q_{\text{edge}} \\ = 0, & q = q_{\text{edge}} \\ > 0, & q > q_{\text{edge}} \end{cases} \tag{3.9}$$

Therefore, regular stern caustics can occur only in the interval $0 < q < q_{\text{edge}}(H)$, where $\kappa < 1$. The boundary q_{edge} is the first positive zero of Ξ_0 and lies below the phase-speed minimum $q_p(H)$, as shown in Figure 2(c). On this interval, $\Xi_1 = \Xi_0 - 2(1 - \kappa)^2/q < \Xi_0 < 0$, so $0 < P_C < 1$ and $1 - \kappa P_C > 0$.

The excluded case $\kappa = 1$ occurs at $q = q_p > q_{\text{edge}}$, where $\Xi_0 > 0$. Hence $\mathcal{N}_\beta = (1 - P)\Xi_0 > 0$, and this point cannot be a regular caustic.

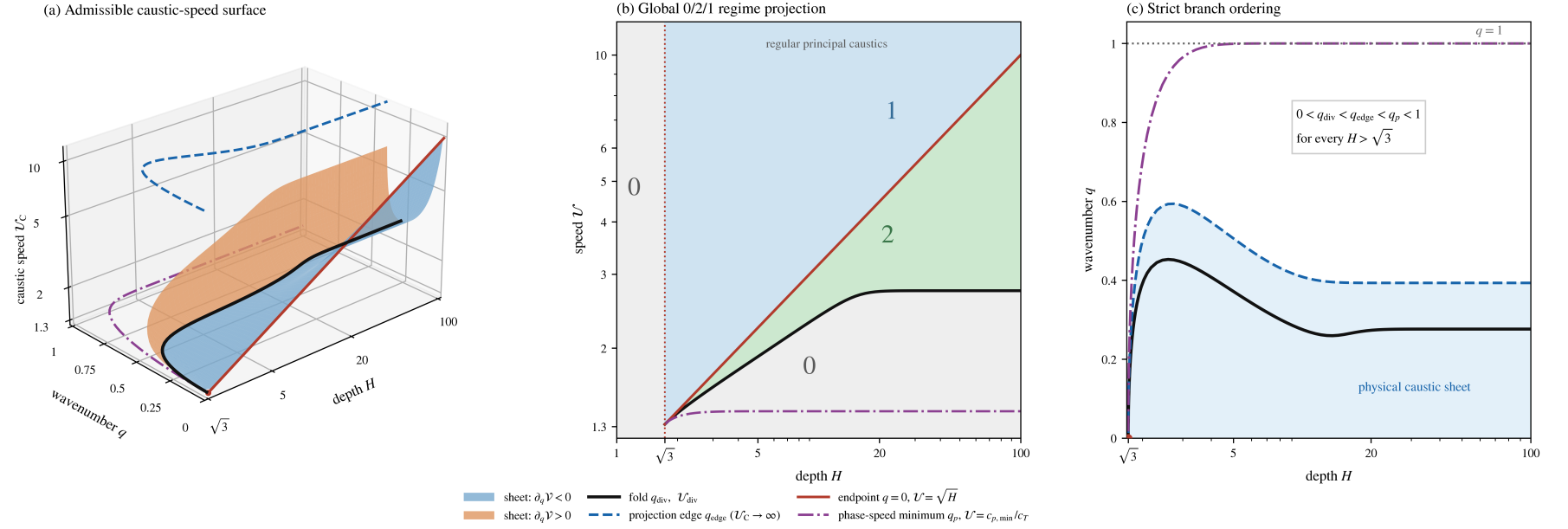


Figure 2: (a) The admissible caustic-speed surface; its blue and orange sheets are the decreasing and increasing portions at fixed depth. (b) The number of regular principal caustics in each stern half-plane under the approximation described below. (c) The ordering $0 < q_{\text{div}} < q_{\text{edge}} < q_p < 1$.

Since $\mathcal{V} = \mathcal{C}/P_C$, its variation depends on both the phase speed and the wavevector direction selected by the caustic condition. Differentiating at fixed H gives

$$\partial_q \mathcal{V} = \frac{\partial_q \mathcal{C}}{P_C} + \mathcal{C} \partial_q \left(\frac{1}{P_C} \right). \quad (3.10)$$

The first term is negative because $\partial_q \mathcal{C} = 2\mathcal{C}(\kappa - 1)/q < 0$ and represents the contribution from decreasing phase speed. The second describes changes in the directional requirement for ray coalescence. At a fixed phase speed, a more oblique wavevector reduces P_C and therefore requires a larger source speed.

At the long-wave end, with $\varepsilon = H^2 - 3 > 0$, the fixed-depth expansions give

$$\mathcal{C} = H \left[1 - \frac{\varepsilon}{3} q^2 + O(q^4) \right], \quad \frac{1}{P_C} = 1 + \frac{\varepsilon}{9} q^2 + O(q^4). \quad (3.11)$$

At this order, the decrease due to phase speed is three times the increase due to obliquity. Their combination yields

$$\mathcal{V}(q, H) = H - \frac{2H(H^2 - 3)}{9} q^2 + O(q^4). \quad (3.12)$$

The curve therefore decreases from its long-wave limit H .

Near the other endpoint, $\Xi_0 \rightarrow 0^-$ while Ξ_1 remains negative by the identity above. Consequently,

$$P_C \rightarrow 0^+, \quad \mathcal{V}(q, H) \rightarrow \infty \quad (q \rightarrow q_{\text{edge}}^-). \quad (3.13)$$

The wavevector direction required for a caustic approaches the direction perpendicular to source motion. The projection of the source velocity onto the wavevector direction must equal the finite phase speed, so the required source speed increases without bound. This kinematic divergence defines the projection edge of the caustic-speed surface. The initial decrease and endpoint divergence guarantee at least one interior minimum.

The principal minimum is set by the balance between phase-speed reduction and the directional requirement for ray coalescence. The capillary-to-gravity ratio increases as q^2 , and the finite-depth factor $\tanh(qH)$ changes smoothly along the same surface-wave dispersion branch. These properties motivate a single dominant balance, which we adopt for the onset description.

Numerical scans at 47 depths, from just above the critical depth ($H^2 - 3 = 1 \times 10^{-4}$) to $H = 1000$, support this description. Scans using 400 and 1040 wavenumber points each detected one stationary point in every checked depth slice. The roots were located by bisection applied to an exact residual for $\partial_q \mathcal{V} = 0$, and the computed second derivative was positive at each root. The root positions were stable under grid refinement and selected higher-precision repetitions.

Figure 3 summarises these checks and illustrates the emergence of two nearby caustic solutions just above onset.

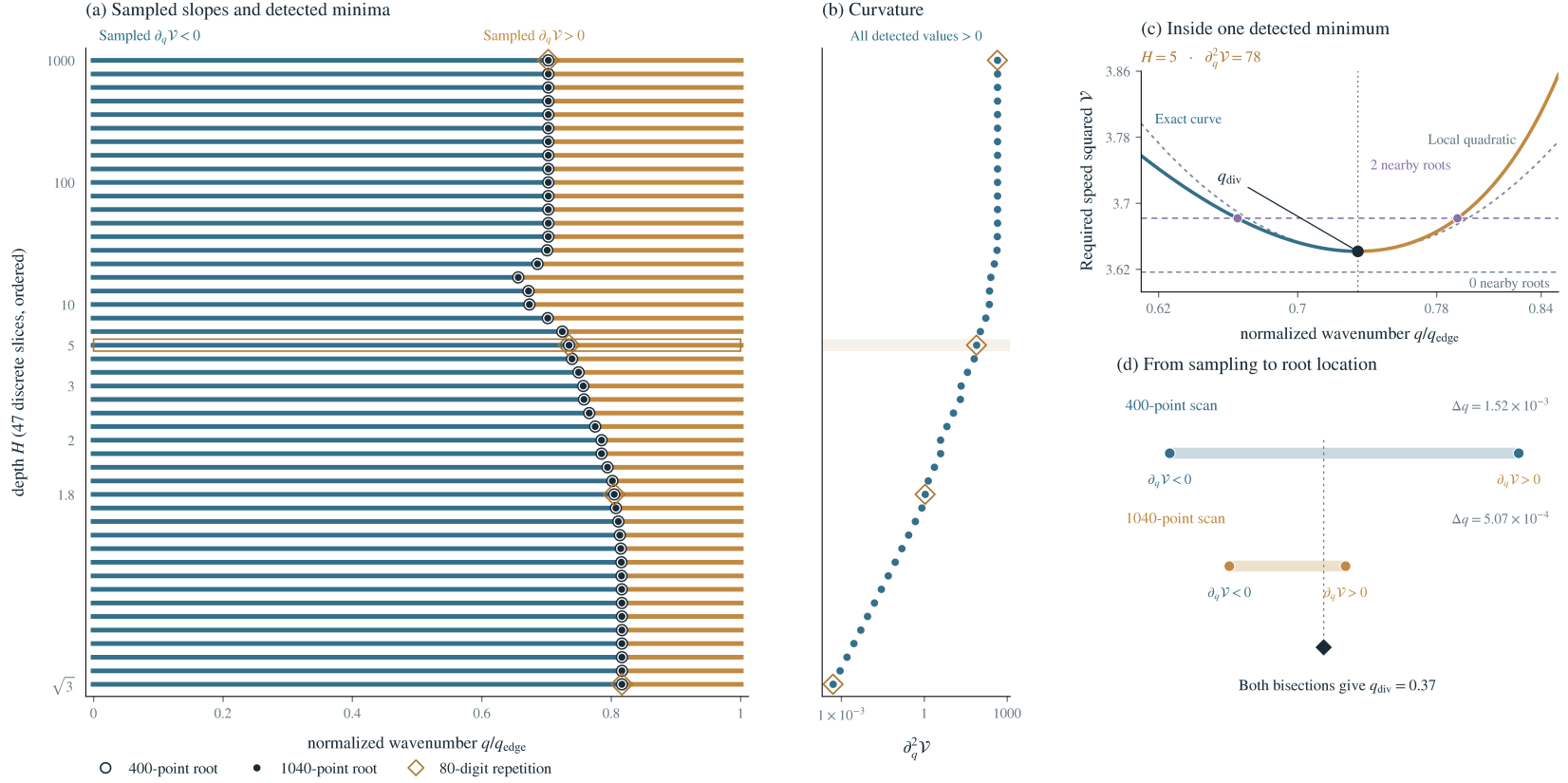


Figure 3: Numerical checks of the caustic onset. (a) Sampled slope signs and detected minima at 47 discrete depths, with results from both grids and selected higher-precision repetitions. (b) Curvature at the detected minima. (c) The local minimum at $H = 5$, its quadratic approximation and nearby speed-level intersections. (d) Initial brackets from the two grids and their common bisection root. The lowest depth row corresponds to $H = \sqrt{3} + 1 \times 10^{-4}$. In panel (d), Δq denotes the width of the initial bisection bracket.

We denote the wavenumber at the principal minimum by $q_{\text{div}}(H)$ and the corresponding onset speed by $\mathcal{U}_{\text{div}}(H)$. The numerically identified minimum satisfies

$$\partial_q \mathcal{V}(q_{\text{div}}, H) = 0, \quad \partial_q^2 \mathcal{V}(q_{\text{div}}, H) > 0, \quad \mathcal{U}_{\text{div}}^2(H) = \mathcal{V}(q_{\text{div}}, H). \quad (3.14)$$

Its positive curvature gives the local form

$$\mathcal{V}(q, H) = \mathcal{U}_{\text{div}}^2(H) + \frac{1}{2} \partial_q^2 \mathcal{V}(q_{\text{div}}, H) (q - q_{\text{div}})^2 + O((q - q_{\text{div}})^3). \quad (3.15)$$

A source speed just below this minimum gives no nearby caustic solution, whereas a speed just above it gives two neighbouring caustic wavenumbers. The two branches meet at an interior fold of the required-speed curve, whose positive curvature gives the nondegeneracy condition.

A geometric caustic can have a weak wave response when its contributing source-weighted spectrum carries little weight. In pressure-generated gravity-wave wakes, the far-field amplitude depends explicitly on the pressure spectrum (Darmon et al., 2014; Pethiyagoda et al., 2021). The source spectrum also influences the visible capillary-gravity wake (Moisy and Rabaud, 2014).

On a regular branch, changing the integration variable in Eq. (2.15) from θ to q gives a band contribution of the form

$$\eta_I(\mathbf{R}) = \text{Re} \left\{ i \int_I \mathcal{A}(q) e^{i\mathbf{k}(q) \cdot \mathbf{R}} dq \right\}, \quad (3.16)$$

Here $\mathcal{A}(q)$ is the complex amplitude density in the q -integral, including the source spectrum, the on-shell weight, the change-of-variable Jacobian and the overall normalisation. The interval I includes the additional roots and the full neighbourhood contributing around each root. For an interval of width w ,

$$|\eta_I| \leq \int_I |\mathcal{A}(q)| dq \leq w \sup_{q \in I} |\mathcal{A}(q)|. \quad (3.17)$$

This estimate includes fully constructive interference. A band with small total spectral weight therefore makes only a small wave contribution, even when it contains several stationary roots.

A caustic changes the phase alignment of wave contributions on the existing dispersion branch. The Green-function pole structure remains unchanged. The regular spectral weight contains no factor $1/\partial_q \mathcal{V}$, so a large local slope of $\mathcal{V}$ does not itself create a singular amplitude factor. Figure 4(a) illustrates a contributing spectral band and different phase arrangements of the same weighted contributions.

The local phase variation provides a further explanation for weak additional caustics. Consider an isolated simple caustic at q_C , away from the onset merger, where $\mathcal{V}(q_C, H) = \mathcal{U}^2$, and let $\beta_C = \beta(q_C; H, \mathcal{U})$. The ray-angle derivative in Eq. (2.34) gives

$$\partial_q^2 \beta(q_C) \propto \partial_q \mathcal{V}(q_C, H), \quad (3.18)$$

with a positive, finite proportionality factor on a regular interior segment. For upper stern rays at fixed $H, \mathcal{U}$ and r , the phase in Eq. (2.24) becomes

$$\Phi(q, \beta) = \mathbf{k}(q) \cdot \mathbf{R} = -\frac{r}{\ell_T} q \cos[\theta(q) + \beta]. \quad (3.19)$$

Along the stationary ray map, the phase derivative taken at fixed observation angle satisfies

$$\partial_q \Phi(q, \beta)|_{\beta=\beta(q; H, \mathcal{U})} = 0. \quad (3.20)$$

Differentiating this identity twice and using $\partial_q \beta(q_C) = 0$ gives

$$\partial_q^3 \Phi(q_C, \beta_C) = -\partial_q \partial_\beta \Phi(q_C, \beta_C) \partial_q^2 \beta(q_C). \quad (3.21)$$

At the caustic direction, $\partial_q \Phi = \partial_q^2 \Phi = 0$. Writing $\delta q = q - q_C$, a small wavenumber displacement gives

$$\Phi(q_C + \delta q, \beta_C) - \Phi(q_C, \beta_C) = \frac{1}{6} \partial_q^3 \Phi(q_C, \beta_C) (\delta q)^3 + O((\delta q)^4). \quad (3.22)$$

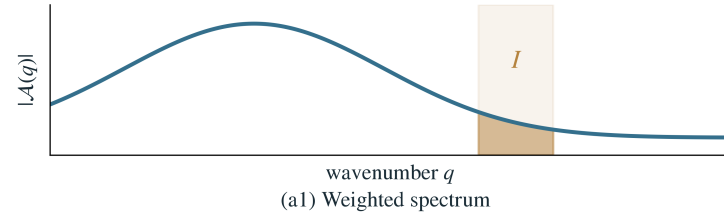
We denote by $\delta q_C > 0$ the characteristic half-width of the locally coherent wavenumber neighbourhood around q_C , so that $|q - q_C| \lesssim \delta q_C$. Taking an order-one phase change as its scale gives

$$\delta q_C \sim \left[\frac{6}{|\partial_q^3 \Phi(q_C, \beta_C)|} \right]^{1/3}. \quad (3.23)$$

We use the far-field cubic approximation across the coherent band, with the band separated from neighbouring caustics and endpoints. For comparable regular geometric factors, a steeper crossing increases the magnitude of the cubic phase coefficient and narrows the coherent band. With a bounded, slowly varying spectral weight, the focused contribution scales with $|\mathcal{A}(q_C)| \delta q_C$, so steep additional crossings can produce weak amplitude ridges. This cubic-phase scaling is consistent with the classical Airy approximation for two coalescing stationary points ([Chester et al., 1957](#); [Berry and Upstill, 1980](#)).

Figure 4(b) schematically compares the additional contributions with the principal ridges on the same amplitude scale. In this schematic, the additional geometric directions remain identifiable, but their response is too weak to form distinct amplitude ridges, leaving the visible principal 0/2/1 caustic pattern unchanged.

(a) Phase focusing



Dispersed phases

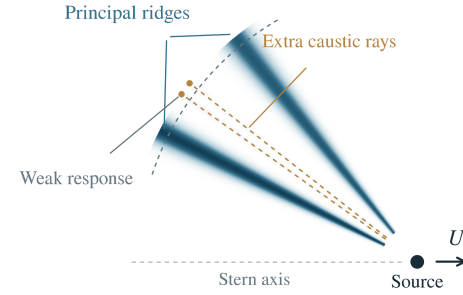


Aligned limit

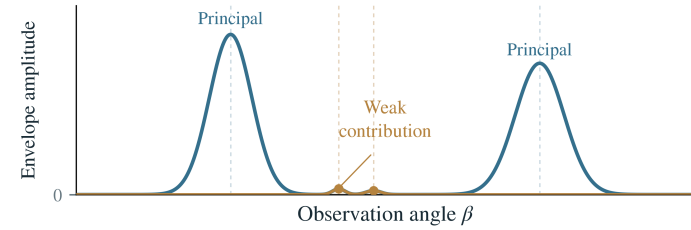


(a2) Phase alignment

(b) Weak additional caustics



(b1) Amplitude ridges (schematic)



(b2) Angular envelope

Figure 4: Phase focusing and weak additional caustics. (a1) A weighted spectrum and contributing band. (a2) Different phase arrangements of contributions with the same individual weights; the aligned arrangement represents fully constructive addition. (b1) Principal amplitude ridges and weak responses along additional caustic directions. (b2) Their angular envelope on a common amplitude scale. The spectra, directions and amplitudes are schematic.

We neglect additional turns whose combined wave contribution is small compared with the principal wake envelope, provided that they do not introduce an appreciably lower competing minimum. Under this weak-contribution approximation, the 0/2/1 classification below describes the principal caustic branches in each stern half-plane and retains the main onset and branch changes.

The principal onset lies within the admissible interval, with the ordering shown in Figure 2(c):

$$0 < q_{\text{div}} < q_{\text{edge}} < q_p < 1. \quad (3.24)$$

3.3 Number of caustics at a prescribed speed

At fixed $H > \sqrt{3}$, we count the principal caustics under the weak-contribution approximation of Section 3.2. In this description, the required-speed curve decreases from its long-wave limit H to $\mathcal{U}_{\text{div}}^2$, then increases without bound. Since $0 < P_C < 1$ at the minimum, the onset speed exceeds the minimum phase speed. The initial decrease also places the minimum below H . Here $c_{p,\text{min}}/c_T = \sqrt{\mathcal{C}(q_p; H)}$ is the minimum dimensionless phase speed at the prescribed depth. Thus,

$$\frac{c_{p,\text{min}}}{c_T} < \mathcal{U}_{\text{div}} < \sqrt{H}. \quad (3.25)$$

Radiating stationary waves are therefore possible for $c_{p,\text{min}}/c_T < \mathcal{U} < \mathcal{U}_{\text{div}}$, although no principal caustic appears in this interval under the adopted approximation. Intersections of the required-speed curve with the horizontal level $\mathcal{U}^2$ give the following number of regular principal caustics in each stern half-plane:

$$\begin{cases} 0 \text{ principal caustics,} & 0 < \mathcal{U} < \mathcal{U}_{\text{div}}, \\ 2 \text{ principal caustics,} & \mathcal{U}_{\text{div}} < \mathcal{U} < \sqrt{H}, \\ 1 \text{ principal caustic,} & \mathcal{U} \geq \sqrt{H}. \end{cases} \quad (3.26)$$

In the two-caustic interval, q_- and q_+ denote the lower- and higher-wavenumber principal solutions, respectively, with

$$0 < q_- < q_{\text{div}} < q_+ < q_{\text{edge}}. \quad (3.27)$$

At $\mathcal{U} = \mathcal{U}_{\text{div}}$, the two principal solutions coincide at q_{div} , giving one degenerate interior caustic.

As the speed approaches $\sqrt{H}$ from below, q_- approaches zero and the lower branch terminates at the long-wave endpoint. The upper branch remains regular, so one principal caustic persists at and above $\mathcal{U} = \sqrt{H}$.

The two changes therefore occur through pair creation at the interior minimum and subsequent loss of the lower branch through the long-wave boundary. Section 4 describes these events locally, and Section 5 identifies the wave sectors separated by the principal branches.

4 Local scaling near caustic transitions

We quantify the local changes in wavenumber and ray angle for the principal caustic branches described in Section 3.

4.1 Separation of the caustic pair

At fixed $H > \sqrt{3}$, define the squared-speed excess above onset by

$$\delta = \mathcal{U}^2 - \mathcal{U}_{\text{div}}^2(H). \quad (4.1)$$

Using the quadratic expansion in Eq. (3.15), the two caustic wavenumbers satisfy

$$q_+ - q_- \sim 2\sqrt{\frac{2\delta}{\partial_q^2 \mathcal{V}}} \quad (\delta \downarrow 0^+), \quad (4.2)$$

where $\partial_q^2 \mathcal{V}$ is evaluated at (q_{div}, H) . Let $\beta_{\pm} = \beta(q_{\pm}; H, \mathcal{U})$. Using Eqs. (2.36) and (2.38), the residual can be written as $\mathcal{N}_{\beta} = \Xi_0(1 - \mathcal{V}/\mathcal{U}^2)$. The quadratic expansion and the smooth positive prefactor in Eq. (2.34) relating $\partial_q \beta$ to this residual give a leading slope profile of order δ between the roots. Integrating over their separation, which is of order $\delta^{1/2}$, gives

$$|\beta_+ - \beta_-| = \left| \int_{q_-}^{q_+} \partial_q \beta dq \right| \propto \delta^{3/2}. \quad (4.3)$$

The angular separation therefore grows more slowly than the wavenumber separation near onset. Figure 5(b,c) shows this local opening at $H = 5$. To display the small angular separation, panel (c) plots the deviations $\beta_{\pm} - \bar{\beta}$ from the pair midpoint $\bar{\beta} = (\beta_- + \beta_+)/2$. The logarithmic insets give fitted separation exponents of 0.5 and 1.5, respectively, consistent with these laws.

4.2 Loss of the lower caustic at $Fr_h = 1$

At fixed $H > \sqrt{3}$, set

$$\mu = H - \mathcal{U}^2. \quad (4.4)$$

The long-wave expansion in Eq. (3.12) gives

$$q_- \sim \frac{3}{\sqrt{2H(H^2 - 3)}} \mu^{1/2} \quad (\mu \downarrow 0^+). \quad (4.5)$$

Combining the long-wave expansions with the ray-angle relation in Eq. (2.29) yields

$$\frac{\pi}{2} - \beta_- \sim \cot \beta_- \sim \frac{4}{3} \sqrt{H^2 - 3} q_- \sim \sqrt{\frac{8}{H}} \mu^{1/2}. \quad (4.6)$$

Thus, the lower-caustic wavenumber and its angular distance from the lateral direction both vanish as $\mu^{1/2}$, while the corresponding wavelength grows as $\mu^{-1/2}$. Figure 5(a) compares the two vanishing quantities with their leading predictions at $H = 5$.

4.3 Narrowing of the two-caustic regime near the critical depth

As $H \downarrow \sqrt{3}$, set $\varepsilon = H^2 - 3$ and consider the joint scaling $q = O(\varepsilon^{1/2})$, with $s = q^2/\varepsilon$. For fixed s in the interior $0 < s < 1/2$, the required-speed curve has the expansion

$$\frac{\mathcal{V}(q, H)}{H} = 1 + \varepsilon^2 \frac{s(54s^2 - 135s + 50)}{225(2s - 1)} + O(\varepsilon^3). \quad (4.7)$$

The leading correction has its minimum at $s = 1/3$, with value $-11/225$. Together with the group-speed and phase-speed expansions, this gives

$$q_{\text{div}}^2 \sim \frac{\varepsilon}{3}, \quad q_{\text{edge}}^2 \sim \frac{\varepsilon}{2}, \quad q_p^2 \sim \frac{5\varepsilon}{6}, \quad (4.8)$$

and

$$H - \mathcal{U}_{\text{div}}^2 \sim \frac{11H}{225} \varepsilon^2. \quad (4.9)$$

Near the critical depth, as $\varepsilon \rightarrow 0^+$, the squared-speed width $H - \mathcal{U}_{\text{div}}^2$ of the two-caustic regime therefore decreases quadratically with ε . Figure 5(d) shows the three characteristic wavenumbers approaching zero and compares them with their leading predictions. Its inset shows the relative squared-speed width $(H - \mathcal{U}_{\text{div}}^2)/H$ and its quadratic decay.

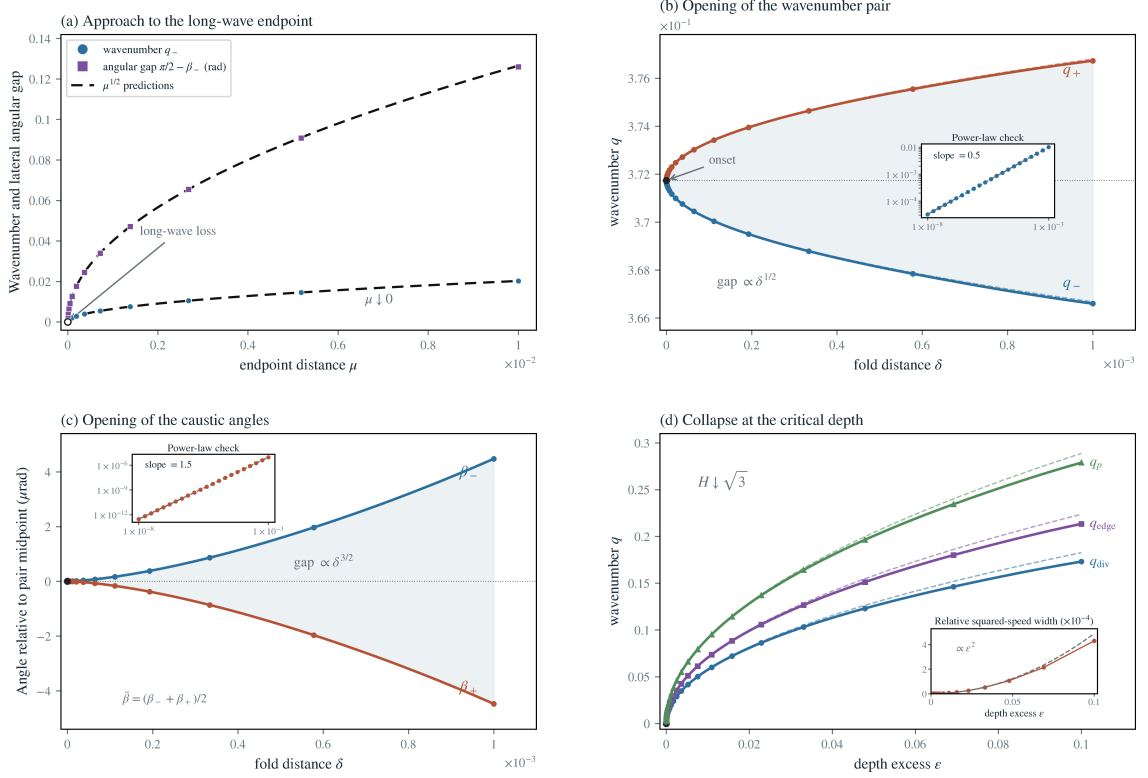


Figure 5: Local caustic transitions, with $H = 5$ in (a–c). (a) Long-wave endpoint approach. (b) Wavenumber splitting. (c) Angular splitting about the pair midpoint. (d) Wavenumber collapse as $H \downarrow \sqrt{3}$; inset: relative squared-speed width. In (d), coloured markers show numerical values, solid lines connect them, and dashed lines show the leading-order predictions of Eqs. (4.8) and (4.9) in the main panel and inset, respectively.

5 Wave patterns across caustic transitions

We examine how the caustic transitions in Sections 3 and 4 change the wave contributions, crest geometry and visible wake. Pressure-forced fields connect the stationary-wave geometry to the resulting surface pattern, while calculations of individual wave contributions resolve the long spatial scales near the critical limits.

5.1 Wave components in a representative wake

Consider the two-caustic wake at $H = 5$ and $\mathcal{U} \approx 2.07$. Following the terminology of Liang and Chen (2018), we label the finite-depth stationary contributions as transverse-connected (T), divergent (D) and fan-connected (F). Within the admissible stationary interval, the T

contribution lies below q_- , the D contribution between q_- and q_+ , and the F contribution above q_+ . Figure 6 shows their ray-angle map and kinematic crestlines. Within $\beta_+ < \beta < \beta_-$, three stationary wave contributions coexist along each observation ray, with different wavelengths and crest orientations.

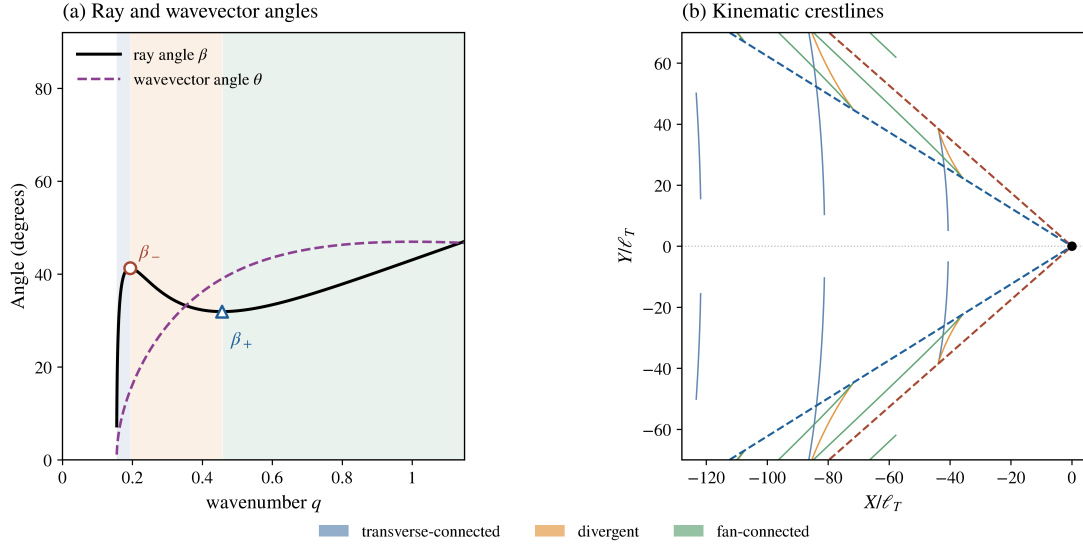


Figure 6: Three stationary wave contributions at $H = 5$ and $\mathcal{U} \approx 2.07$. (a) Ray and wavevector angles. (b) Kinematic crestlines.

We use an axisymmetric Gaussian pressure source for the following calculations. In laboratory coordinates, its pressure head is

$$\frac{p(X, Y, t)}{\rho g} = a_p \exp \left[-\frac{(X - X_s(t))^2 + Y^2}{2\sigma^2} \right], \quad (5.1)$$

where $X_s(t)$ is the source position, a_p is the peak pressure head and σ is the pressure-weighted root-mean-square width in either horizontal direction.

With the spatial Fourier convention of Section 2.2, the pressure spectrum is

$$\tilde{p}(\mathbf{k}, t) = 2\pi\rho g a_p \sigma^2 \exp \left[-\frac{\sigma^2 k^2}{2} - ik_X X_s(t) \right]. \quad (5.2)$$

Source translation changes the Fourier phase, while the Gaussian factor weights the contributing wavenumbers.

For the pressure-forced fields in Figures 7 and 9, we evaluate the finite-time retarded response in Eq. (2.7) together with the initial-data contribution. The calculations start from static equilibrium under the Gaussian load, with zero initial surface velocity. The source accelerates smoothly from rest over $35\ell_T/c_T$, and the fields are evaluated at $t = 120\ell_T/c_T$. The pressure transform is evaluated at the periodic Fourier wavenumbers and divided by the domain area. SciPy's `quad_vec` evaluates the acceleration-stage convolution with maximum-norm absolute and relative tolerances of 10^{-14} and 10^{-11} ; the constant-speed part is integrated analytically. An inverse Fourier sum reconstructs the surface elevation. The fields are plotted in source-centred coordinates, denoted below by X and Y , with $X < 0$ behind the source.

The periodic domains for Figures 7 and 9 are $650\ell_T \times 400\ell_T$ and $1800\ell_T \times 400\ell_T$, using 512×320 and 1792×384 Fourier modes, respectively. For the two-caustic case in Figure 7(c) and the $\mathcal{U} = 7$ case in Figure 9(b), separate mode-count and domain-refinement checks changed the

surface fields by at most 2.2×10^{-4} and 2.8×10^{-5} in relative L^2 norm, respectively. The relative RMS differences were below 3.4×10^{-6} , and the sampled peak directions were unchanged.

The selected wave contributions in Figure 8 and Section 5.4, together with the interference diagnostic in Section 5.3, are evaluated from the steady pole representation in Eq. (2.15). These calculations retain the same Gaussian form, with the spectral intervals, widths and load normalisations stated for each comparison.

5.2 Wave-pattern transitions at fixed depth

At fixed depth $H = 3$, we examine how caustic onset and long-wave endpoint loss change the surface pattern. The Gaussian source has fixed width $\sigma \approx 1.4\ell_T$ and peak pressure head $a_p = 0.05\ell_T$. Figure 7 compares five speeds across these transitions.

For surface elevation in source-centred coordinates, we define the angular RMS envelope as

$$\mathcal{E}(\beta) = \left\langle \frac{\eta^2(-r \cos \beta, r \sin \beta) + \eta^2(-r \cos \beta, -r \sin \beta)}{2} \right\rangle_r^{1/2}. \quad (5.3)$$

The average covers $25 \leq r/\ell_T \leq 125$. This quantity measures the strength of the surface response along each observation direction. Figures 7–9 use the same fixed a_p to normalise elevation and RMS.

Below the onset speed $\mathcal{U}_{\text{div}} \approx 1.62$, the calculated wake has curved crests and a broad RMS distribution. At onset, the response is more concentrated in an oblique sector, with a higher RMS peak: $\max_{\beta} \mathcal{E}/a_p \approx 0.35$, compared with 0.27 below onset in Figure 7(a,b).

At the representative speed $\mathcal{U} = (\mathcal{U}_{\text{div}} + \sqrt{H})/2 \approx 1.67$, the three stationary contributions described in Section 5.1 overlap between $\beta_+ \approx 46.8^\circ$ and $\beta_- \approx 54.5^\circ$. Figure 7(c) shows their combined surface pattern. The RMS maximum at 48.9° lies within this interval and identifies the strongest wake direction, close to the divergent–fan caustic β_+ . The larger RMS at β_+ than at β_- makes the divergent–fan boundary more prominent in this surface pattern. Finite-distance focusing and interference shift the amplitude maximum relative to the caustic.

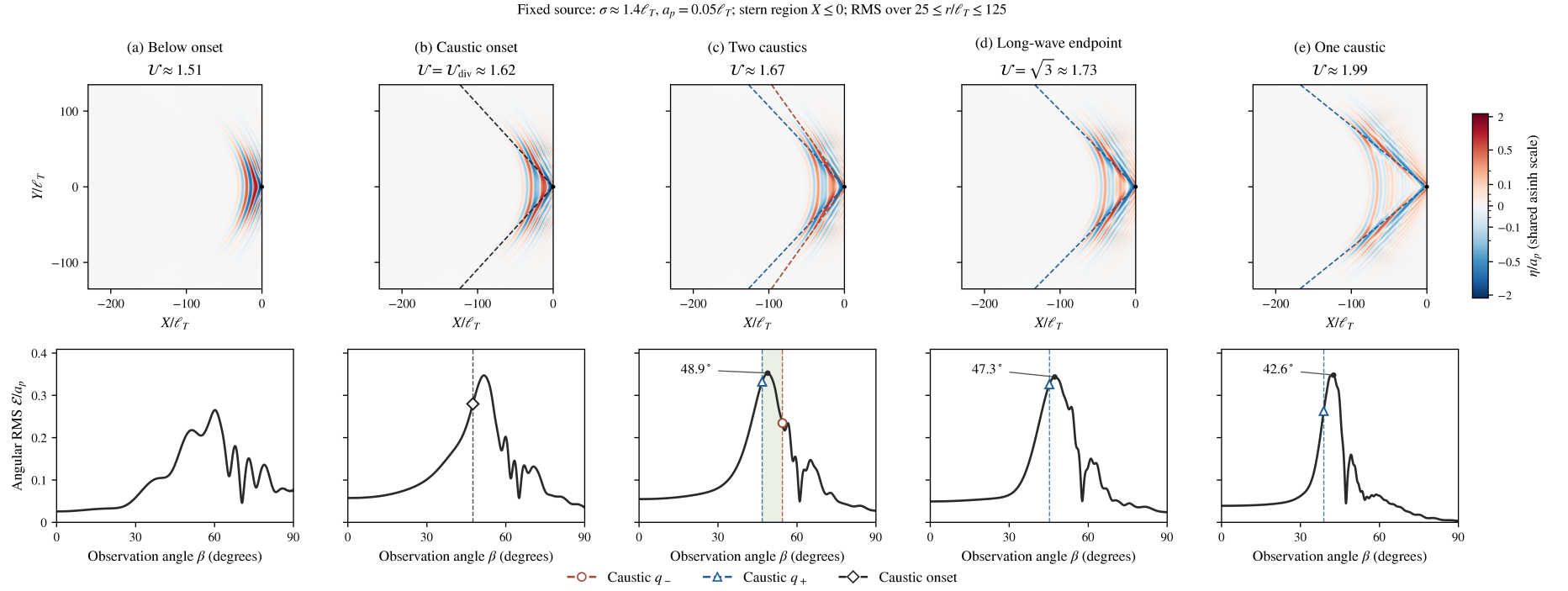


Figure 7: Stern wake transitions at $H = 3$: (a) below onset, (b) onset, (c) two caustics, (d) long-wave endpoint and (e) one caustic. The upper row shows surface elevation and the lower row angular RMS. Dashed rays mark caustics.

As $\mathcal{U}$ approaches $\sqrt{H}$ from below at fixed $H = 3$, the transverse-connected crests become more widely spaced and approach the direction perpendicular to the source track. We use $\mu = H - \mathcal{U}^2$ as defined in Eq. (4.4). To isolate this contribution, denoted by η_T , we evaluate the steady pole integral over its admissible interval below q_- , including both signs of k_Y . Its envelope $\mathcal{E}_T$ uses the same RMS definition.

The transverse pattern loses strength as its admissible spectral interval contracts towards $q = 0$. The fading surface bands and smaller centreline extrema in Figure 8(a–d) indicate a reduction in the local displacement on the common pressure-head scale: decreasing μ from 0.04 to 2.5×10^{-3} reduces $\max_{\beta} \mathcal{E}_T/a_p$ from approximately 6.6×10^{-2} to 1.7×10^{-2} . Meanwhile, the RMS maximum shifts towards the source track even as the q_- caustic guide approaches the lateral direction, as shown in Figure 8(e). The long-wave phase varies mainly with distance behind the source. Nearly lateral rays stay close to $X = 0$, where the transverse pole field vanishes, so their response is suppressed within the fixed RMS annulus. The strongest local response can therefore move away from the geometric caustic as the transverse pattern fades.

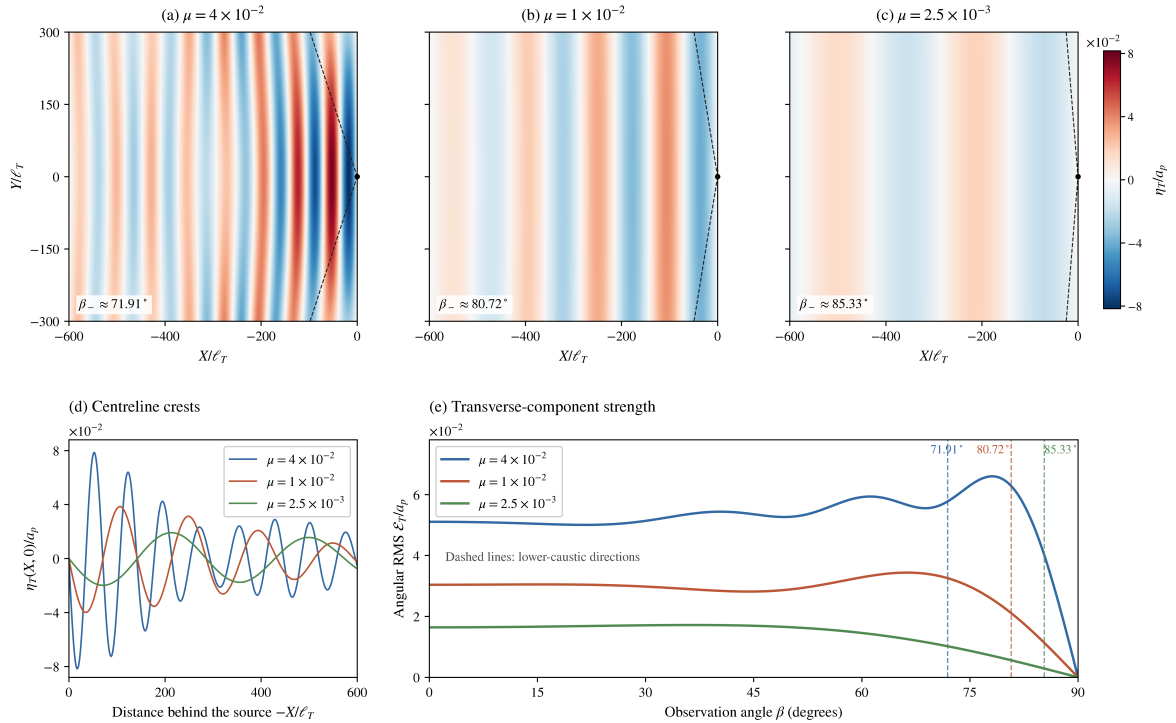


Figure 8: Long-wave loss at $H = 3$. (a–c) Transverse-component elevation at decreasing μ . (d) Centreline profiles. (e) Angular RMS. Dashed lines mark the q_- caustic directions.

At $Fr_h = 1$, corresponding to $\mathcal{U} = \sqrt{H}$, the transverse-connected interval collapses at the long-wave endpoint described in Section 4.2. At and above this threshold, Figure 7(d,e) retains oblique divergent and fan-connected crests. The RMS maximum shifts from 47.3° to 42.6° , while $\max_{\beta} \mathcal{E}/a_p$ remains approximately 0.34–0.35. A prominent oblique wake therefore persists after the transverse contribution is lost, with its strongest direction turning towards the source track.

5.3 The high-speed Mach-like wake

At fixed $H = 3$, increasing the source speed to $\mathcal{U} = 4, 7$ and 10 turns the oblique crests towards the source track, forming a narrower Mach-like wake in Figure 9(a–c). The Gaussian source is the same as in Section 5.2. The RMS maximum moves from 19.8° to 10.6° and 7.05° , while

$\max_{\beta} \mathcal{E}/a_p$ falls from 0.18 to 9.4×10^{-2} and 6.3×10^{-2} in Figure 9(d). At fixed contributing wavenumber, the radiation pole weight decreases approximately as $\mathcal{U}^{-1}$, contributing to the weaker response.

The narrowing accompanies $q_+ \rightarrow q_{\text{edge}}$. In this limit, the ray relation in Eq. (2.29) gives

$$\beta_+ \sim \frac{c_{g,\min}}{c_T \mathcal{U}} \approx \frac{1.26}{\mathcal{U}}, \quad (5.4)$$

with angles in radians and $c_{g,\min}$ the minimum intrinsic group speed at $H = 3$. This is the group-speed selection mechanism described in deep water by Moisy and Rabaud (2014). The wavelength normal to the caustic-selected crests tends to $2\pi\ell_T/q_{\text{edge}} \approx 10.67\ell_T$, while their spacing along the caustic ray increases from about $194\ell_T$ to $516\ell_T$ across these speeds.

Compared with Figure 7(e), the RMS curves show much deeper minima between β_+ and the long-wave Mach direction $\beta_M = \arcsin(\sqrt{H}/\mathcal{U})$, marked by triangles and squares, respectively. The minima occur near 21.45° , 12.06° and 8.42° . To interpret these features, we use twice the steady pole term as a local stern-response approximation and split its spectrum at q_+ into divergent-side and fan-side contributions η_D and η_F . In this decomposition, the contributions nearly cancel before the total displacement is squared.

For fixed contributing wavenumber q , the high-speed, small-angle phase in the upper wake is

$$\Phi(q, \beta) \simeq \frac{r}{\ell_T \mathcal{U}} q \left[\mathcal{U}\beta - \sqrt{\mathcal{C}(q; H)} \right]. \quad (5.5)$$

The $r/(\ell_T \mathcal{U})$ factor reduces phase variation across the fixed radial interval. When the divergent and fan contributions are nearly opposite, their cancellation can persist over much of the sampled ray, producing a low-amplitude direction within the annulus. Finite-distance phase sampling also shifts the RMS maximum relative to the geometric caustic: at $\mathcal{U} = 10$, the maximum lies slightly inside $\beta_+ \approx 7.26^\circ$.

The RMS at β_M decreases more slowly, from approximately $5.5 \times 10^{-2}a_p$ to $4.4 \times 10^{-2}a_p$. Let D_M and F_M be the RMS values of η_D and η_F along this direction, using the same radial average, and write $\mathcal{E}_M = \mathcal{E}(\beta_M)$. The local response approximation gives

$$\mathcal{E}_M^2 \simeq D_M^2 + F_M^2 + 2\gamma_M D_M F_M, \quad \gamma_M = \frac{\langle \eta_D \eta_F \rangle_r}{D_M F_M}. \quad (5.6)$$

The normalised cross term is $\gamma_M \approx -0.28$, 3.5×10^{-2} and 0.61 for $\mathcal{U} = 4, 7$ and 10 , respectively. Thus destructive overlap becomes constructive and partly offsets the decrease in $D_M^2 + F_M^2$. This explains the slower RMS decrease at the Mach direction within the observation window.

5.4 Wave-pattern transitions near the critical depth

Near $H = \sqrt{3}$, the speed interval between caustic onset and long-wave endpoint loss narrows. To resolve the changes across $Fr_h = 1$, we introduce a rescaled speed offset v , motivated by the squared-speed width in Eq. (4.9):

$$\varepsilon = H^2 - 3 \rightarrow 0^+, \quad Fr_h^2 = \frac{\mathcal{U}^2}{H} = 1 + \varepsilon^2 v. \quad (5.7)$$

The calculations use $\varepsilon = 2.5 \times 10^{-2}$ and $v = -2.5 \times 10^{-2}$, 0 and 2.5×10^{-2} , placing the cases below, at and above $Fr_h = 1$, respectively. The negative case lies within the leading two-caustic range $-11/225 < v < 0$ and retains both caustics. At $v = 0$, the q_- branch reaches $q = 0$, while the q_+ caustic persists across the endpoint.

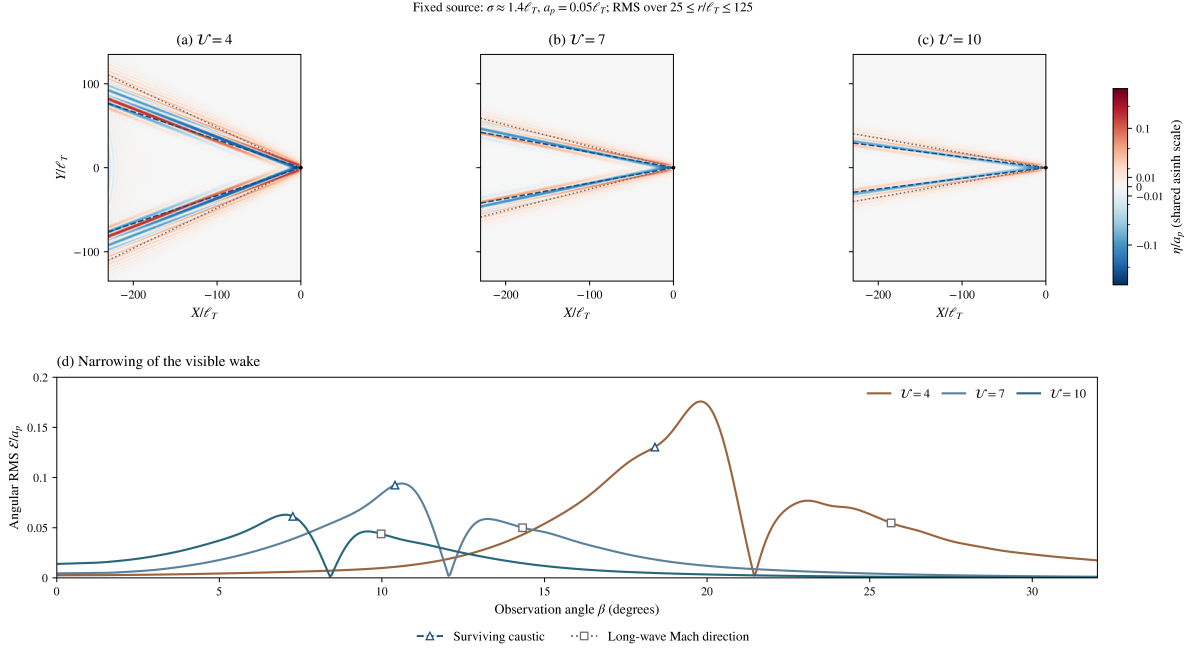


Figure 9: High-speed wakes at $H = 3$. (a–c) Stern surface elevation. (d) Angular RMS.

The critical wave pattern extends farther laterally than along the source track. With $s = q^2/\varepsilon$ and $a(s) = v + s/3 - s^2/5$, the long-wave dispersion relation and Doppler condition give $k_X \ell_T \sim \varepsilon^{1/2} \sqrt{s}$ and $k_Y \ell_T \sim \varepsilon^{3/2} \sqrt{s a(s)}$ where $a(s) > 0$. We therefore compare the fields in the coordinates

$$X_* = \varepsilon^{1/2} \frac{X}{\ell_T}, \quad Y_* = \varepsilon^{3/2} \frac{Y}{\ell_T}. \quad (5.8)$$

The corresponding longitudinal and lateral length scales are $\ell_T \varepsilon^{-1/2}$ and $\ell_T \varepsilon^{-3/2}$. The dimensionless integrated pressure-head load is $Q = 2\pi(a_p/\ell_T)(\sigma/\ell_T)^2$. We set $Q = 1$ and $\sigma/\ell_T = 1$. The fields are calculated from the finite-depth pole representation in Eq. (2.15), retaining the positive-frequency modes with $k_Y > 0$ and their negative-wavevector conjugates. We denote this directional contribution by η_+ .

In the scaled coordinates, the oblique ray $X_* = -BY_*$, with $B = 0.47$, passes through the overlapping wave regions in all three cases, as shown in Figure 10. Below the endpoint it lies between the two caustics, where the transverse, divergent and fan contributions coexist; at and above the endpoint it samples the surviving divergent–fan overlap. Holding this direction fixed allows their interference to be compared across the transition. The local maps in Figure 11 use the same scaled lateral coordinate Y_* and the offset $Z_{\text{off}} = X_* + BY_*$. The white dashed line $Z_{\text{off}} = 0$ follows the green ray in Figure 10.

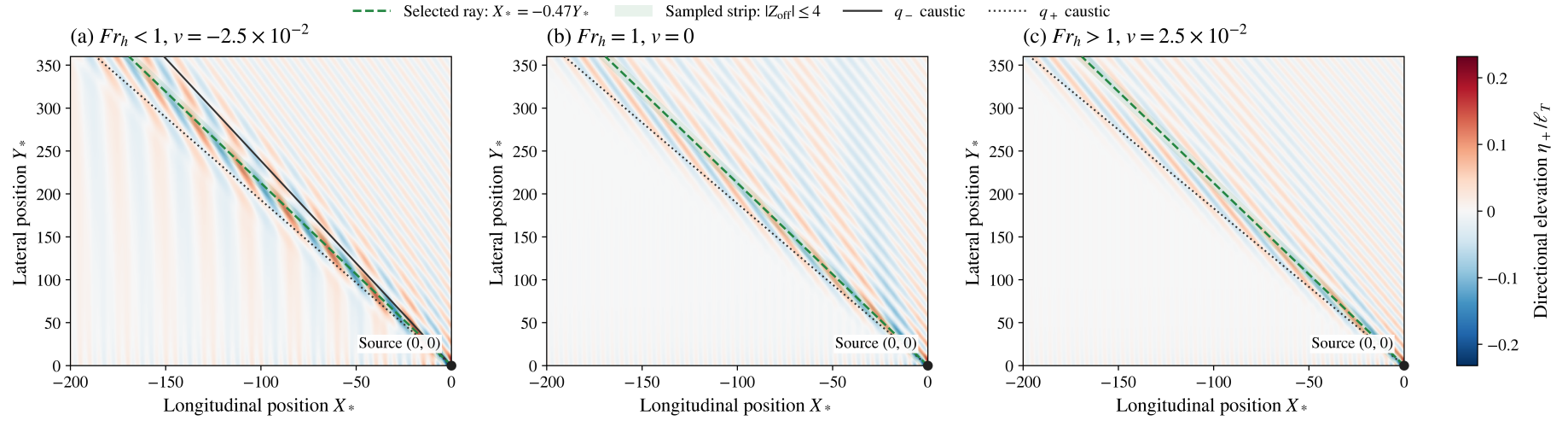


Figure 10: Sampling ray at $\varepsilon = 2.5 \times 10^{-2}$. Columns: $v = -2.5 \times 10^{-2}$, 0 and 2.5×10^{-2} . Green dashed lines mark $X_* = -0.47Y_*$; faint green shading indicates the strip $|Z_{\text{off}}| \leq 4$. Solid and dotted guides mark the q_- and q_+ caustics.

Below the endpoint, three contributions produce a modulation with several phase differences, as shown in Figure 11(a,d,g). As $v \uparrow 0$, the transverse interval contracts towards $s = 0$, giving an $O(\sqrt{-v})$ upper bound on its complex amplitude in the leading long-wave model. At the endpoint, the transverse contribution is absent, while the surviving divergent and fan amplitudes become comparable and produce deeper interference minima in Figure 11(b,e,h). Both contributions persist above the endpoint, and the spacing between successive interference minima decreases in Figure 11(c,f,i).

Let A_j , with $j = T, D, F$, denote the dimensionless complex pole contribution of each band at an observation point, so that $\eta_{+,j}/\ell_T = \text{Re } A_j$. The solid and dashed curves in Figure 11(g–i) show the coherent magnitude $|\sum_j A_j|$ and the phase-blind reference $(\sum_j |A_j|^2)^{1/2}$, respectively. The reference omits interference between the bands. The coherent curve lies above it where the net interference is constructive and below it where it is destructive. Their comparison measures interference in the surface-displacement amplitude.

For the two surviving contributions, let $K_j = \partial_{Y_*} \Phi_j$ denote their finite-depth stationary-phase gradients along the ray. The approximate spacing between interference minima is

$$\Delta Y_* \simeq \frac{2\pi}{|K_D - K_F|}. \quad (5.9)$$

The successive minima farther along the ray in Figure 11(h,i) are separated by approximately 470 at $v = 0$ and 325 at $v = 2.5 \times 10^{-2}$, consistent with the predictions 469.5 and 323.7. The transition therefore changes both the contributing spectral intervals and their relative phases.

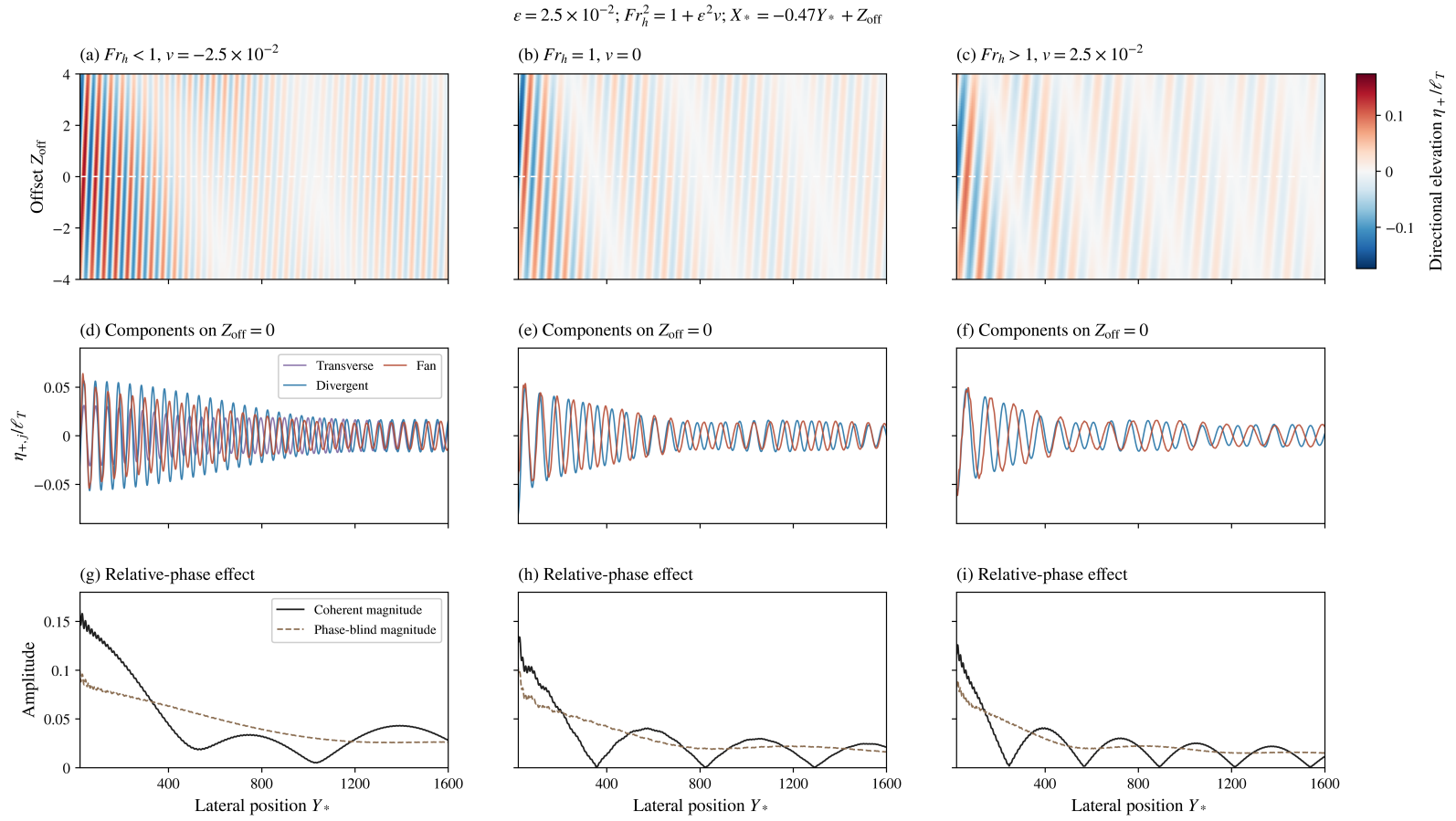


Figure 11: Near-critical transition at $\varepsilon = 2.5 \times 10^{-2}$. Columns: $v = -2.5 \times 10^{-2}$, 0 and 2.5×10^{-2} . (a–c) Directional elevation in the sampled strip. (d–f) Separate transverse, divergent and fan elevations along $Z_{\text{off}} = 0$. (g–i) Coherent and phase-blind magnitudes.

6 Discussion

6.1 Physical meaning of the finite-depth caustic transitions

Finite depth separates two mechanisms: for $H > \sqrt{3}$, the principal pair is created at an interior wavenumber as the speed exceeds $\mathcal{U}_{\text{div}}$, whereas its lower branch is lost through the long-wave endpoint at $\mathcal{U} = \sqrt{H}$. The latter removes the transverse–divergent boundary as it approaches the lateral direction, while the divergent–fan boundary remains regular.

The speed ordering in Eq. (3.25) distinguishes stationary radiation from caustic formation. Stationary waves can exist below $\mathcal{U}_{\text{div}}$, and the radiating wake persists after the lower caustic is lost. The two caustic thresholds describe changes in the organisation of existing wave contributions.

The ordering $q_{\text{edge}} < q_p < 1$ also clarifies the role of surface tension. All regular stern caustics select wavenumbers for which the gravitational restoring contribution exceeds the capillary contribution. Surface tension influences their positions through the variation of phase and group speeds with wavenumber. This restriction applies to the caustic-selected waves; it does not restrict the complete wake spectrum to $q < 1$.

As $H \downarrow \sqrt{3}$, the onset and endpoint-loss speeds approach one another. Eqs. (4.8) and (4.9) show that the squared-speed width of the two-caustic regime decreases as $(H^2 - 3)^2$, while its characteristic wavenumbers approach zero. The two transitions therefore become close in parameter space while retaining their different mechanisms.

6.2 Caustic geometry and the visible wave pattern

A caustic direction locates the coalescence of stationary wave contributions. The direction of maximum amplitude additionally depends on their source weights, relative phases and propagation distance. The fixed-depth calculations in Section 5.2 illustrate this distinction: the two caustics have unequal prominence, and the angular RMS maximum need not coincide with either boundary. Near onset, the angular separation grows as $\delta^{3/2}$, whereas the wavenumber separation grows as $\delta^{1/2}$, with δ defined in Eq. (4.1). Newly formed caustics can therefore remain close in direction even when their wavenumbers have begun to separate.

Near the long-wave endpoint, the transverse contribution's strongest RMS response moves towards the source track within the fixed observation window, even as its caustic approaches the lateral direction. The divergent and fan contributions retain a prominent oblique wake after the transverse interval closes. Loss of the geometric boundary therefore changes the wave composition without requiring the total response to become weak.

At high speed, the angle of the surviving divergent–fan caustic decreases, and the calculated wake becomes Mach-like. This narrowing is a continuous limit of the surviving branch. The angular minima in Section 5.3 show that destructive interference can suppress the response between geometric directions while both stationary contributions remain present. The visible wake therefore contains amplitude variations that cannot be inferred from the caustic directions alone.

The coordinate scaling in Eq. (5.8) shows that the lateral length scale, $\ell_T \varepsilon^{-3/2}$, grows faster than the longitudinal scale, $\ell_T \varepsilon^{-1/2}$, where $\varepsilon = H^2 - 3$. Endpoint loss changes the modulation of this extended pattern. The surviving divergent and fan contributions continue to interfere, with their relative phase advance setting the beat spacing and their amplitudes and phases determining local enhancement and suppression.

6.3 Scope of the analysis

The results concern linear waves on an inviscid fluid of uniform depth. The numerical scans support the dominant minimum of the required-speed curve. Section 3 establishes the principal caustic classification under the weak-contribution approximation of Section 3.2. The pressure-forced calculations in Section 5 show how these caustic transitions affect the wave patterns for the Gaussian sources and observation ranges considered here.

The finite-time Green-function calculations retain startup contributions, whereas the steady pole integrals isolate the stationary contributions used to examine long-wave stretching and interference. The near-critical interference minima refer to the selected directional field and sampled observation window. The calculations provide surface elevations and RMS amplitudes for prescribed loads, but do not give a universal relation between caustic directions and maximum-amplitude directions. Viscous attenuation, nonlinear interactions and varying bathymetry are outside the present model.

7 Conclusions

We have analysed how finite depth affects the principal stern caustics of linear capillary-gravity waves from a steadily moving pressure source. The range $0 < H \leq \sqrt{3}$ admits no regular stern caustics. For $H > \sqrt{3}$, the approximation described in Section 3.2 gives zero, then two, and finally one regular principal caustic per stern half-plane as the source speed increases. A pair appears at a finite-wavenumber onset. Its lower branch subsequently reaches the long-wave endpoint at $Fr_h = 1$. The transverse-divergent boundary is then removed, and the divergent-fan boundary remains regular.

The local analysis quantifies the different rates of wavenumber and angular separation near onset, the approach of the lower branch to zero wavenumber, and the narrowing of the two-caustic speed interval near the critical depth. The joint critical scaling also identifies the increasingly extended lateral structure of the associated long waves.

The pressure-forced and directional long-wave calculations connect these geometric results to the surface pattern. Before its interval closes, the transverse contribution stretches and weakens. Interference persists between the surviving divergent and fan contributions. Their relative phases and spectral weights set the prominence of the boundaries and the location of the largest response. Finite depth therefore controls both the creation and loss of principal caustics and the changes in wave contributions that shape the visible wake.

AI assistance

AI tools were used to assist with figure preparation and language editing. The author takes full responsibility for the scientific content and the final manuscript.

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