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Abstract

We present **antasena**, an open-source Python library for calculations that underlie diving with nitrox, oxygen-nitrogen mixtures richer in oxygen than air. It computes depth limits, equivalent depths, inert-gas uptake, no-decompression limits, staged decompression schedules, central-nervous-system and pulmonary oxygen exposure, real-gas cylinder contents, and partial-pressure blending, and its command-line and desktop interfaces display the derivation of every result with the user's inputs substituted. Each working equation is derived from the three-dimensional momentum balance of a fluid at rest, the advection-diffusion equation for a dissolved gas in perfused tissue, and the Helmholtz free energy of a gas mixture, with every reducing assumption stated. Because a dive plan is a sequence of segments of fixed gas and linearly varying depth, tissue tensions, oxygen doses, and gas consumption admit closed forms without time stepping. An independent re-implementation reproduces the closed-form tensions to within relative differences of 5×10^{-13} against fourth-order Runge-Kutta integration and of order 10^{-15} against 60-digit arithmetic, whereas the textbook form loses digits to cancellation on short segments. The analysis indicates that the classical equivalent air depth is shallower than its alveolar-consistent counterpart by 6 to 15 cm, that the textbook blending formula yields fills about 0.3 percentage points richer than intended for a 32% mixture at 200 bar, and that ideal-gas planning overstates the usable gas of an air cylinder filled to 232 bar by about 5%. All 79 independent checks pass. The library serves teaching and planning and does not predict individual physiological risk.

Keywords: Bühlmann ZH-L16C model; enriched air nitrox; multiparameter equation of state; perfusion-limited gas exchange; Schreiner equation

1 Introduction

Breathing compressed gas under water places a diver inside a chain of physical processes that are elementary when taken one at a time and subtle when taken together. The ambient pressure increases by roughly one atmosphere for every ten meters of sea water, so that the partial pressure of each component of the breathing gas, and with it the amount of that component dissolved in blood and tissue, rises in proportion to depth [21]. Nitrogen taken up at depth must be eliminated during and after the ascent, and an elimination that is too rapid allows dissolved gas to come out of solution as bubbles, the process that underlies decompression sickness [41]. Oxygen, which sustains life at the surface, becomes toxic to the central nervous system (CNS) and to the lungs when its partial pressure is raised far enough or held long enough [11, 10]. The gas stored in a cylinder departs measurably from ideal behavior at filling pressures, so that the reading of a pressure gauge is not proportional to the amount of gas available [31]. Enriched air nitrox (EAN), a mixture of oxygen and nitrogen with a larger oxygen fraction than air, reduces the nitrogen load of a dive at the cost of a shallower oxygen limit, and it therefore brings all of these processes into a single planning problem. Mixtures are designated by their oxygen percentage, so that EAN32 contains 32% oxygen by mole.

The quantitative treatment of these processes has a long history. The first staged decompression schedules were derived by representing the body as a set of compartments that exchange nitrogen exponentially, with half-times between a few minutes and a few hours [5]. The exponential compartment was later placed on a physiological footing by deriving it from the mass balance of an inert gas in a perfused tissue [29]. The tolerable excess of tissue tension over ambient pressure was expressed as a linear function of depth, the M-value [45], and that idea was subsequently generalized to sixteen compartments with coefficients fitted to experimental exposures, producing the ZH-L16 family of models that remains widely used in dive computers and planning software [7, 8]. The exact solution of the compartment equation for a linearly varying ambient pressure removed the need to approximate descents and ascents by steps [38]. Alongside these deterministic models, bubble nucleation models and probabilistic models fitted to large sets of recorded dives address the formation of the gas phase and the likelihood of injury, which the compartment models leave outside their scope [46, 44]. The physiology of oxygen

toxicity followed a parallel path, beginning with the observation of convulsions in divers breathing oxygen under pressure [11]. Dose-response measurements then quantified the development of pulmonary toxicity [9, 10], and later statistical models treated both forms of toxicity [1]. The thermodynamics of the gases themselves rests on reference equations of state for nitrogen, oxygen, and air, combined for arbitrary mixtures by multiparameter formulations [39, 37, 31, 30].

Taken together, the calculations that govern nitrox diving form a compact and unusually concrete application of hydrostatics, gas thermodynamics, mass transport, and linear differential equations. In diver training they are nevertheless usually met as tables and rules of thumb, such as a fixed number of meters per bar, a fixed formula for the oxygen fill pressure, or an equivalent depth that disregards the water vapor in the lungs, and their derivations and ranges of validity are rarely shown. Dive computers and planning programs implement the same models with considerable care, but they generally do not expose the intermediate quantities or the assumptions on which those quantities rest. A student or an instructor who wishes to see why a depth limit takes a particular value, how a decompression ceiling arises from a set of tissue tensions, or how much a real-gas correction changes a blending instruction has few transparent tools with which to do so. The numerical side of the problem also deserves attention. Expressions that are algebraically identical can differ substantially in floating-point arithmetic [16, 24], and the textbook form of the Schreiner solution is an instance of this behavior that arises in a physically meaningful setting.

Compact, single-purpose scientific libraries whose numerical core is legible and whose results are checked against independent implementations have proven effective in comparable settings, including idealized shear-flow instability, collective animal motion, and nonlinear dispersive waves [23, 22, 26]. The approach suits nitrox diving particularly well, since every quantity of interest can be traced to a short chain of physical laws and empirical models, and since the empirical parts can be isolated and labeled as such. A verified library of this kind can serve at once as a planning aid, as a reference implementation against which other tools may be compared, and as teaching material in which each number is accompanied by its derivation.

This study presents **antasena**, an open-source Python library for nitrox diving calculations. The library computes the maximum operating depth (MOD), the best mix for a planned depth, the equivalent air depth (EAD), the tissue tensions and the no-decompression limit (NDL) of the ZH-L16C model, decompression schedules with gradient factors (GFs) and gas switches, the CNS clock and the pulmonary dose expressed in oxygen tolerance units (OTUs), real-gas cylinder contents and cooling, and partial-pressure blending, together with bounds on the MOD and the NDL that are guaranteed under analyzer and gauge uncertainty. It is distributed with a command-line interface (CLI) and a graphical user interface (GUI), both of which display the derivation of each result with the user’s numbers substituted. The governing equations are derived below from the three-dimensional conservation laws and from the thermodynamics of gas mixtures, each modeling assumption is stated where it enters, and the closed forms that the library evaluates are obtained from them step by step. The library is then exercised on a set of planning problems and on four canonical dives, and every result is compared with an independent re-implementation written for this purpose, including the numbers printed by the CLI and displayed by the GUI. The aim is a verified and transparent computational aid for teaching and planning. The decompression and oxygen-exposure models it implements are established empirical models, and the library makes no claim about the physiological risk to an individual diver.

2 Methods

2.1 Model Description

The quantities that a nitrox diver and a gas blender require follow from four physical ingredients, namely the hydrostatic pressure of the water column, the partial pressures of the components of the breathing gas, the transport of dissolved inert gas between the lungs and the tissues, and the equation of state of the gas stored in a cylinder. Two empirical models complete the description, one for the supersaturation that tissues are assumed to tolerate and one for the cumulative effect of oxygen exposure. Throughout, pressures are written p in pascals (Pa) within derivations and P in bar ($1 \text{ bar} = 10^5 \text{ Pa}$) in working formulas, following diving practice. Depths d are in meters (m), temperatures T in kelvin (K), amounts of substance n in moles (mol), and times t in seconds (s) within derivations and in minutes (min) in working formulas. Mole fractions are dimensionless, and a subscript names the species or the compartment to which a quantity refers.

The derivation begins with the water column. Let $\mathbf{x} = (x, y, z)$ denote position in a Cartesian frame fixed to the Earth, with z directed vertically upward and the mean free surface at $z = 0$. The water is described by its velocity $\mathbf{u}(\mathbf{x}, t)$ in m s^{-1} , its pressure $p(\mathbf{x}, t)$ in Pa, and its density $\rho_w(\mathbf{x}, t)$ in kg m^{-3} . Conservation of linear momentum for a continuum under gravity is expressed by the Cauchy momentum equation [3],

$$\rho_w \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \nabla \cdot \boldsymbol{\tau} + \rho_w \mathbf{g}, \quad (1)$$

where $\boldsymbol{\tau}$ is the deviatoric (viscous) stress tensor in Pa, $\mathbf{g} = -g \mathbf{e}_z$ is the gravitational acceleration with magnitude g in m s^{-2} , and $\mathbf{e}_z$ is the upward unit vector. Equation (1) contains the full dynamics of the water, but the quantity that matters to the diver is only the pressure at the diver’s position. The first assumption is therefore that this pressure is the hydrostatic pressure of water at rest. Its justification is a comparison of magnitudes. Currents and wave orbital motion produce dynamic pressure fluctuations of order $\rho_w |\mathbf{u}|^2/2$, which for a brisk current of

$|\mathbf{u}| = 1 \text{ m s}^{-1}$ amounts to 5×10^{-3} bar, whereas the hydrostatic increment is close to 0.1 bar per meter of depth. Setting $\mathbf{u} = \mathbf{0}$ in (1) removes the local acceleration, the advective acceleration, and the viscous stress, which vanishes for a fluid at rest, and leaves the hydrostatic balance

$$\nabla p = -\rho_w g \mathbf{e}_z. \quad (2)$$

Written in components, (2) reads

$$\frac{\partial p}{\partial x} = 0, \quad \frac{\partial p}{\partial y} = 0, \quad \frac{\partial p}{\partial z} = -\rho_w g. \quad (3)$$

The first two components state that the pressure does not vary horizontally, so that $p = p(z)$ and the three-dimensional problem reduces to a single ordinary differential equation in z . The second assumption is that the density of the water is uniform over the depth range of a recreational or technical dive. The density change produced by a pressure change Δp is $\Delta \rho_w / \rho_w = \Delta p / K$, where K is the isothermal bulk modulus, close to 2.2 GPa near 20 °C [43]. With $\Delta p = 4 \times 10^5$ Pa at 40 m, the relative change is about 2×10^{-4} , smaller than the difference between the density conventions used in practice. Introducing the depth $d = -z \geq 0$, so that $dz = -dd$, the third component of (3) becomes $dp/dd = \rho_w g$. Integrating from the surface, where the pressure equals the atmospheric pressure p_s , to the depth d gives

$$\int_{p_s}^{p(d)} dp' = \int_0^d \rho_w g dd' \implies p(d) = p_s + \rho_w g d. \quad (4)$$

Dividing by 10^5 Pa bar^{-1} yields the working form

$$P(d) = P_s + \kappa d, \quad \kappa = \frac{\rho_w g}{10^5 \text{ Pa bar}^{-1}}, \quad (5)$$

where P is the absolute ambient pressure in bar, $P_s = p_s/10^5$ is the surface pressure in bar, and κ is the hydrostatic gradient in bar m^{-1} . The library uses the standard acceleration $g = 9.80665 \text{ m s}^{-2}$ and three density conventions, namely sea water with $\rho_w = 1025 \text{ kg m}^{-3}$, the value of 1020 kg m^{-3} adopted for depth gauges under the European standard EN 13319 [13], and fresh water with $\rho_w = 1000 \text{ kg m}^{-3}$. A fourth, rounded convention sets $P_s = 1 \text{ bar}$ and $\kappa = 0.1 \text{ bar m}^{-1}$ exactly, which reproduces the tables taught in diver training.

The surface pressure p_s is itself the hydrostatic pressure of the atmosphere at the altitude h of the dive site, measured in meters above mean sea level. The same balance (3) applies to the air column, now with the height h in place of z and with a density that varies with pressure. Treating dry air as an ideal gas gives $\rho_a = pM_a/(RT_a)$, where M_a is the molar mass of dry air in kg mol^{-1} , R is the molar gas constant in $\text{J mol}^{-1} \text{K}^{-1}$, and T_a is the air temperature in K. Taking the temperature to decrease linearly with height, $T_a(h) = T_0 - Lh$ with sea-level temperature T_0 and lapse rate L in K m^{-1} , the vertical balance becomes a separable equation,

$$\frac{dp}{dh} = -\frac{gM_a}{R} \frac{p}{T_0 - Lh} \implies \int_{p_0}^{p_s} \frac{dp}{p} = -\frac{gM_a}{R} \int_0^h \frac{dh'}{T_0 - Lh'}, \quad (6)$$

where p_0 is the sea-level pressure. Both integrals are logarithms, $\ln(p_s/p_0) = [gM_a/(RL)] \ln(1 - Lh/T_0)$, and exponentiation gives the barometric formula

$$p_s(h) = p_0 \left(1 - \frac{Lh}{T_0}\right)^{gM_a/(RL)}. \quad (7)$$

The library evaluates (7) with the constants of the International Standard Atmosphere (ISA) for the troposphere [27], $p_0 = 101325 \text{ Pa}$, $T_0 = 288.15 \text{ K}$, $L = 0.0065 \text{ K m}^{-1}$, and $M_a = 0.0289644 \text{ kg mol}^{-1}$, together with $R = 8.314462618 \text{ J mol}^{-1} \text{K}^{-1}$ [40], for which the exponent $gM_a/(RL)$ equals 5.2558. At altitude the diver is assumed to be acclimatized, so that the tissues begin the dive in equilibrium with the local surface pressure.

The pressure at depth acts on the breathing gas, whose composition determines how that pressure is shared among its components. Consider a mixture of $\mathcal{N}$ species with amounts n_j , mole fractions $x_j = n_j/\sum_k n_k$, temperature T , and volume V in m^3 . All of its equilibrium properties follow from the Helmholtz free energy $A(T, V, \{n_j\})$ in joules, which is conveniently split into an ideal-gas part and a residual part, $A = A^{\text{ig}} + A^{\text{r}}$, the latter carrying the effect of intermolecular forces. The pressure follows from the thermodynamic identity $p = -(\partial A/\partial V)_{T, \{n_j\}}$. In the ideal-gas part the molecules do not interact, and statistical mechanics gives

$$A^{\text{ig}} = \sum_j n_j \left[\mu_j^\circ(T) + RT \ln \frac{n_j RT}{V p^\circ} - RT \right], \quad (8)$$

where $\mu_j^\circ(T)$ is the standard chemical potential of species j in J mol^{-1} at the reference pressure p° . Only the logarithm depends on V , with $\partial \ln(n_j RT/(V p^\circ))/\partial V = -1/V$, so that differentiation term by term yields

$$p^{\text{ig}} = -\left(\frac{\partial A^{\text{ig}}}{\partial V}\right)_{T, \{n_j\}} = \sum_j \frac{n_j RT}{V} = \sum_j p_j, \quad p_j \equiv \frac{n_j RT}{V} = x_j p. \quad (9)$$

The partial pressure p_j is the pressure that species j would exert alone in the volume V , and (9) is Dalton's law. Its use for the breathed gas rests on the assumption that the residual part is negligible at ambient pressure. This assumption is well satisfied, since the compressibility factor $Z = pV/(nRT)$ of air at 37 °C and 6 bar, the ambient pressure at 50 m, is 0.9987 in the reference equation of state [31], so that the ideal-mixture partial

pressures are accurate to about one part in a thousand. Writing f_{O_2} for the oxygen mole fraction of a nitrox mixture and $f_{\text{N}_2} = 1 - f_{\text{O}_2}$ for its inert fraction, with argon counted as nitrogen, the inspired partial pressures at depth d follow from (5) and (9) as

$$P_{\text{O}_2}(d) = f_{\text{O}_2}(P_s + \kappa d), \quad P_{\text{N}_2}(d) = f_{\text{N}_2}(P_s + \kappa d). \quad (10)$$

Dry air contains an oxygen mole fraction close to 0.209 [35], and the library adopts the value $f_{\text{O}_2}^{\text{air}} = 0.2095$ customary in diving practice.

Because (5) is an affine and strictly increasing map from depth to pressure, every depth limit of nitrox diving is the pre-image of a pressure limit. The MOD is defined as the depth at which the inspired oxygen partial pressure reaches a chosen ceiling $P_{\text{O}_2}^{\text{max}}$, set conventionally at 1.4 bar for the working phase of a dive and at 1.6 bar as a contingency or decompression limit. Setting $P_{\text{O}_2}(d_{\text{MOD}}) = P_{\text{O}_2}^{\text{max}}$ in (10), dividing by f_{O_2} , and solving the resulting linear equation for the depth gives

$$P_s + \kappa d_{\text{MOD}} = \frac{P_{\text{O}_2}^{\text{max}}}{f_{\text{O}_2}} \implies d_{\text{MOD}} = \frac{1}{\kappa} \left(\frac{P_{\text{O}_2}^{\text{max}}}{f_{\text{O}_2}} - P_s \right). \quad (11)$$

The inverse question, which mixture to breathe at a planned depth d , reverses the roles of the variables. The best mix is the largest oxygen fraction whose partial pressure at d does not exceed the ceiling, $f^* = P_{\text{O}_2}^{\text{max}}/P(d)$, which the library rounds down to whole percent so that the rounding never raises the oxygen partial pressure. A lower bound follows in the same way from a minimum oxygen partial pressure $P_{\text{O}_2}^{\text{min}}$ below which the mixture cannot sustain consciousness, and the corresponding hypoxic floor, $(P_{\text{O}_2}^{\text{min}}/f_{\text{O}_2} - P_s)/\kappa$, is positive, and therefore restrictive, only when the surface pressure is low.

The partial pressure of nitrogen in the lungs drives its uptake by the tissues, and the description of that uptake starts from the local conservation of the dissolved gas. Let $c(\mathbf{x}, t)$ denote the concentration of dissolved nitrogen in mol m^{-3} at a point $\mathbf{x}$ of the body, which may lie in flowing blood or in the stationary extravascular tissue. Nitrogen is physiologically inert, so it is neither produced nor consumed, and its conservation is expressed by the advection-diffusion equation

$$\frac{\partial c}{\partial t} + \nabla \cdot (c \mathbf{v}) = \nabla \cdot (D \nabla c), \quad (12)$$

where $\mathbf{v}(\mathbf{x}, t)$ is the blood velocity in m s^{-1} , which vanishes outside the vessels, and $D(\mathbf{x})$ is the molecular diffusivity of dissolved nitrogen in $\text{m}^2 \text{s}^{-1}$, with Fick's law $\mathbf{j} = -D \nabla c$ giving the diffusive flux in $\text{mol m}^{-2} \text{s}^{-1}$ [15]. The dissolved concentration is related to the partial pressure, or tension, of the gas phase with which the liquid would be in equilibrium by Henry's law [21],

$$c = \sigma(\mathbf{x}) p_g, \quad (13)$$

where $p_g(\mathbf{x}, t)$ is the local nitrogen tension in Pa and σ is the solubility in $\text{mol m}^{-3} \text{Pa}^{-1}$, which takes the value σ_b in blood and σ_t in tissue. Because nitrogen does not bind to hemoglobin, (13) with a constant solubility applies to blood as well as to tissue. The tension, unlike the concentration, is continuous across the interface between blood and tissue, and it is therefore the natural variable for exchange between the two.

Equation (12) is reduced to a lumped description by integrating it over a finite region. Consider a region Ω of the body with tissue volume V_t in m^3 , supplied by an arterial inlet Γ_a , drained by a venous outlet Γ_v , and enclosed otherwise by a lateral surface Γ_ℓ across which no blood flows, so that $\partial\Omega = \Gamma_a \cup \Gamma_v \cup \Gamma_\ell$. Integrating (12) over Ω and applying the divergence theorem to the two flux terms gives

$$\frac{d}{dt} \int_{\Omega} c dV = - \oint_{\partial\Omega} c \mathbf{v} \cdot \mathbf{n} dS + \oint_{\partial\Omega} D \nabla c \cdot \mathbf{n} dS, \quad (14)$$

where $\mathbf{n}$ is the outward unit normal. Four assumptions, each standard in the physiology of inert-gas exchange [29], reduce (14) to an ordinary differential equation. The first is that diffusive exchange across the boundary is negligible. On Γ_a and Γ_v the advective flux dominates because the Péclet number of vascular flow is large, and on Γ_ℓ diffusion into neighboring regions is neglected, so that each region communicates with the rest of the body only through its blood supply and the second surface integral vanishes. The second is that exchange is perfusion-limited. The diffusion time ℓ^2/D across the intercapillary distance ℓ is taken to be short compared with the transit time of blood through the capillary bed, so that the tension is uniform within the region, $p_g = p_t(t)$, and venous blood leaves in equilibrium with the tissue it drains, $p_v = p_t$. The third is that arterial blood arrives in equilibrium with the alveolar gas, so that its tension equals the alveolar nitrogen partial pressure, $p_a = p_{\text{alv}}(t)$. The fourth is that the gas remains in solution, so that no free-gas phase forms in the region during the exposure.

Each term of (14) can now be evaluated. On the inlet and outlet the concentration is uniform over the cross section and equal to $\sigma_b p_a$ and $\sigma_b p_v$, respectively, while $\mathbf{v} \cdot \mathbf{n} = 0$ on Γ_ℓ . With the volumetric blood flow $\dot{Q}$ in $\text{m}^3 \text{s}^{-1}$ defined by $\int_{\Gamma_v} \mathbf{v} \cdot \mathbf{n} dS = - \int_{\Gamma_a} \mathbf{v} \cdot \mathbf{n} dS = \dot{Q}$, the advective integral becomes

$$- \oint_{\partial\Omega} c \mathbf{v} \cdot \mathbf{n} dS = \dot{Q} \sigma_b p_a - \dot{Q} \sigma_b p_v = \dot{Q} \sigma_b (p_{\text{alv}} - p_t). \quad (15)$$

Under the uniform tension of the second assumption, the amount of nitrogen in the region is $\int_{\Omega} c dV = \sigma_t V_t p_t$, where the small amount held in the blood within the region is absorbed into the effective capacity $\sigma_t V_t$. Equating

the rate of change of this amount with (15) and dividing by $\sigma_t V_t$ gives

$$\frac{dp_t}{dt} = k(p_{\text{alv}}(t) - p_t), \quad k = \frac{\dot{Q} \sigma_b}{V_t \sigma_t}, \quad (16)$$

where the rate constant k in s^{-1} is the ratio of the nitrogen-carrying capacity of the blood flow to the nitrogen capacity of the tissue. For a constant alveolar pressure the solution of (16) is $p_t(t) = p_{\text{alv}} + (p_t(0) - p_{\text{alv}})e^{-kt}$, and the departure from equilibrium halves in the time $\tau_{1/2}$ that satisfies $e^{-k\tau_{1/2}} = 1/2$, that is, $\tau_{1/2} = \ln 2/k$. Equation (16) is the perfusion-limited, well-mixed compartment that underlies classical decompression practice [5, 29]. The body is represented by a set of such compartments with half-times spread over two orders of magnitude, each a model of tissues that share a perfusion-to-capacity ratio rather than an identifiable organ. The library adopts the sixteen nitrogen compartments of the ZH-L16C model, with half-times τ_i from 5 to 635 min [7, 8]. In what follows, the tension of compartment i is written $P_i(t)$ in bar, its rate constant $k_i = \ln 2/\tau_i$ in min^{-1} , and time is measured in minutes.

The alveolar nitrogen partial pressure that drives (16) differs from the inspired value of (10), because the alveolar gas is saturated with water vapor at body temperature and contains the carbon dioxide produced by metabolism. Applying Dalton's law (9) to the alveolar gas, whose total pressure equals the ambient pressure, gives $P = p_{\text{H}_2\text{O}} + P_{\text{CO}_2} + P_{\text{O}_2}^{\text{alv}} + P_{\text{alv}}$, and the alveolar gas equation expresses the inert partial pressure in terms of the dry-gas fraction, the water vapor pressure, and the respiratory exchange [14]. The library follows the convention of the ZH-L16 model, in which the carbon dioxide and the difference between oxygen uptake and carbon dioxide output are neglected, so that the dry alveolar gas has the inert fraction of the inspired gas and only water vapor is subtracted [7],

$$P_{\text{alv}} = (P - p_{\text{H}_2\text{O}}) f_{\text{N}_2}, \quad p_{\text{H}_2\text{O}} = 0.0627 \text{ bar}, \quad (17)$$

where $p_{\text{H}_2\text{O}}$ is the alveolar water vapor pressure, close to the saturation pressure of 0.0628 bar at 37 °C [43]. The omission of carbon dioxide raises the alveolar inert pressure slightly. It is a property of the adopted model and is retained for consistency with its published coefficients. Before a dive at the surface pressure P_s every compartment is in equilibrium with air, so that $P_i = (P_s - p_{\text{H}_2\text{O}}) f_{\text{N}_2}^{\text{air}}$, which equals 0.7514 bar at sea level.

A dive plan is a sequence of segments on each of which the breathing gas is fixed and the depth varies linearly in time, as in a descent or an ascent at a constant rate or a stay at constant depth. Let t be the time measured from the start of a segment and $\dot{d}$ the rate of depth change in m min^{-1} , positive during descent. Substituting $d(t) = d_0 + \dot{d}t$ into (5) and the result into (17) shows that the alveolar pressure on the segment is linear in time,

$$P_{\text{alv}}(t) = P_{a0} + R_a t, \quad P_{a0} = (P_s + \kappa d_0 - p_{\text{H}_2\text{O}}) f_{\text{N}_2}, \quad R_a = f_{\text{N}_2} \kappa \dot{d}, \quad (18)$$

where P_{a0} is the alveolar pressure at the start of the segment in bar and R_a its rate of change in bar min^{-1} . With this forcing, (16) becomes a linear first-order equation with constant coefficients, $\dot{P}_i + k_i P_i = k_i (P_{a0} + R_a t)$, subject to the initial tension $P_i(0) = P_0$. Multiplication by the integrating factor $e^{k_i t}$ turns the left side into an exact derivative,

$$\frac{d}{dt} (P_i e^{k_i t}) = k_i e^{k_i t} (P_{a0} + R_a t), \quad (19)$$

and integration from 0 to t , using $\int_0^t k_i e^{k_i s} ds = e^{k_i t} - 1$ and, by parts, $\int_0^t k_i s e^{k_i s} ds = t e^{k_i t} - (e^{k_i t} - 1)/k_i$, gives

$$P_i(t) e^{k_i t} - P_0 = P_{a0} (e^{k_i t} - 1) + R_a \left[t e^{k_i t} - \frac{e^{k_i t} - 1}{k_i} \right]. \quad (20)$$

Multiplying by $e^{-k_i t}$ and collecting the terms that do not decay yields

$$P_i(t) = P_{a0} + R_a \left(t - \frac{1}{k_i} \right) - \left(P_{a0} - P_0 - \frac{R_a}{k_i} \right) e^{-k_i t}, \quad (21)$$

which is the Schreiner equation in the form usually quoted [38]. For $R_a = 0$ it reduces to the exponential approach $P_i(t) = P_{\text{alv}} + (P_0 - P_{\text{alv}})e^{-k_i t}$ of a constant-depth exposure [5]. The same result can be grouped differently. Collecting instead the terms of (20) that are proportional to $P_{a0} - P_0$ and to R_a , and factoring $R_a t$ out of the latter, gives the algebraically identical expression

$$P_i(t) = P_0 + (P_{a0} - P_0) (1 - e^{-k_i t}) + R_a t \phi(k_i t), \quad \phi(s) = 1 - \frac{1 - e^{-s}}{s}, \quad (22)$$

which the library evaluates. The two forms differ in their behavior under rounding. Consider a segment that begins in equilibrium, $P_{a0} = P_0$, as at the start of a descent from the surface. Expanding $e^{-k_i t}$ for $k_i t \ll 1$ shows that the uptake is then $P_i(t) - P_0 \approx R_a k_i t^2 / 2 = (R_a / k_i) (k_i t)^2 / 2$. In (21) this small quantity emerges as the difference between terms of order R_a / k_i , each carrying a rounding error of order $u_r R_a / k_i$, where $u_r \approx 1.1 \times 10^{-16}$ is the unit roundoff of double precision. The relative error of the uptake is therefore of order $2u_r / (k_i t)^2$, the factor $2 / (k_i t)^2$ being the condition number of that subtraction [16, 24]. This situation arises on short and steep segments for fast compartments. In (22) the uptake is formed directly. The factor $1 - e^{-k_i t} = -\text{expm1}(-k_i t)$ is computed without cancellation, and ϕ admits the alternating series

$$\phi(s) = \sum_{m=1}^{\infty} \frac{(-1)^{m+1} s^m}{(m+1)!} = \frac{s}{2} - \frac{s^2}{6} + \frac{s^3}{24} - \cdots, \quad (23)$$

obtained by expanding e^{-s} in the definition, which is evaluated for small s [24]. Because (16) is linear, the solution operator over a segment of duration $t_1 + t_2$ equals the composition of the operators over t_1 and t_2 . The tensions at the end of a plan are therefore obtained exactly by applying (22) segment by segment, with no time-step error.

When the ambient pressure falls below the tension of a compartment, the tissue is supersaturated and dissolved gas may come out of solution. Equation (16) cannot describe bubble formation, which its fourth assumption excludes, so the decompression model instead limits the supersaturation through an empirical tolerance. In the ZH-L16 model the largest tension tolerated by compartment i at ambient pressure P_{amb} , the M-value, is a straight line in the ambient pressure [7, 8],

$$M_i(P_{\text{amb}}) = a_i + \frac{P_{\text{amb}}}{b_i}, \quad (24)$$

where a_i in bar and b_i , dimensionless and smaller than unity, are fitted coefficients, a form that generalizes the linear M-values of earlier United States Navy practice [45]. The GF construction, used widely in technical diving, admits only a fraction $\gamma \in (0, 1]$ of the supersaturation $M_i - P_{\text{amb}}$ that the M-value line allows [2],

$$P_i \leq P_{\text{amb}} + \gamma [M_i(P_{\text{amb}}) - P_{\text{amb}}]. \quad (25)$$

The ceiling of a compartment is the smallest ambient pressure at which (25) still holds. Substituting (24) and collecting the terms in P_{amb} gives

$$P_i \leq \gamma a_i + P_{\text{amb}} \left(1 - \gamma + \frac{\gamma}{b_i}\right) \iff P_{\text{amb}} \geq \frac{P_i - \gamma a_i}{1 - \gamma + \gamma/b_i}, \quad (26)$$

where the direction of the inequality is preserved because the coefficient $1 - \gamma + \gamma/b_i$ is positive for $\gamma \in (0, 1]$ and $b_i > 0$. Every compartment must satisfy its own bound, so the ceiling of the diver is the largest of these over the compartments,

$$P_{\text{ceil}} = \max_i \frac{P_i - \gamma a_i}{1 - \gamma + \gamma/b_i}, \quad d_{\text{ceil}} = \frac{P_{\text{ceil}} - P_s}{\kappa}, \quad (27)$$

and an ascent is permitted only to depths at or below d_{ceil} . The GF is not held constant during the ascent. It takes a low value γ_{lo} at and below the first decompression stop and rises linearly in ambient pressure to a high value γ_{hi} at the surface,

$$\gamma(P_{\text{amb}}) = \gamma_{\text{hi}} + (\gamma_{\text{lo}} - \gamma_{\text{hi}}) \frac{P_{\text{amb}} - P_s}{P_{\text{first}} - P_s}, \quad P_s \leq P_{\text{amb}} \leq P_{\text{first}}, \quad (28)$$

where P_{first} is the ambient pressure at the first stop. A choice $\gamma_{\text{lo}} < \gamma_{\text{hi}} < 1$ places the first stop deeper and lengthens the shallow stops relative to the unmodified model. The GFs are a planning convention and carry no physiological content beyond the scaling in (25).

The NDL follows from the same tolerance evaluated at the surface. A direct ascent is permitted when every compartment satisfies (25) at $P_{\text{amb}} = P_s$ with $\gamma = \gamma_{\text{hi}}$, that is, when $P_i \leq M_i^\gamma$ with the surfacing limit $M_i^\gamma = P_s + \gamma_{\text{hi}}[M_i(P_s) - P_s]$. At constant depth the tension of each compartment approaches P_{alv} monotonically, so the time t_i at which it first reaches M_i^γ follows by setting the constant-depth solution equal to the limit and solving for the time,

$$P_{\text{alv}} + (P_0 - P_{\text{alv}}) e^{-k_i t_i} = M_i^\gamma \implies t_i = \frac{1}{k_i} \ln \frac{P_{\text{alv}} - P_0}{P_{\text{alv}} - M_i^\gamma}, \quad P_{\text{alv}} > M_i^\gamma > P_0. \quad (29)$$

When $P_{\text{alv}} \leq M_i^\gamma$ the compartment never reaches its limit and $t_i = \infty$. The NDL is the longest stay from which a direct ascent is permitted and is therefore the smallest of these times, $t_{\text{NDL}} = \min_i t_i$. The compartment that attains the minimum is decided by a competition between its rate constant k_i , which favors the fast compartments, and its tolerance M_i^γ , which favors the slow ones, since both decrease with the half-time.

The tissue model also yields a precise meaning for the EAD, the depth at which air would impose the same nitrogen load as a nitrox mixture at depth d . The classical definition equates inspired nitrogen partial pressures, $P_{\text{ead}}^{\text{cl}} f_{\text{N}_2}^{\text{air}} = P(d) f_{\text{N}_2}$. The tissue tensions are driven, however, by the alveolar pressure (17), and equating alveolar rather than inspired pressures gives $(P_{\text{ead}} - p_{\text{H}_2\text{O}}) f_{\text{N}_2}^{\text{air}} = (P(d) - p_{\text{H}_2\text{O}}) f_{\text{N}_2}$, whose solution is

$$P_{\text{ead}} = \frac{(P(d) - p_{\text{H}_2\text{O}}) f_{\text{N}_2}}{f_{\text{N}_2}^{\text{air}}} + p_{\text{H}_2\text{O}}, \quad d_{\text{ead}} = \frac{P_{\text{ead}} - P_s}{\kappa}. \quad (30)$$

The map $P \mapsto P_{\text{ead}}$ in (30) is affine, which has a strong consequence. If the air diver follows the depth profile obtained by applying (30) at every instant to the nitrox profile, then the alveolar pressures of the two divers coincide at every instant by construction. Equation (16) is driven by P_{alv} alone, both divers start from the same surface tensions, and a linear equation with given forcing and initial value has a unique solution. Every compartment of the air diver therefore follows the same tension history as that of the nitrox diver, so that the alveolar-consistent EAD makes an entire nitrox profile equivalent to an air profile, compartment by compartment and at every instant. Subtracting the classical pressure $P(d) f_{\text{N}_2} / f_{\text{N}_2}^{\text{air}}$ from (30) shows that the two definitions differ by the constant

$$\Delta P_{\text{ead}} = P_{\text{ead}} - P_{\text{ead}}^{\text{cl}} = p_{\text{H}_2\text{O}} \left(1 - \frac{f_{\text{N}_2}}{f_{\text{N}_2}^{\text{air}}}\right) > 0, \quad (31)$$

which is independent of depth and positive whenever $f_{N_2} < f_{N_2}^{\text{air}}$, so that the classical value is always the shallower of the two for nitrox. The consequence of the classical choice for the tissues follows from the same linearity. Under the classical map the alveolar pressure of the air diver is $P(d)f_{N_2} - p_{H_2O}f_{N_2}^{\text{air}}$, lower than that of the nitrox diver by the constant $\delta = p_{H_2O}(f_{N_2}^{\text{air}} - f_{N_2})$. Subtracting the two instances of (16) shows that the tension difference $D_i = P_i^{\text{nx}} - P_i^{\text{air}}$ obeys $\dot{D}_i = k_i(\delta - D_i)$ with $D_i(0) = 0$, whose solution

$$D_i(t) = \delta (1 - e^{-k_i t}) \quad (32)$$

holds for any profile on a single gas. The equivalent narcotic depth (END) is computed either as the depth itself, when oxygen is counted as narcotic as in common practice, or as the classical EAD otherwise.

The toxic effect of oxygen is summarized by two dose indices, both integrals of a function of the oxygen partial pressure along the dive. The CNS clock compares the time spent at each oxygen partial pressure with a single-exposure limit $T_{\text{lim}}(P_{O_2})$ in minutes, tabulated by the United States National Oceanic and Atmospheric Administration (NOAA) at 0.1 bar intervals from 0.6 to 1.6 bar [28] and interpolated linearly between nodes, and accumulates the fraction of the limit consumed,

$$\mathcal{C} = \int_0^{t_{\text{end}}} \frac{dt}{T_{\text{lim}}(P_{O_2}(t))}. \quad (33)$$

Below 0.6 bar the library continues the limit linearly to 900 min at 0.5 bar and stops accrual at 0.5 bar, a common convention, and above 1.6 bar no limit is defined and the plan is rejected. Consider one interpolation interval traversed by a linear segment of duration Δt , on which the oxygen partial pressure varies as $P_{O_2}(t) = p_a + \dot{p}t$ from p_a to p_b , with rate $\dot{p} = (p_b - p_a)/\Delta t$ in bar min^{-1} . The limit is then a linear function of time, $T_{\text{lim}}(t) = T_a + (T_b - T_a)t/\Delta t$, where T_a and T_b are its values at the ends of the interval. Changing the variable of integration from t to T_{lim} , with $dt = \Delta t dT_{\text{lim}}/(T_b - T_a)$, gives

$$\Delta \mathcal{C} = \frac{\Delta t}{T_b - T_a} \int_{T_a}^{T_b} \frac{dT_{\text{lim}}}{T_{\text{lim}}} = \Delta t \frac{\ln T_b - \ln T_a}{T_b - T_a}, \quad (34)$$

so that the dose on the interval is the duration divided by the logarithmic mean of the limits at its ends. The expression is symmetric in T_a and T_b , and it reduces to $\Delta t/T_a$ as $T_b \rightarrow T_a$, which covers the constant-depth case. The library splits each segment at the table nodes and sums (34) over the pieces, and at the surface it reduces the clock with a 90 min half-time, $\mathcal{C}(t) = \mathcal{C}(0) 2^{-t/90}$, again a planning convention.

The pulmonary index rests on the observation that the decrement of vital capacity during oxygen breathing grows as a power of the excess of the oxygen partial pressure over about 0.5 bar [9, 10]. The OTU integrates the corresponding dose rate,

$$\mathcal{U} = \int_0^{t_{\text{end}}} u(t)^{5/6} dt, \quad u = \max\left(0, \frac{P_{O_2} - 0.5 \text{ bar}}{0.5 \text{ bar}}\right), \quad (35)$$

in units of minutes of breathing oxygen at 1 bar. On a linear segment with $u > 0$ throughout, $u = 2P_{O_2} - 1$ with P_{O_2} in bar, so that $du = 2\dot{p} dt$. Changing the variable of integration to u and integrating the power gives

$$\Delta \mathcal{U} = \frac{1}{2\dot{p}} \int_{u_a}^{u_b} u^{5/6} du = \frac{1}{2\dot{p}} \frac{6}{11} \left(u_b^{11/6} - u_a^{11/6}\right) = \frac{3}{11} \frac{\Delta t}{p_b - p_a} \left(u_b^{11/6} - u_a^{11/6}\right), \quad (36)$$

with $\Delta \mathcal{U} = u_a^{5/6} \Delta t$ at constant depth. The antiderivative $(6/11) \max(0, u)^{11/6}$ is continuous and vanishes at $u = 0$, so (36) remains exact, with u clipped at zero, for ramps that cross 0.5 bar. Both indices are bookkeeping devices fitted to population data and do not predict the response of an individual [1].

The consumption of breathing gas follows from Boyle's law. A diver's breathing is characterized by the surface air consumption (SAC) $\dot{V}_s$ in L min^{-1} , the volume of gas per minute that the diver would draw if the same ventilation were supplied at the reference pressure $P_{\text{atm}} = 1.01325$ bar. The assumptions are that the volume ventilated per minute does not change with depth and that the breathed gas behaves ideally at ambient pressure, as established above. A volume $\dot{V}_s$ drawn at ambient pressure P then corresponds to a free volume $\dot{V}_s P/P_{\text{atm}}$ at the reference pressure, and the free gas used on a linear segment of duration Δt is

$$\Delta G = \frac{\dot{V}_s}{P_{\text{atm}}} \int_0^{\Delta t} P(t) dt = \dot{V}_s \frac{P_0 + P_1}{2 P_{\text{atm}}} \Delta t, \quad (37)$$

in liters at P_{atm} , where P_0 and P_1 are the ambient pressures at the ends of the segment, and the integral of a linear function equals the mean of its end values times the duration.

The gas that supplies this consumption is stored at pressures where the ideal-gas law is no longer adequate. A cylinder of internal volume V_c in liters, filled to absolute pressure P at temperature T , holds an amount of gas

$$n = \frac{PV_c}{Z(T, P, x) RT}, \quad (38)$$

which is the definition of the compressibility factor Z , with $x = f_{O_2}$ and the units of PV_c converted to joules. The compressibility follows from the residual Helmholtz energy, as Dalton's law followed from the ideal part. Writing the molar Helmholtz energy as $a = A/n = RT \alpha(\delta, \tau_T)$, where the reduced Helmholtz energy $\alpha = \alpha^{\text{ig}} + \alpha^{\text{r}}$ is a function of the reduced density $\delta = \rho/\rho_r(x)$ and the inverse reduced temperature $\tau_T = T_r(x)/T$, with molar density $\rho = n/V$ in mol m^{-3} and composition-dependent reducing functions ρ_r and T_r , the pressure identity

takes the form $p = \rho^2(\partial a/\partial \rho)_{T,x}$. By the chain rule $\partial/\partial \rho = \rho_r^{-1}\partial/\partial \delta$, so that

$$p = \rho^2 RT \frac{1}{\rho_r} \left(\frac{\partial \alpha}{\partial \delta} \right)_{\tau_T, x} = \rho RT \delta \left(\frac{\partial \alpha}{\partial \delta} \right)_{\tau_T, x}. \quad (39)$$

For the ideal part $\alpha^{\text{ig}} = \ln \delta + f(\tau_T, x)$, as follows from (8), so that $\delta \partial \alpha^{\text{ig}}/\partial \delta = 1$. Dividing (39) by ρRT then gives

$$Z = \frac{p}{\rho RT} = 1 + \delta \left(\frac{\partial \alpha^r}{\partial \delta} \right)_{\tau_T, x}, \quad (40)$$

which shows that the entire departure from ideality is carried by the residual part. The reference model in the library evaluates (40) with a multiparameter mixture formulation that combines the reference equations of state for nitrogen and oxygen through composition-dependent reducing functions and a departure function [39, 37, 30], as implemented in the `CoolProp` library [4]. For air this model agrees with the dedicated air formulation to within about 0.25% in Z up to 300 bar [31], and treating argon as nitrogen shifts Z by less than about 0.15% at 300 bar.

For teaching, the library also provides the Peng-Robinson cubic equation of state [34], which corresponds to a simple closed-form model of the residual free energy and reads

$$p = \frac{RT}{v - b} - \frac{a_{\text{PR}}(T)}{v^2 + 2bv - b^2}, \quad (41)$$

where $v = 1/\rho$ is the molar volume in $\text{m}^3 \text{mol}^{-1}$, b is the co-volume in $\text{m}^3 \text{mol}^{-1}$, and a_{PR} is the attraction parameter in $\text{Pa m}^6 \text{mol}^{-2}$. For each pure component these are

$$b = 0.07780 \frac{RT_c}{p_c}, \quad a_{\text{PR}} = 0.45724 \frac{R^2 T_c^2}{p_c} \left[1 + \mu \left(1 - \sqrt{T/T_c} \right) \right]^2, \quad \mu = 0.37464 + 1.54226 \omega - 0.26992 \omega^2, \quad (42)$$

with critical temperature T_c , critical pressure p_c , and acentric factor ω . The mixture parameters follow from the van der Waals one-fluid rules $a_{\text{PR}} = \sum_{jk} x_j x_k \sqrt{a_j a_k}$ and $b = \sum_j x_j b_j$. Substituting $v = ZRT/p$ into (41), multiplying by $v/(RT)$, and introducing the dimensionless groups $A' = a_{\text{PR}} p/(RT)^2$ and $B' = b p/(RT)$ gives

$$Z = \frac{Z}{Z - B'} - \frac{A' Z}{Z^2 + 2B' Z - B'^2}, \quad (43)$$

and clearing the denominators yields the cubic

$$Z^3 - (1 - B')Z^2 + (A' - 3B'^2 - 2B')Z - (A'B' - B'^2 - B'^3) = 0. \quad (44)$$

A physical root must satisfy $v > b$, that is, $Z > B'$. Both oxygen and nitrogen lie well above their critical temperatures at diving conditions, and throughout the range considered here (44) has exactly one real root with $Z > B'$, which is the compressibility of the gas. Any other real roots lie below B' and are discarded.

Two cylinder problems follow from conserving the amount n . When a cylinder filled hot to pressure P_1 at temperature T_1 cools to T_2 at constant volume, n from (38) at (T_1, P_1) is unchanged, and the settled pressure P_2 is the root of $P_2 V_c - nRT_2 Z(T_2, P_2, x) = 0$. The ideal-gas estimate $P_2 = P_1 T_2/T_1$ follows by setting $Z = 1$ at both states. The usable free gas behind a gauge reading is the difference between the amounts at the absolute fill pressure and at one atmosphere, the gas that remains when the gauge reads zero, expressed as an ideal volume at P_{atm} and the cylinder temperature. Gauge pressures are converted to absolute pressures by adding P_{atm} .

Blending is likewise a statement about amounts of substance. A cylinder holding n_0 of gas with oxygen fraction x_0 receives n_A of a gas with oxygen fraction x_A and then n_B of a gas with fraction x_B , typically oxygen followed by air, to reach n_f at the target fraction x_t . Conservation of the total amount and of the amount of oxygen gives the linear system

$$n_0 + n_A + n_B = n_f, \quad n_0 x_0 + n_A x_A + n_B x_B = n_f x_t. \quad (45)$$

Eliminating $n_B = n_f - n_0 - n_A$ from the second equation gives $n_0 x_0 + n_A x_A + (n_f - n_0 - n_A)x_B = n_f x_t$, in which n_A appears only with the coefficient $x_A - x_B$. Collecting terms yields the unique solution

$$n_A = \frac{n_f(x_t - x_B) - n_0(x_0 - x_B)}{x_A - x_B}, \quad n_B = n_f - n_0 - n_A, \quad (46)$$

in which n_0 and n_f follow from (38) at the starting and target pressures and compositions. The final mixture is a convex combination of the three gases, and a target is reachable without draining only if $n_A \geq 0$ and $n_B \geq 0$. With $x_B \leq x_t \leq x_A$, the two conditions are upper bounds on the residual amount, $n_0(x_0 - x_B) \leq n_f(x_t - x_B)$ and $n_0(x_A - x_0) \leq n_f(x_A - x_t)$, so that when either is violated the largest admissible residual is

$$n_0^{\text{max}} = n_f \min \left(\frac{x_t - x_B}{x_0 - x_B}, \frac{x_A - x_t}{x_A - x_0} \right), \quad (47)$$

which the library converts into a drain-to pressure by inverting (38) at the residual composition. The textbook formula is recovered as a special case. With $Z = 1$ at a common temperature, n is proportional to P , and with pure oxygen as the first gas, $x_A = 1$, and gauge pressures treated as absolute, (46) reduces to the oxygen

increment

$$\Delta P_{O_2} = \frac{P_f(x_t - x_B) - P_0(x_0 - x_B)}{1 - x_B}, \quad (48)$$

with which the real-gas solution is compared.

The inputs to all of these formulas are measured with finite accuracy. An oxygen analyzer reading f with accuracy $\pm\delta f$ places the true fraction in $[f - \delta f, f + \delta f]$, and a depth gauge may read up to δd shallow. Differentiating (11) with respect to the oxygen fraction gives

$$\frac{\partial d_{\text{MOD}}}{\partial f_{O_2}} = -\frac{P_{O_2}^{\text{max}}}{\kappa f_{O_2}^2} < 0, \quad (49)$$

so the MOD is strictly decreasing in the oxygen fraction, its exact range over the analyzer interval is attained at the ends of the interval, and the guaranteed MOD is $d_{\text{MOD}}(f + \delta f) - \delta d$. For the NDL, differentiating (29) with respect to the alveolar pressure gives

$$\frac{\partial t_i}{\partial P_{\text{alv}}} = \frac{1}{k_i} \left(\frac{1}{P_{\text{alv}} - P_0} - \frac{1}{P_{\text{alv}} - M_i^\gamma} \right) < 0, \quad (50)$$

which is negative because $M_i^\gamma > P_0$ makes the second denominator the smaller. The alveolar pressure (17) increases with depth and with f_{N_2} , and the minimum over compartments preserves the ordering, so the guaranteed NDL is attained at the deepest depth and the leanest mixture in the uncertainty box. These corner bounds are exact in real arithmetic. In floating point they carry rounding errors of the order of the machine precision [33].

2.2 Numerical Implementation

The formulation above is implemented in the **antasena** library, a pure-Python library with no runtime dependencies beyond the Python standard library, so that it installs and runs wherever Python 3.9 or later is available, including the GUI, which is built on the **tkinter** library. All arithmetic is carried out in IEEE 754 double precision [16]. The library is organized in modules that follow the structure of the model: **environment** for the hydrostatic map (5) and the standard atmosphere (7); **gas** for the partial pressures (10), the MOD (11), the best mix, and the equivalent depths (30); **buhlmann** for the compartment kinetics (22), the ceiling (27), and the NDL (29); **toxicity** for the oxygen indices (34) and (36); **planner** for complete dive plans; **eos**, **cylinder**, and **blending** for (38)–(46); and **uncertainty** for the guaranteed bounds. An **explain** module rebuilds each headline result through the public interface and returns a **Derivation** object that lists the governing equation, every intermediate value with the user’s inputs substituted, the assumptions, and the range of validity; the test suite asserts that the final value of each derivation equals the library result, so that the explanation cannot drift from the computation.

Tissue tensions are advanced by (22) through the method **TissueState.expose**, which takes the ambient pressures at the two ends of a segment, its duration, and the inert fraction of the gas, forms the alveolar pressures by (17), and updates all sixteen compartments at once. The function $\phi(s)$ is evaluated as $1 + \expm1(-s)/s$ for $s \geq 0.05$ and by the first ten terms of the alternating series (23) for $s < 0.05$, where the truncation error is below $s^{11}/12! < 10^{-22}$. In the first branch the addition of unity to $\expm1(-s)/s \approx -1 + s/2$ retains a mild cancellation, with a condition number of about $2/s$ that does not exceed 40, so that ϕ carries a relative rounding error of at most a few units of 10^{-15} just above the switch and approaches the unit roundoff as s grows. The NDL is computed from (29) without iteration, and the ceiling from (27) as a maximum over compartments.

The **plan.dive** function assembles a plan from a list of levels, each a depth and a duration that excludes transit. Descents and ascents between levels are taken at constant rates, by default 18 and 9 m min⁻¹. Every segment is applied in turn to the tissue state, and its oxygen doses (34) and (36) and its gas use (37) are accumulated. At the end of the last level the dive is classified as a decompression dive if the ceiling at γ_{hi} lies below the surface, which is the same criterion that defines the NDL. In that case the first stop is placed at the ceiling for γ_{lo} , rounded up to the stop grid of 3 m, and the GF varies with ambient pressure according to (28). The ascent then proceeds one grid depth at a time. Before each move the transit to the next grid depth is simulated at the ascent rate on a copy of the tissue state, and the move is taken if the ceiling at the GF of the destination lies at or above the destination; otherwise the diver waits in whole minutes until it does. On arrival at each grid depth the richest available decompression gas whose oxygen partial pressure does not exceed 1.6 bar is selected, with one minute allowed for the switch. A safety stop of three minutes at 5 m is added until the first mandatory wait. Plans whose bottom gas exceeds 1.6 bar are rejected, and plans that exceed the working limit of 1.4 bar, approach the CNS limit, exceed 300 OTU, the level to which the daily dose of the REPEX procedures settles over an extended exposure [18, 19], or leave a cylinder below its reserve produce warnings. Each cylinder in the plan is debited with its gas use, and the end pressure is reported both from the reference equation of state and from the ideal-gas law.

The reference compressibility is stored as a table of $Z(T, P, x)$ generated once from the multiparameter mixture model through **CoolProp** [30, 4], on a uniform grid with 21 temperatures from 253.15 to 353.15 K, 41 pressures from 0 to 400 bar, and 21 oxygen fractions from 0 to 1, so that the library itself does not depend on **CoolProp**. Between grid nodes Z is interpolated by four-point Lagrange polynomials along each axis, shifted inward at the edges of the grid. Because Z is analytic in all three variables in the supercritical region covered by the table, the interpolation error decreases as the fourth power of the grid spacing. The Peng-Robinson cubic (44) is solved

by Newton’s method started from $Z = 1$, with a bracketing fallback. The inverse problems, the pressure that holds a given amount of gas and the composition after a top-off, are solved by Brent’s method [6], which retains a sign-changing bracket at every step and therefore converges for any continuous function with a sign change in the bracket; the function $pV_c - nRTZ(T, p, x)$ is strictly increasing in p throughout the gas region considered, so its root is unique.

The CLI is invoked as `antasena` followed by one of the subcommands `mod`, `bestmix`, `ead`, `ndl`, `blend`, `topoff`, `cylinder`, and `plan`, each of which accepts a `--why` flag that prints the corresponding derivation. Run without arguments, or with the subcommand `gui`, it opens the GUI, which provides four tabs: gas limits, with an operating-window chart of oxygen fraction against depth; blending and cylinders, with the fill path and the compressibility of the three models; dive planning, with the profile, the decompression ceiling, and the tissue loading at the end of the bottom phase; and a Learn tab of derivations computed live from the values on the other tabs. Units can be switched between metric and imperial without loss, since all values are held internally in metric units. The library is accompanied by a test suite of 113 tests, many of them property-based through the `Hypothesis` library [32], which check, among other properties, that the closed-form tensions satisfy the semigroup property under segment splitting, that the NDL agrees with bisection, that blending conserves total and oxygen amounts for all three equations of state, that guaranteed bounds enclose sampled true values, and that no decompression schedule places the diver above the ceiling.

2.3 Numerical Experiments

The library is exercised on two kinds of problem. The first kind consists of planning quantities evaluated over ranges of their inputs: the MOD, best mix, and equivalent depths for oxygen fractions from 0.21 to 1 and depths to 60 m under the four density conventions; the NDL for air, EAN32, and EAN36 at depths from 9 to 45 m and γ_{hi} of 1.0 and 0.85; the oxygen indices on random linear ramps; the compressibility, usable gas, cooling, and blending of cylinders at temperatures from 0 to 60 °C and pressures to 300 bar; and the guaranteed bounds for analyzer accuracies to ± 2 percentage points and gauge errors to ± 1 m.

The second kind consists of four canonical dives, which span the regimes of the planner and recur in several of the analyses. All four use sea water at sea level ($\rho_w = 1025 \text{ kg m}^{-3}$, $P_s = 1.01325 \text{ bar}$), the library defaults for descent and ascent rates and stop placement, GFs of 0.30 and 0.85, and an SAC of 20 L min^{-1} . Dive P1 is a stay of 20 min at 30 m on EAN32, close to the NDL of that mixture. Dive P2 is a stay of 25 min at 40 m on air with EAN50 and pure oxygen as decompression gases. Dive P3 is a stay of 30 min at 35 m on EAN28 with EAN50 as the decompression gas. Dive P4 is a multilevel dive on EAN32 of 12 min at 30 m, 20 min at 18 m, and 25 min at 12 m. The bottom gas of P1 is carried in a 12 L cylinder at 200 bar, that of P2 and P3 in 24 L at 232 bar, and that of P4 in 18 L at 232 bar, each decompression gas in 7 L at 200 bar, all at 20 °C. Dive P2 is additionally planned with GF pairs of (0.50, 0.80) and (1.00, 1.00) and on a grid of γ_{lo} from 0.20 to 1.00 and γ_{hi} from 0.60 to 1.00 in steps of 0.05. Dive P1 is repeated after a surface interval of 60 min to illustrate residual nitrogen, and the NDL of a repeat dive is computed as a function of the surface interval for three first-dive profiles. The NDL of EAN32 is finally computed in fresh water at site altitudes of 0, 1000, 2000, and 3000 m.

2.4 Data Analyses

Every result of the library is compared with an independent computation written for this study without reusing the mathematics of the library. The model constants of the independent code, including the sixteen ZH-L16C coefficient triples, the water vapor pressure, and the oxygen-limit table, are typed again from their sources, so that a transcription error in either code would appear as a disagreement. The independent references are of six kinds.

Tissue tensions are integrated by the classical fourth-order Runge-Kutta (RK4) method with a step of one second along every segment [17], and the convergence of that integration to the closed form is measured as the step is halved from 240 s to 0.47 s. The Schreiner solution is also evaluated in 60-digit arithmetic with the `mpmath` library, with the double-precision inputs converted to their exact binary values, to measure the rounding error of (21) and (22) for the 5-min compartment at the start of an 18 m min^{-1} descent on air over $k_i t$ from 10^{-8} to 3. The NDL is recomputed by bisection on the surfacing condition, with the tensions of all sixteen compartments evaluated directly and the bracket halved to 10^{-10} min. The decompression schedules are audited by integrating the tensions independently at one-second resolution along every ascent and evaluating the ceiling (27) at the planned GF at every second.

The oxygen indices are recomputed by adaptive Gauss–Kronrod quadrature [36], through `scipy.integrate.quad` [42], with breakpoints at the table nodes and at 0.5 bar and with the CNS limit written as piecewise-linear bands rather than read from the library table, on 2000 random linear ramps with end pressures drawn uniformly from 0.3 to 1.6 bar and durations log-uniformly from 0.1 to 160 min. The same ramps are used to test time-reversal symmetry and additivity under a random split. The compressibility is compared with `CoolProp` at 400 random points with temperatures from 253 to 353 K, pressures from 1 to 400 bar, and oxygen fractions from 0 to 1. Blending instructions are tested by simulating, in a real-gas cylinder, a fill that follows each instruction: the amount in the cylinder when the gauge reads the stop value is found by solving $n = n(P, x(n))$ with Brent’s method for each phase of the fill, using either the library table or `CoolProp` directly, and the resulting oxygen fraction is compared with the target. The guaranteed bounds are tested by Monte Carlo sampling of 20,000 points uniformly distributed in the uncertainty box.

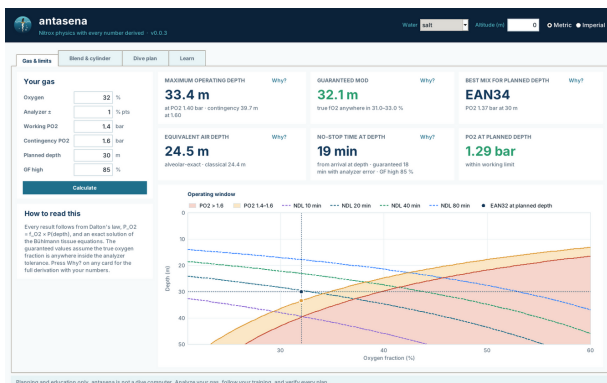
The CLI is run in a fresh interpreter for every subcommand, and the numbers it prints are parsed and compared with library calls for the same inputs, rounded to the printed precision. The GUI is opened under a virtual display, each tab is rendered and captured at a window size of 1440×900 pixels, and the values shown on its result cards are read from the interface and compared with library calls for the inputs shown in its fields, including a conversion to imperial units and back. Each comparison is recorded as passing or failing in a plain-text report, and a driver script runs all analyses and tallies the verdicts. All analyses were run with Python 3.12 and `antasena` 0.0.3 using the `NumPy`, `SciPy`, and `Matplotlib` libraries [20, 42, 25].

3 Results

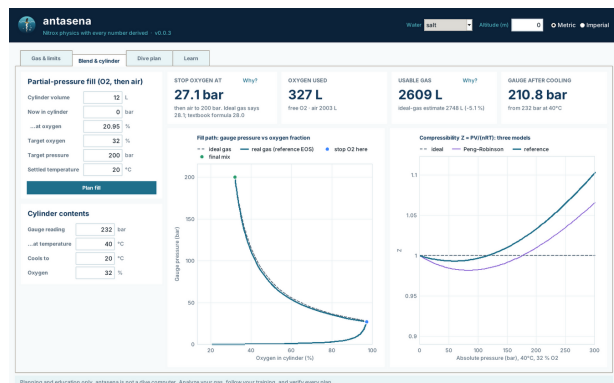
All values reported below are taken directly from the plain-text reports written by the supplementary analysis scripts. Pressures are absolute unless they are described as gauge readings, and depths refer to sea water at sea level unless another convention is stated. The results are presented in the order of the model, beginning with the interfaces through which a user meets the library and proceeding from hydrostatics through tissue kinetics, decompression, oxygen exposure, and cylinder thermodynamics to the effect of measurement uncertainty and of repeated exposure.

The three interfaces of the library return the same numbers. The GUI, shown in Figure 1, opened with its default inputs of EAN32 at a planned depth of 30 m and an analyzer accuracy of ± 1 percentage point. Its result cards displayed an MOD of 33.4 m, a guaranteed MOD of 32.1 m, a best mix of EAN34, an alveolar-consistent EAD of 24.5 m, an NDL of 19 min at $\gamma_{hi} = 0.85$, and an oxygen partial pressure of 1.29 bar. The blending tab displayed an oxygen stop of 27.1 bar for EAN32 to 200 bar in a 12 L cylinder, 327 L of oxygen to be added, 2609 L of usable EAN32 behind a reading of 232 bar at 40 °C, and a settled reading of 210.8 bar after cooling to 20 °C. The planning tab displayed a run time of 28 min, a CNS clock of 12%, and 33 bar of bottom gas remaining after 20 min at 30 m. Each of the fifteen interface checks agreed with the library to the displayed precision, including the conversion of the MOD to 109.7 ft and back to 33.4 m, which confirms that the imperial display is a pure presentation layer over metric internal values. The Learn tab, shown in Figure 1(d), reproduces the chain from (5) to (11) with the current inputs substituted, so that a reader can follow each intermediate number on the screen. The CLI, of which a recorded session is shown in Figure 2, passed all seventeen of its checks. These included the real-gas and ideal-gas oxygen stops of 27.1 and 28.1 bar, a usable volume of 3168 L for 12 L of air at 300 bar and 45 °C, a complete staged plan with gas switches, and a nonzero exit status for a plan whose bottom gas exceeds 1.6 bar, which allows the CLI to be used inside scripts that must stop on an unsafe input.

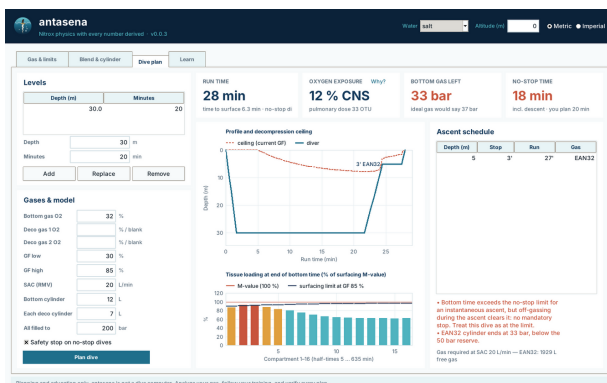
(a) Gas & limits



(b) Blend & cylinder



(c) Dive plan



(d) Learn

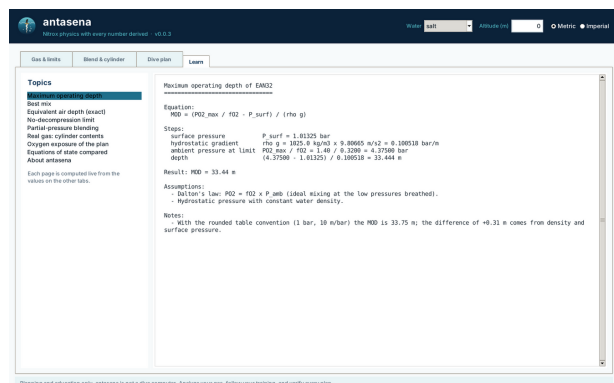


Figure 1: The **antasena** GUI at its default inputs. (a) Gas and limits tab, with result cards for the MOD, guaranteed MOD, best mix, EAD, NDL, and oxygen partial pressure, and the operating-window chart of oxygen fraction against depth. (b) Blending and cylinder tab, with the fill path and the compressibility of the three equations of state. (c) Dive planning tab, with the profile and decompression ceiling, the tissue loading at the end of the bottom phase, and the ascent schedule. (d) Learn tab, here showing the derivation of the MOD with the current inputs substituted.

(a)

```
$ antasena mod 32 --analyzer 1
EAN32: MOD at P02 1.40 bar = 33.4 m
with analyzer error +/-1.0 %: guaranteed MOD 32.1 m (range 32.1 to 34.8 m)

$ antasena bestmix 30
Best mix for 30 m at P02 1.40 bar: EAN34 (P02 1.370 bar)

$ antasena ndl 32 30 --analyzer 1
EAN32 at 30 m, GF high 85 %: NDL 19.1 min
analyzer +/-1.0 %: NDL between 18.1 and 20.1 min

$ antasena blend 32 200
[real gas ] add O2 to 27.1 bar, top with air to 200 bar 02 used 327 L
[ideal gas] add O2 to 28.1 bar, top with air to 200 bar 02 used 333 L

$ antasena cylinder 300 --temp 45 --cool-to 20
12 L at 300 bar, 45 °C: 3168 usable free liters (ideal gas 3553 L, -10.8 %)
at 20 °C the gauge will read 265.5 bar
```

(b)

```
$ antasena plan 40:25 --gas 20.95 --deco 50 100 --volume 24 --fill 232
Plan: 40:25 on Air; GF 30/85
NDL at depth: 5.7 min
  Stop   Time   Run   Gas
  21 m   1'    30.3' EAN50
  12 m   1'    32.3' EAN50
   9 m   2'    34.7' EAN50
   6 m   5'    40.0' EAN50
   3 m   7'    47.3' O2
Run time 47.7 min, TTS 20.4 min, CNS 18.8 %, OTU 51
Air: 2787 L used, ends at 102 bar (ideal-gas estimate 114 bar)
EAN50: 428 L used, ends at 164 bar (ideal-gas estimate 170 bar)
O2: 189 L used, ends at 204 bar (ideal-gas estimate 205 bar)

$ antasena plan 40:10 --gas 40
plan rejected: EAN40 reaches P02 2.01 bar at 40 m, beyond the 1.6 bar limit of the
oxygen-toxicity model
```

Figure 2: Recorded session of the `antasena` CLI, run in a fresh interpreter for each command. (a) Planning quantities: MOD with analyzer uncertainty, best mix, NDL with analyzer uncertainty, oxygen-first blending with real-gas and ideal-gas stops, and the usable volume and settled pressure of a hot cylinder. (b) A staged decompression plan on air with EAN50 and oxygen, and the rejection of a plan whose bottom gas exceeds the oxygen limit. The `--fill` value in (b) applies to every cylinder, so the 7 L decompression cylinders start from 232 bar rather than the 200 bar of dive P2, and their end pressures are correspondingly higher than those of the planned dive.

The depth limits follow the affine structure of (5), as Figure 3 shows. The hydrostatic gradient κ is 0.1005182, 0.1000278, and 0.0980665 bar m⁻¹ in sea water, under EN 13319, and in fresh water, respectively, so that the depth equivalent of one bar ranges from 9.95 to 10.20 m. The MOD of EAN32 at 1.4 bar is accordingly 33.44 m in sea water, 33.61 m under EN 13319, 34.28 m in fresh water, and 33.75 m under the rounded convention (Table 1). Because (11) scales as $1/\kappa$, the convention matters most for lean mixtures with deep limits. Relative to sea water, fresh water lengthens the MOD of air by 1.41 m, whereas the rounded convention lengthens it by between 0.15 and 0.42 m across the range of mixtures, partly because it takes the surface pressure as 1 bar. In the plane of depth and oxygen fraction, the curves of constant oxygen partial pressure are hyperbolae, $f_{O_2} = P_{O_2}/(P_s + \kappa d)$, and the band between 1.4 and 1.6 bar narrows in depth as the oxygen fraction rises, from 9.50 m for air to 1.99 m for pure oxygen (Figure 3(c)). The MOD of pure oxygen at 1.6 bar is 5.84 m in sea water, shallower than the 6 m stop that the rounded convention allows, so that under a strict 1.6 bar limit the library schedules the switch to oxygen at 3 m. The continuous optimum f^* at 30 m is 0.3475, and rounding down gives EAN34 at an oxygen partial pressure of 1.370 bar. Across depths from 10 to 40 m the rounding of the best mix to whole percent lowered the oxygen partial pressure by between 0.007 and 0.041 bar below the ceiling, the largest reduction occurring at 40 m, where the optimum 0.2781 is rounded to EAN27. In fresh water at altitude the MOD of EAN32 increases from 34.28 m at sea level to 37.46 m at 3000 m, where the surface pressure is 0.7011 bar, and the hypoxic floor of air for a minimum of 0.16 bar becomes positive only near the highest altitude, at 0.64 m. The library values agreed exactly with an independent evaluation of (11), and the oxygen partial pressure recomputed at each MOD equaled 1.4 bar to within 2.2×10^{-16} bar, one unit in the last place of a double-precision number of that magnitude.

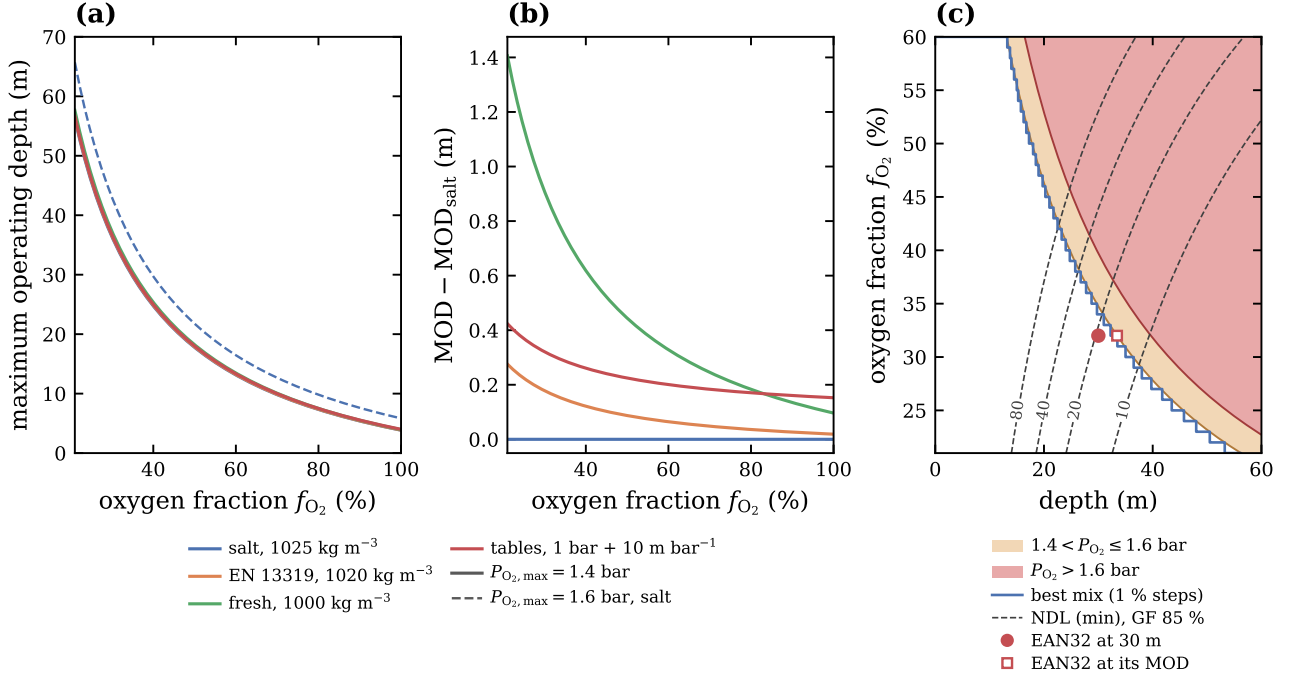


Figure 3: Hydrostatics and the nitrox operating window. (a) MOD against oxygen fraction at $P_{O_2}^{\max} = 1.4$ bar for four density conventions, and at 1.6 bar in sea water. (b) Difference between the MOD under each convention and the sea-water value at 1.4 bar. (c) Operating window in the plane of depth and oxygen fraction in sea water, with the bands of oxygen partial pressure between 1.4 and 1.6 bar and above 1.6 bar, the best mix rounded down to whole percent, contours of the NDL at $\gamma_{hi} = 0.85$ labeled in minutes, and EAN32 at 30 m and at its MOD.

Table 1: MOD in meters at $P_{O_2}^{\max} = 1.4$ bar, and at 1.6 bar in parentheses, under the four density and surface-pressure conventions of the library.

Gas	Sea water	EN 13319	Fresh water	Rounded tables
Air	56.40 (65.90)	56.68 (66.22)	57.81 (67.55)	56.83 (66.37)
EAN28	39.66 (46.77)	39.86 (47.00)	40.65 (47.94)	40.00 (47.14)
EAN32	33.44 (39.66)	33.61 (39.86)	34.28 (40.65)	33.75 (40.00)
EAN36	28.61 (34.14)	28.75 (34.30)	29.32 (34.99)	28.89 (34.44)
EAN40	24.74 (29.71)	24.86 (29.86)	25.36 (30.46)	25.00 (30.00)
EAN50	17.78 (21.75)	17.86 (21.86)	18.22 (22.30)	18.00 (22.00)
O ₂	3.85 (5.84)	3.87 (5.87)	3.94 (5.98)	4.00 (6.00)

The two definitions of the EAD differ by a constant offset, as Figure 4 shows. For EAN32 at 30 m the alveolar-consistent EAD of (30) is 24.485 m and the classical value is 24.397 m. The offset (31) is 5.56, 8.72, 11.88, and 15.03 cm for EAN28, EAN32, EAN36, and EAN40, respectively, the same at every depth examined from 15 to 30 m, and it grows linearly with the oxygen fraction, as (31) predicts, since $1 - f_{N_2}/f_{N_2}^{\text{air}}$ is linear in f_{O_2} . The library values agreed with the analytic offset to 1.1×10^{-14} m. The equivalence property was tested along the multilevel dive P4, a profile of twelve segments that includes descent, three levels, ascents between them, and a safety stop. The nitrox profile and the air profile obtained through (30) produced tensions that were identical in every compartment and at every sample to the last bit, so that the equivalence holds in floating point as well as in exact arithmetic for this profile. Under the classical map the tension difference grew to a maximum relative value of 5.5×10^{-3} at the end of the dive, and it followed (32) with $\delta = 6.928$ mbar to within 2.7×10^{-15} bar in every compartment. The fast compartments reached the asymptote δ within the first level, whereas the slowest compartments had attained only a small fraction of it by the end of the dive (Figure 4(c)), which is the behavior expected from the factor $1 - e^{-k_i t}$. The END of EAN32 at 30 m equals the depth, 30.000 m, when oxygen is counted as narcotic and the classical EAD of 24.397 m otherwise.

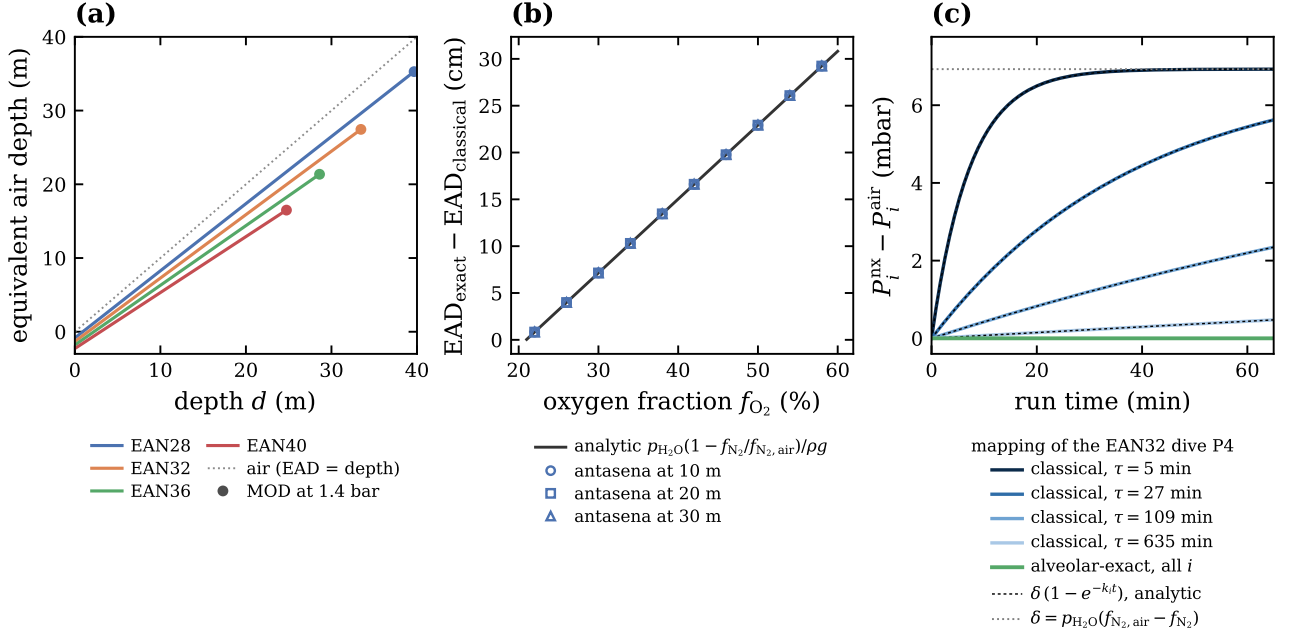


Figure 4: Equivalent air depth. (a) Alveolar-consistent EAD against depth for four mixtures up to their 1.4 bar MOD, with the air line for reference. (b) Difference between the alveolar-consistent and classical EAD against oxygen fraction, from the analytic offset (31) and from the library at three depths. (c) Difference between the tensions of the EAN32 dive P4 and of its air equivalent under the classical map for four compartments, with the analytic solution (32) and its asymptote δ , and under the alveolar-consistent map, for which the difference is zero.

The closed-form tissue tensions agree with numerical integration to the limits of floating point, as Figure 5 shows. Along the staged dive P2, a profile of 32 segments, the RK4 integration with a step of one second reproduced the library tensions to a maximum relative difference of 4.6×10^{-13} at every sample of all sixteen compartments, and the final tensions of the fastest and slowest compartments, 0.5356626564 and 0.8406281416 bar, agreed to all ten printed digits. The final tensions illustrate the spread of time scales that the model encodes. At the end of the dive the tension of the 5-min compartment had fallen to 0.536 bar, below its surface-saturation value of 0.751 bar, as a consequence of oxygen breathing at the last stop, whereas the compartments with half-times between 18.5 and 38.3 min retained the largest tensions, of about 1.58 bar or more, and the slowest compartment had risen by only about 0.09 bar above its initial value. Halving the step from 60 s to 0.94 s reduced the error by factors whose base-two logarithms lay between 3.91 and 4.14, the fourth-order convergence expected of the method [17], down to 1.5×10^{-14} at 0.47 s, while the two largest steps, 240 and 120 s, lay outside the asymptotic range. Against 60-digit arithmetic for the 5-min compartment on an 18 m min^{-1} descent, the rearranged form (22) computed the uptake $P_i(t) - P_0$ with a maximum relative error of 8.7×10^{-16} at the sampled values of $k_i t$ from 10^{-8} to 3, consistent with the bound of a few units of 10^{-15} that the evaluation of ϕ admits near $k_i t = 0.05$. The textbook form (21) lost accuracy approximately in proportion to $(k_i t)^{-2}$, as the condition-number argument predicts, with relative errors of 10^{-14} at $k_i t = 0.08$, 10^{-8} near $k_i t = 3 \times 10^{-5}$, and up to 0.49 near $k_i t = 10^{-8}$, at which point about half of the computed uptake was rounding error. Splitting a segment at random points changed the final tensions by at most 2.2×10^{-16} bar, which confirms the semigroup property of the exact solution, and the Schreiner form with zero rate reproduced the exponential approach to 4.4×10^{-16} bar.

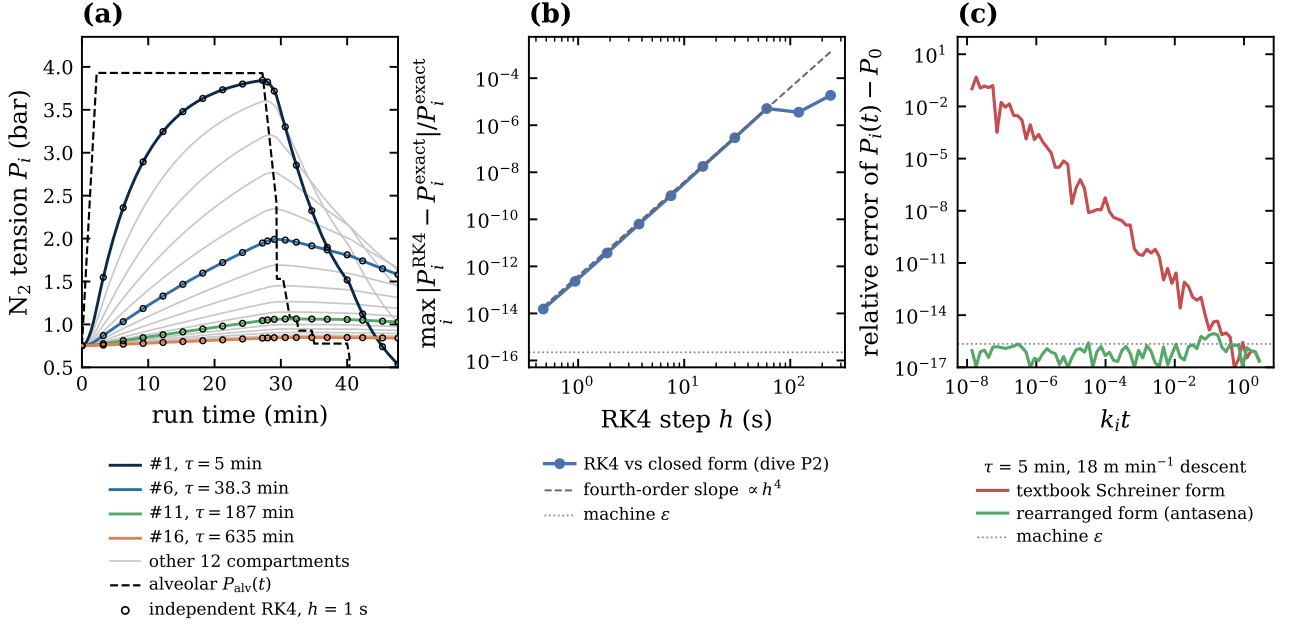


Figure 5: Perfusion-limited tissue kinetics along dive P2. (a) Nitrogen tensions of four representative compartments and of the remaining twelve, with the alveolar pressure and the independent RK4 integration. (b) Maximum relative difference between the RK4 and closed-form tensions at the end of the dive against the step, with a reference slope of fourth order and machine precision. (c) Relative error of the uptake computed by the textbook and rearranged Schreiner forms against 60-digit arithmetic, for the 5-min compartment on an 18 m min $^{-1}$ descent.

The NDL computed from (29) agreed with bisection to 4.3×10^{-11} min over all depths and mixtures, as Figure 6(a) shows. At 30 m the NDL is 16.32 min on air and 27.13 min on EAN32 at $\gamma_{hi} = 1$, and 12.11 and 19.07 min at $\gamma_{hi} = 0.85$. The gain from nitrox grows as the depth decreases. At 18 m the NDL at $\gamma_{hi} = 0.85$ is 42.94 min on air, 77.88 min on EAN32, and 100.55 min on EAN36, and at 12 m it exceeds 400 min on EAN32, because the alveolar pressure approaches the surfacing limits of the slow compartments, so that the logarithm in (29) grows without bound. Lowering γ_{hi} from 1 to 0.85 shortened the NDL by 22 to 41% across the tabulated depths and mixtures, the largest reductions occurring at the shallowest depths on nitrox, where the controlling compartments are slow and their limits lie close to the alveolar pressure. The compartment that sets the limit is neither the fastest nor fixed (Figure 6(b)). Every one of the sixteen compartments controls the NDL somewhere in the plane of depths from 9 to 45 m and oxygen fractions from 0.21 to 0.40, with the fast compartments controlling deep exposures and the slow compartments controlling shallow ones. At 30 m on air the 5-min compartment reaches its limit only after 20.00 min, because its M-value of 2.986 bar at the surface is the most tolerant, and the limit of 16.32 min is set by the 8-min compartment (Figure 6(c) and Table 2). On EAN32 the alveolar pressure of 2.697 bar lies below the surfacing limit of the 5-min compartment, which therefore never reaches it, and the limit of 27.13 min is set by the 12.5-min compartment. Table 2 also shows that the times t_i of neighboring compartments are close together near the minimum, the second and third compartments on air at 30 m reaching their limits within 2 min of each other, so that a small change in depth or mixture can transfer control from one compartment to another. Counting the descent at 18 m min $^{-1}$, as the planner does, shortens the NDL measured from arrival at the bottom by 0.50 to 1.08 min for the cases examined, from 19.07 to 18.29 min for EAN32 at 30 m and from 6.79 to 5.71 min for air at 40 m.

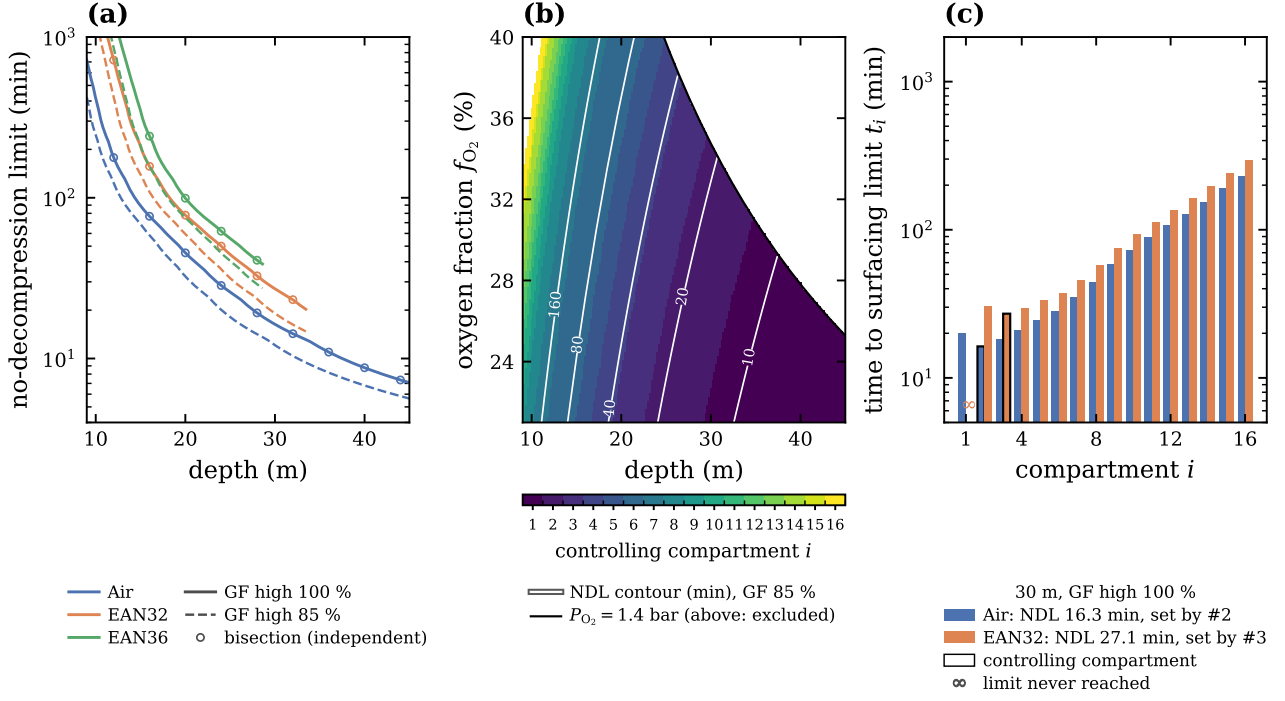


Figure 6: Structure of the NDL in sea water. (a) NDL against depth for three mixtures at two values of γ_{hi} , with the independent bisection. (b) Controlling compartment over depth and oxygen fraction at $\gamma_{hi} = 0.85$, with NDL contours in minutes and the 1.4 bar oxygen limit. (c) Time for each compartment to reach its surfacing limit at 30 m and $\gamma_{hi} = 1$ for air and EAN32, with the controlling compartment outlined.

Table 2: ZH-L16C nitrogen coefficients adopted by the library [8]. The table also lists the M-value at the sea-level surface pressure and the time for each compartment to reach it from surface saturation at 30 m ($\gamma_{hi} = 1$). An infinite time marks a compartment that never reaches its limit. The minimum of each column, the NDL, is set in bold.

i	τ_i (min)	a_i (bar)	b_i	$M_i(P_s)$ (bar)	t_i , air (min)	t_i , EAN32 (min)
1	5.0	1.1696	0.5578	2.9861	20.00	∞
2	8.0	1.0000	0.6514	2.5555	16.32	30.25
3	12.5	0.8618	0.7222	2.2648	18.17	27.13
4	18.5	0.7562	0.7825	2.0511	21.03	29.43
5	27.0	0.6200	0.8126	1.8669	24.58	33.18
6	38.3	0.5043	0.8434	1.7057	28.26	37.26
7	54.3	0.4410	0.8693	1.6066	34.81	45.36
8	77.0	0.4000	0.8910	1.5372	44.43	57.47
9	109.0	0.3750	0.9092	1.4894	58.26	75.01
10	146.0	0.3500	0.9222	1.4487	72.89	93.48
11	187.0	0.3295	0.9319	1.4168	88.30	112.92
12	239.0	0.3065	0.9403	1.3841	106.35	135.62
13	305.0	0.2835	0.9477	1.3527	127.90	162.67
14	390.0	0.2610	0.9544	1.3227	154.15	195.58
15	498.0	0.2480	0.9602	1.3032	189.18	239.66
16	635.0	0.2327	0.9653	1.2824	230.84	291.97

The GFs control the depth and length of the stops, as Figure 7 shows for dive P2. In what follows, the time to surface (TTS) denotes the time from leaving the bottom to reaching the surface. With $(\gamma_{lo}, \gamma_{hi}) = (0.30, 0.85)$ the first stop computed from the ceiling lay at 21 m, the 8-min compartment led the ascent, and the plan comprised stops of 1, 1, 2, 5, and 7 min at 21, 12, 9, 6, and 3 m, the first and last including one-minute gas switches to EAN50 and oxygen, for a run time of 47.67 min and a TTS of 20.44 min. The pair $(0.50, 0.80)$ placed the first ceiling-derived stop at 15 m and gave a run time of 48.67 min and a TTS of 21.44 min, one minute longer than the pair with the lower γ_{lo} , because the lower γ_{hi} added one minute at 9 m. The unmodified model $(1.00, 1.00)$

gave 40.67 min and 13.44 min. In the plane of ambient pressure and tissue tension (Figure 7(a)) the leading compartment approaches the GF envelope from below during the ascent and follows it closely through the stops, the tension falling during each stop and the envelope moving toward the M-value line as the ambient pressure decreases. Over the grid of GFs the TTS ranged from 13.4 to 30.4 min and depended mainly on γ_{hi} , changing by 1 to 3 min for every 0.05 in γ_{hi} and by at most 2 min across the whole range of γ_{lo} at fixed γ_{hi} . Pairs with $\gamma_{lo} > \gamma_{hi}$, which the construction (28) does not admit, are absent from the grid. In all three plans the realized supersaturation remained below the planned GF during the ascent, the largest values of the realized supersaturation minus the planned GF being -5.1×10^{-4} , -1.7×10^{-3} , and -3.8×10^{-2} . The independent one-second audit found the ceiling below the diver at every second of every ascent, with smallest clearances of 5.1×10^{-4} , 1.0×10^{-3} , and 2.4×10^{-2} bar over 1227, 1287, and 807 audited seconds. The small clearance for (0.30,0.85) shows that the planner holds the diver close to the permitted limit without crossing it, which is the intended behavior of a schedule built on the ceiling.

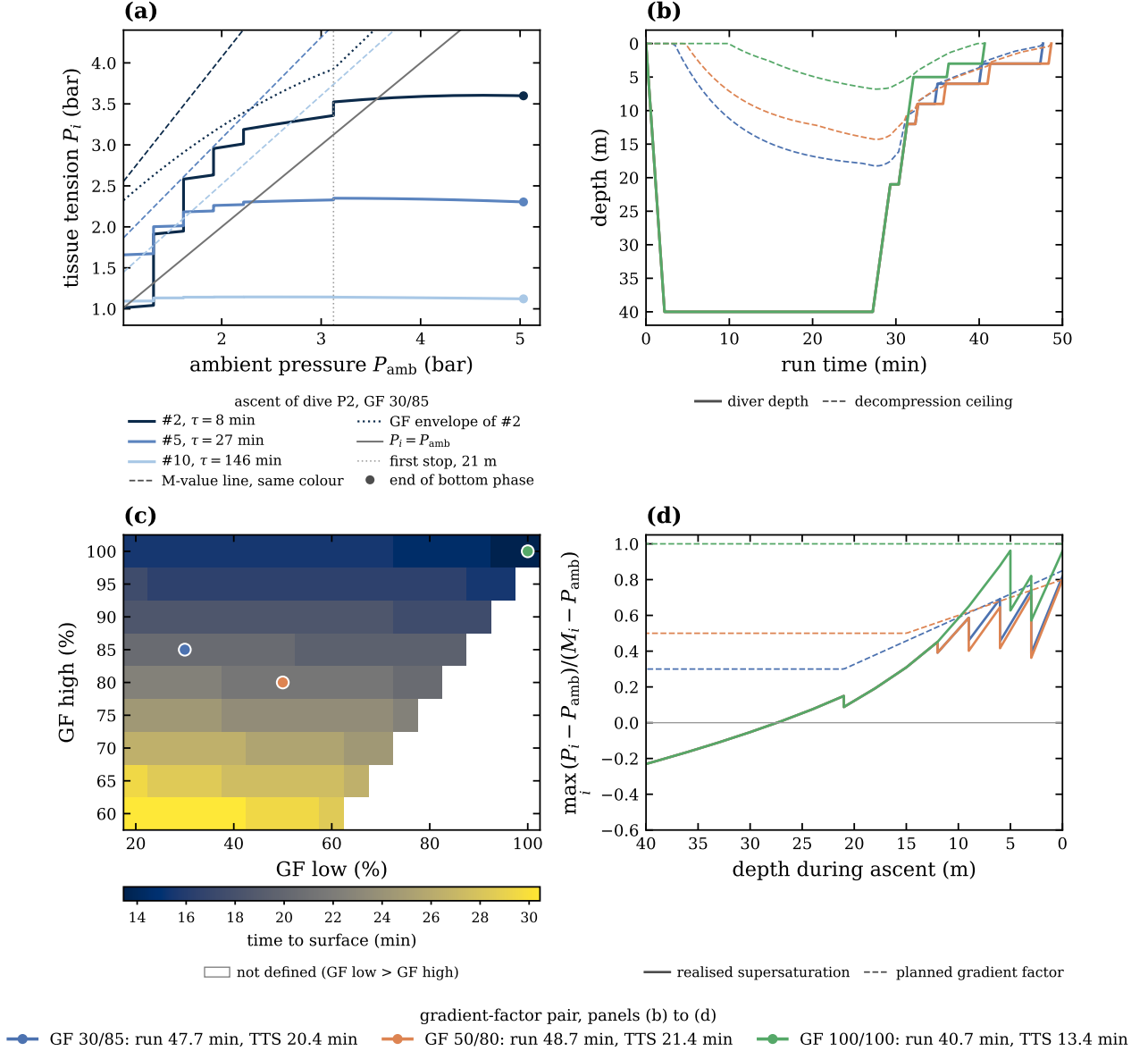


Figure 7: GFs and staged decompression for dive P2. (a) Ascent in the plane of ambient pressure and tissue tension for three compartments at $(\gamma_{lo}, \gamma_{hi}) = (0.30, 0.85)$, with their M-value lines, the GF envelope of the leading compartment, the line $P_i = P_{amb}$, and the first stop. (b) Depth and decompression ceiling against run time for three GF pairs. (c) TTS over the plane of γ_{lo} and γ_{hi} , with the three pairs marked. (d) Realized supersaturation and planned GF against depth during the ascent for the three pairs.

The four canonical dives are summarized in Figure 8 and Table 3. Dive P1 exceeds the NDL of 18.29 min computed with the descent, yet off-gassing during the ascent at 9 m min^{-1} cleared the ceiling before the diver reached it, so the planner required no mandatory stop, added the 5 m safety stop, and reported the dive as being

at the limit. Dives P2 and P3 required staged decompression with first stops computed at 21 and 15 m. Dive P3, with a longer and shallower bottom phase on EAN28, accumulated a TTS of 15.89 min, about 4.5 min less than P2 despite a bottom time longer by 5 min, which reflects the lower inert fraction of its bottom gas. Dive P4 accumulated the longest bottom time but the shortest TTS, because its final ascent begins at 12 m and its tensions at that point lie within the surfacing limits. In every plan the computed ceiling lay at or above the diver at every sample of the trace. The gas budgets show the effect of the bottom-gas volume. Dive P2 used 2787 L of the 5229 L of usable air, and P3 used 2907 L of the 5275 L of usable EAN28, whereas the bottom gas of P1 and P4 ends below the customary reserve of 50 bar, a condition for which the planner issues a warning. The ideal-gas end pressures exceed the real-gas values by 3.6 to 12.4 bar, an overestimate that is largest for the largest cylinders at the highest fill pressures. The decompression gases were used lightly, with EAN50 ending between 129 and 135 bar and oxygen at 174 bar from 200 bar fills.

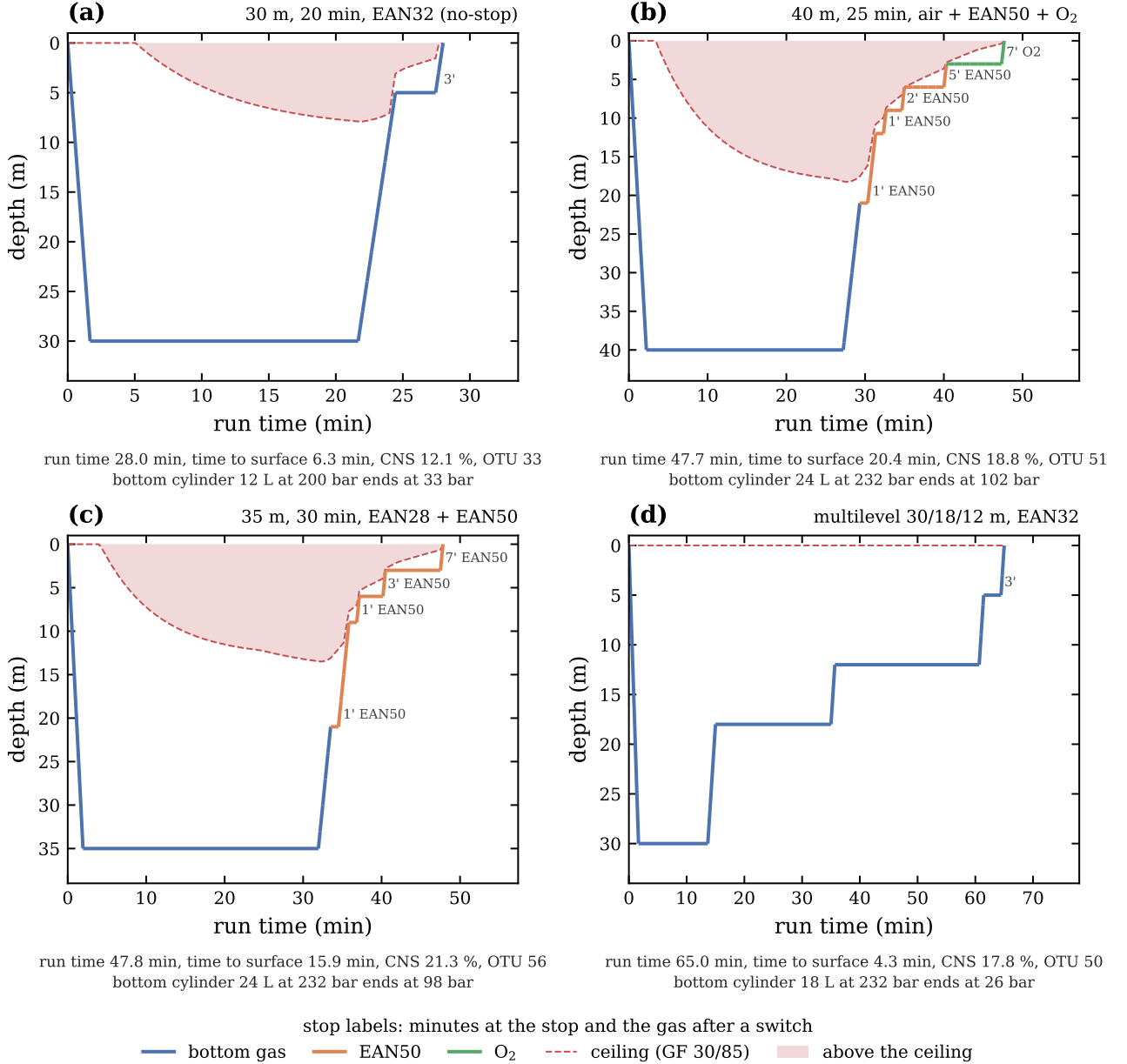


Figure 8: The four canonical dive plans at $(\gamma_{lo}, \gamma_{hi}) = (0.30, 0.85)$ in sea water. (a) P1, 30 m for 20 min on EAN32. (b) P2, 40 m for 25 min on air with EAN50 and oxygen. (c) P3, 35 m for 30 min on EAN28 with EAN50. (d) P4, multilevel at 30, 18, and 12 m on EAN32. The profile is colored by breathing gas, the dashed curve is the decompression ceiling, the shaded region lies above it, stop labels give the stop time in minutes and the gas beneath each panel summarizes the plan.

Table 3: Summary of the four canonical dives. Run time and TTS are in minutes; the bottom-gas end pressure is the gauge reading from the reference equation of state, with the ideal-gas estimate in parentheses.

Dive	Run time	TTS	CNS (%)	OTU	Bottom cylinder	End pressure (bar)
P1	28.00	6.33	12.11	32.5	12 L, 200 bar	33.5 (37.1)
P2	47.67	20.44	18.80	51.2	24 L, 232 bar	102.0 (114.4)
P3	47.83	15.89	21.30	55.7	24 L, 232 bar	98.4 (109.3)
P4	65.00	4.33	17.81	49.7	18 L, 232 bar	26.4 (34.8)

The closed-form oxygen doses agreed with quadrature on all 2000 random ramps, as Figure 9 shows, with maximum relative differences of 1.3×10^{-13} for the CNS clock and 7.2×10^{-14} for the OTU, and median differences of 1.6×10^{-16} and 1.7×10^{-16} . Reversing a ramp left both doses unchanged to the last bit, in agreement with the symmetry of (34) and (36) in their end values, and splitting it at a random point changed them by at most 3.6×10^{-14} and 9.5×10^{-15} in relative terms. The library table matched the independent piecewise-linear bands at all eleven nodes, and the library rejected a ramp that rose above 1.6 bar. The CNS limit falls steeply near its upper end, from 150 min at 1.4 bar to 45 min at 1.6 bar, so that 10 min at 1.6 bar consume 22.22% of the clock, more than the 20.00% consumed by 30 min at 1.4 bar, whereas the corresponding OTU values, 19.29 and 48.96, reverse that ordering. The two indices therefore weight short exposures at high partial pressure and long exposures at moderate partial pressure differently. Dive P2 accumulated 18.80% and 51.2 OTU. Its bottom phase on air at 1.055 bar contributed 9.36% and 27.3 OTU, the one-minute switch to EAN50 at 1.562 bar contributed 1.36%, and the seven minutes on oxygen at 3 m, at 1.315 bar and including the switch, contributed 3.99% and 10.5 OTU. The clock fell to 11.84% after 60 min and to 7.46% after 120 min at the surface. Dive P3, whose bottom phase on EAN28 lay at 1.269 bar, accumulated 21.30% and 55.7 OTU, of which the bottom phase contributed 15.84% and 42.9 OTU, so that the choice of bottom gas, rather than the decompression gas, dominated the oxygen dose of that dive.

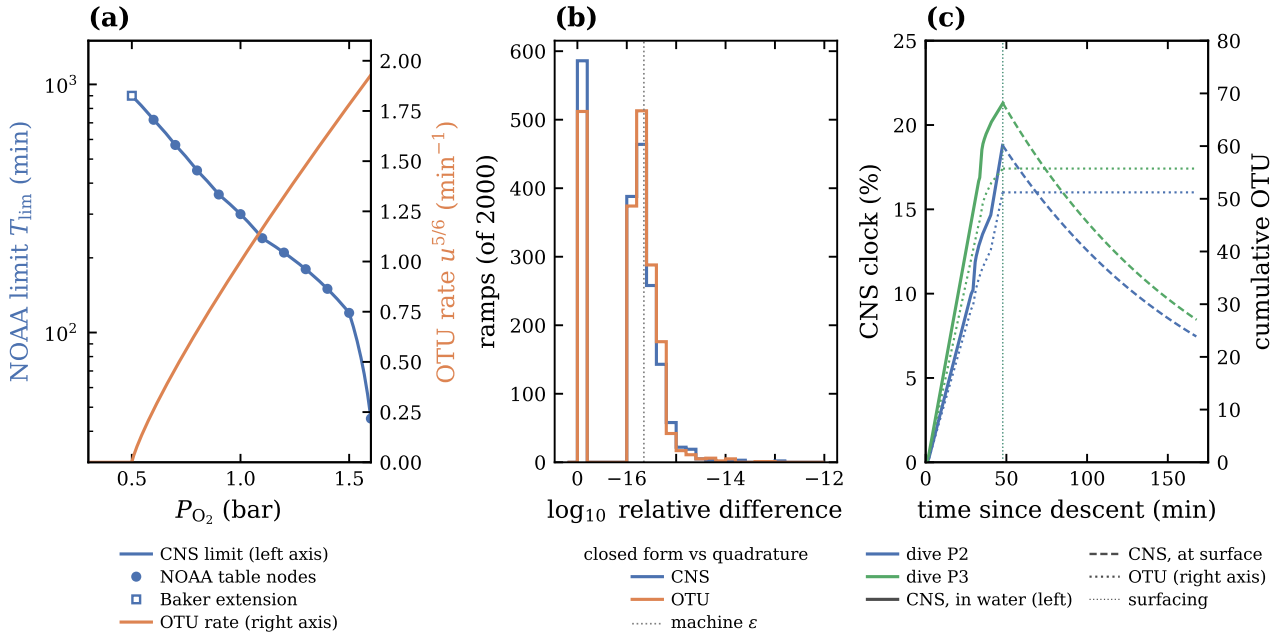


Figure 9: Oxygen exposure. (a) Single-exposure CNS limit against oxygen partial pressure, with the tabulated nodes, the linear continuation to 0.5 bar, and the OTU dose rate. (b) Distribution of the relative difference between the closed-form and quadrature doses over 2000 random ramps, with identical results collected in the bin marked zero. (c) Cumulative CNS clock and OTU through dives P2 and P3 and the following surface interval.

The compressibility of the breathing gases departs from unity by several percent at fill pressures, as Figure 10 shows. At 20 °C the reference model gives $Z = 1.0282$, 1.0499, and 1.1070 for air at 200, 232, and 300 bar, 1.0158, 1.0358, and 1.0896 for EAN32, and 0.9391, 0.9476, and 0.9802 for oxygen, which remains below unity throughout the range. Air and EAN32 pass through a shallow minimum of Z between about 50 and 100 bar, where attractive forces dominate, before the repulsive part of the interaction takes over at higher densities. The interpolated table agreed with CoolProp at 400 random points to a maximum relative error of 1.9×10^{-6} and a median of 3.8×10^{-8} , whereas the Peng-Robinson equation underestimated Z by up to 3.9%, with a median of 2.0%, below 300 bar. The inverse problem recovered the pressure to 2.5×10^{-11} bar. A 12 L cylinder of air holds

2302, 2615, and 3206 L of usable gas at 200, 232, and 300 bar, which is 2.8, 4.8, and 9.8% less than the ideal-gas estimate. For EAN32 the corresponding deficits are 1.6, 3.5, and 8.3%, smaller because the oxygen fraction lowers Z . At lower pressures, where $Z < 1$, the cylinder holds more than the ideal estimate, by up to about 1.3% for air and 1.8% for EAN32 at 20 °C and by up to about 3% for air at 0 °C. A cylinder filled to 232 bar at 45 °C settles to 206.5 bar at 20 °C, 7.2 bar below the ideal-gas prediction of 213.7 bar, because Z decreases together with the temperature. The extra drop grew with both the fill pressure and the fill temperature, from 3.8 bar for 200 bar at 35 °C to 13.7 bar for 300 bar at 55 °C. When the bottom gas of dive P4 is carried in a 12 L cylinder of EAN32 at 200 bar instead of the 18 L cylinder at 232 bar of the canonical plan, the dive requires 3502 L against 2330 L of usable gas. The reference model reaches the 50 bar reserve at 25.80 min and empties the cylinder at 36.92 min of the 65 min plan, against 26.65 and 37.54 min in the ideal-gas estimate.

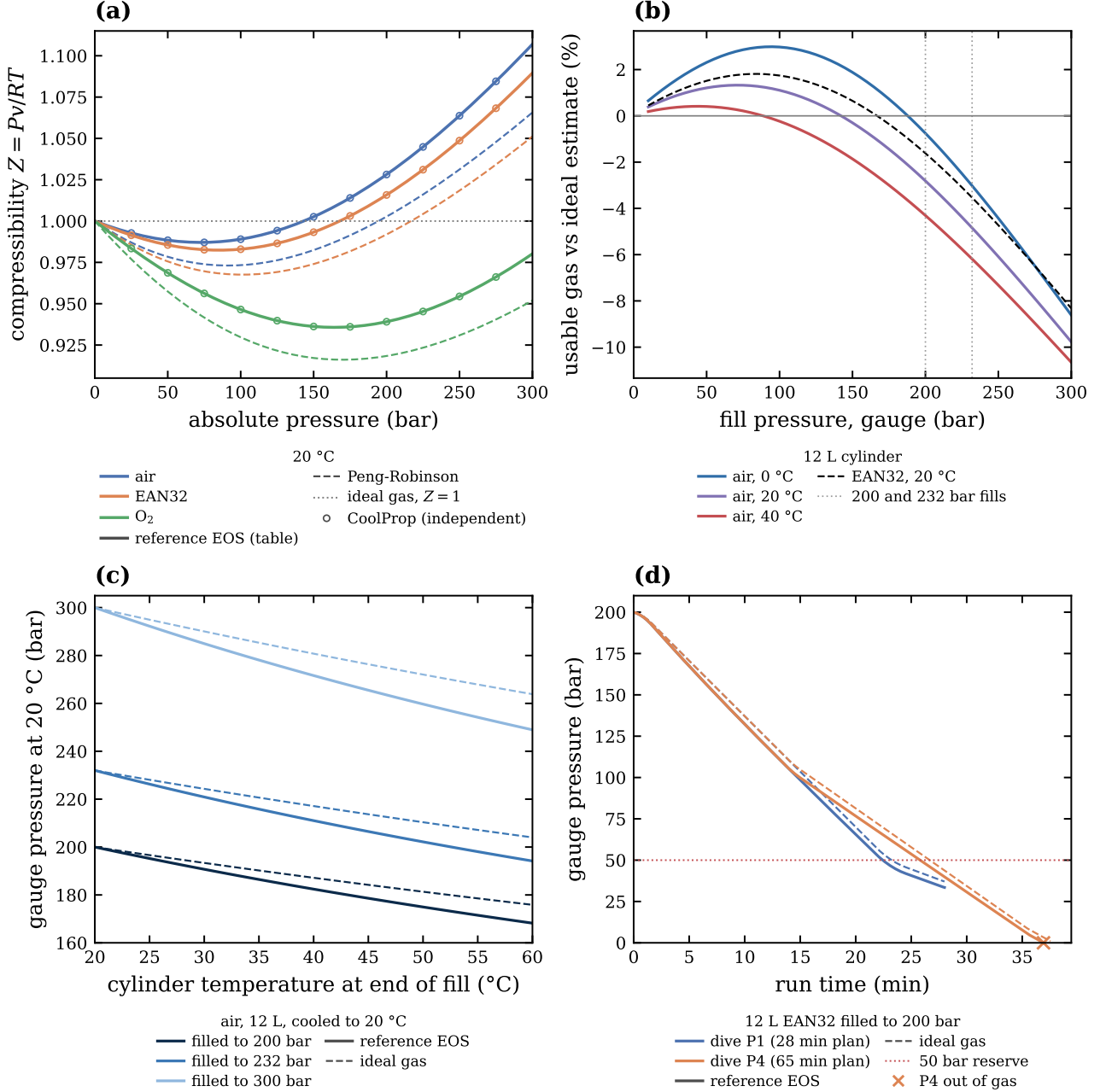


Figure 10: Real-gas thermodynamics of breathing-gas cylinders. (a) Compressibility factor against pressure at 20 °C for air, EAN32, and oxygen from the reference table, the Peng-Robinson equation, and CoolProp. (b) Usable gas of a 12 L cylinder relative to the ideal-gas estimate against fill pressure. (c) Gauge pressure at 20 °C of an air cylinder filled at an elevated temperature, from the reference model and the ideal-gas law. (d) Gauge pressure of a 12 L EAN32 cylinder filled to 200 bar through dives P1 and P4, with the 50 bar reserve and the point at which P4 exhausts the cylinder.

The real-gas correction changes blending instructions by amounts that matter to a blender, as Figure 11 and Table 4 show. For EAN32 to 200 bar from an empty cylinder, the library places the oxygen stop at 27.13 bar, whereas the ideal-gas solution with gauge and absolute pressures kept distinct gives 28.10 bar and the textbook formula (48) gives 27.96 bar. Both simplified instructions call for more oxygen than the real-gas solution, because oxygen with $Z < 1$ packs more densely than the ideal estimate assumes while the air top-off with $Z > 1$ packs less densely. Simulated in a real-gas cylinder, the textbook instruction yields 32.34% oxygen instead of 32%, and the ideal-gas instruction 32.40%, while the library instruction yields the target to 7.4×10^{-14} , or to 4.0×10^{-8} when the fill is simulated with **CoolProp** directly. The bias grows with the target fraction and the fill pressure, from 0.03 percentage points for EAN22 at 200 bar to 1.10 points for EAN40 at 232 bar under the textbook instruction. The textbook formula is slightly less biased than the ideal-gas solution with distinct gauge and absolute pressures, because treating gauge pressures as absolute partly offsets the neglect of Z . The amounts in the library solution satisfied both balances (45) to a relative imbalance of 1.5×10^{-16} for all three equations of state. Partial fills behave as (46) predicts. From a residual of 50 bar of EAN32 the library adds oxygen to 69.4 bar before the air top-off, whereas from a residual of 150 bar of EAN36 the target of EAN32 at 200 bar cannot be reached without draining, and the library prescribes draining to 142.5 bar according to (47).

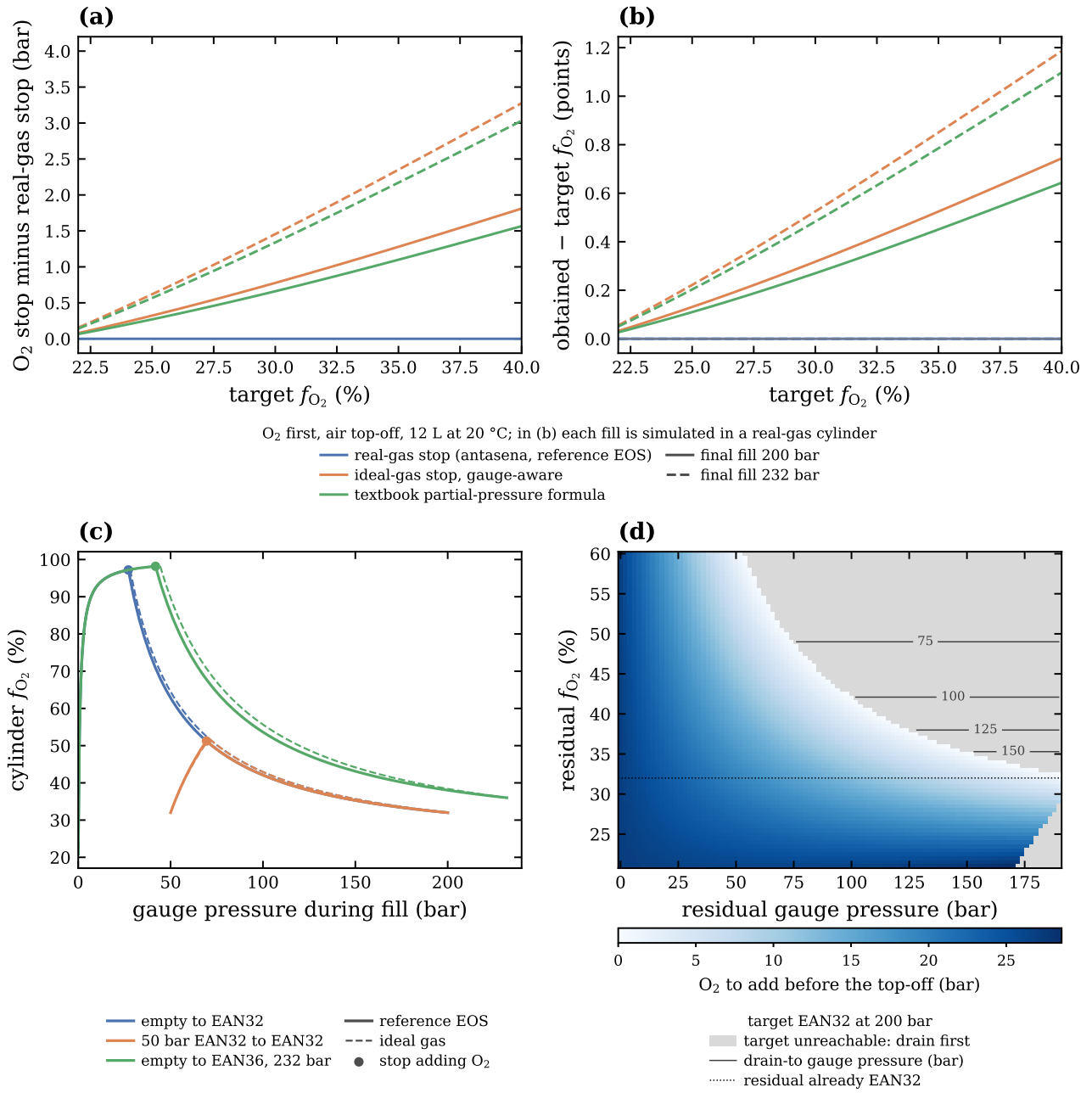


Figure 11: Partial-pressure blending with oxygen first and an air top-off in a 12 L cylinder at 20 °C. (a) Oxygen stop pressure from the ideal-gas solution and the textbook formula, relative to the real-gas stop, against target fraction. (b) Oxygen fraction obtained, minus the target, when each instruction is followed in a real-gas cylinder. (c) Cylinder oxygen fraction against gauge pressure along three fills, from the reference model and the ideal-gas law. (d) Oxygen to be added for EAN32 at 200 bar against the residual pressure and fraction, with the region that requires draining and contours of the drain-to pressure in bar.

Table 4: Oxygen stop pressure (gauge, bar) for oxygen-first blending from an empty 12 L cylinder at 20 °C, and the oxygen fraction (%) obtained when each instruction is followed in a real-gas cylinder.

Target	Oxygen stop (bar)			Obtained (%)		
	Real gas	Ideal gas	Textbook	Real gas	Ideal gas	Textbook
EAN28, 200 bar	17.34	17.93	17.84	28.00	28.24	28.20
EAN32, 200 bar	27.13	28.10	27.96	32.00	32.40	32.34
EAN36, 200 bar	36.89	38.27	38.08	36.00	36.57	36.49
EAN32, 232 bar	30.76	32.57	32.43	32.00	32.65	32.60
EAN40, 232 bar	52.88	56.15	55.91	40.00	41.19	41.10

The guaranteed limits enclose every sampled true value, as Figure 12 shows. For EAN32 analyzed to ± 1 percentage point the MOD at 1.4 bar lies between 32.13 and 34.85 m around the nominal 33.44 m. The interval is asymmetric, extending 1.31 m below and 1.41 m above the nominal value, as the convexity of (11) in the oxygen fraction implies. A gauge that may read 0.5 m shallow lowers the guaranteed value to 31.63 m, and an analyzer accuracy of ± 2 points lowers it further to 30.38 m. The sensitivity is smaller for richer mixtures in absolute terms, the guaranteed MOD of EAN40 at ± 1 point and 0.5 m being 1.35 m below its nominal value, compared with 1.81 m for EAN32, because the derivative (49) scales as $f_{O_2}^{-2}$. For EAN32 at 30 m with ± 1 point and 0.5 m of uncertainty, the NDL at $\gamma_{hi} = 0.85$ lies between 17.40 and 21.15 min around the nominal 19.07 min, and the interval widens rapidly at shallow depths, to between 60.68 and 103.81 min at 18 m with ± 2 points and 1 m. All 20,000 Monte Carlo samples fell inside both intervals. The sampled values covered 99.996% of the MOD interval and 99.08% of the NDL interval, and the lower NDL bound equaled the value at the corner of the box to within rounding, which indicates that the bounds are tight as well as valid.

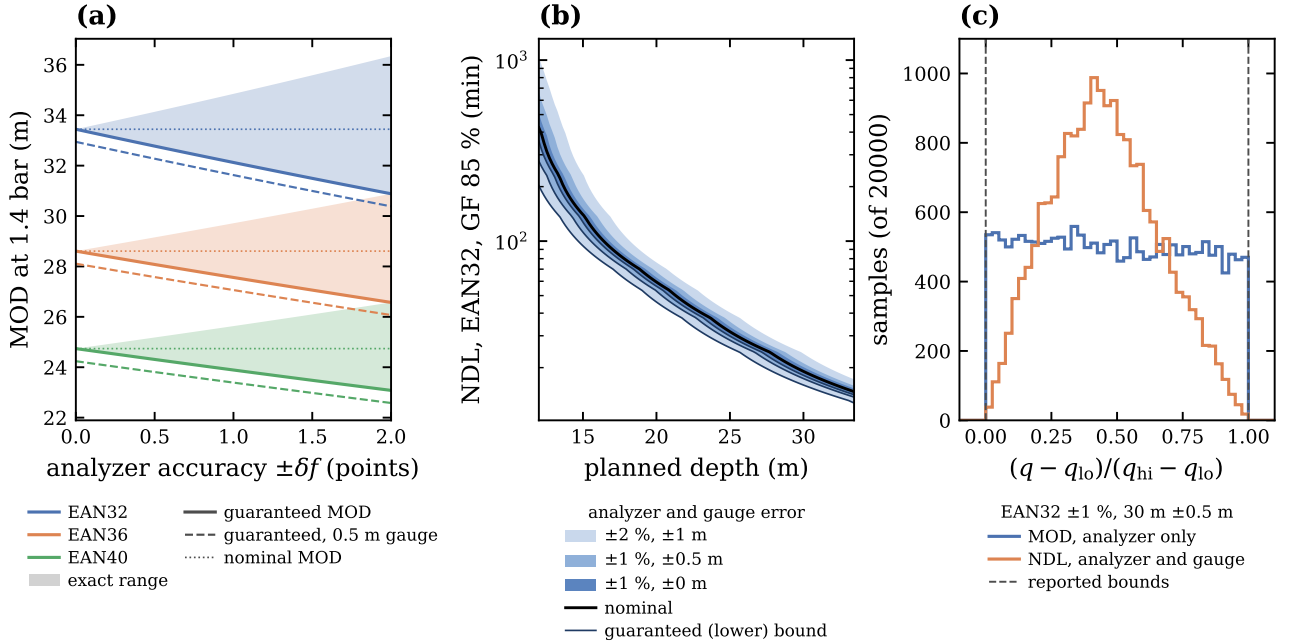


Figure 12: Guaranteed limits under measurement uncertainty. (a) MOD at 1.4 bar against analyzer accuracy for three mixtures, with the exact range, the guaranteed MOD with and without a 0.5 m gauge error, and the nominal value. (b) NDL of EAN32 at $\gamma_{hi} = 0.85$ against depth for three combinations of analyzer and gauge error, with the nominal value and the guaranteed bounds. (c) Position of 20,000 sampled true values within the reported intervals for EAN32 analyzed to ± 1 percentage point at 30 ± 0.5 m.

Residual nitrogen and altitude modify the limits as Figure 13 shows. After dive P1 and a surface interval of 60 min, the tensions of the fast compartments had returned close to surface saturation, whereas the slow compartments retained most of their uptake, and the CNS clock had fallen from 12.11% to 7.63%. The repeat of P1 then accumulated a clock of 19.73% and 65.0 OTU, the latter being a daily total that is not reduced at the surface, and its NDL counted with the descent was 18.26 min against 18.29 min for the first dive. The NDL of a repeat dive recovers within about an hour for the deeper profile and more slowly for the shallower ones. After 20 min of EAN32 at 30 m it grows from 5.61 min with no interval to 18.53, 19.03, and 19.07 min after 30, 60, and 120 min. After 20 min of EAN32 at 24 m it grows from 18.50 to 35.06 min after 60 min and to within

0.02 min of the first-dive value of 35.90 min after 120 min. After 40 min of air at 18 m it grows from 8.79 min to 39.54 min after 60 min and to the first-dive value of 42.94 min after about four hours. The slower recovery of the shallow profiles follows from the structure of the NDL, since shallow limits are controlled by slow compartments, which also release their excess slowly. In fresh water at altitude, with the diver acclimatized, the NDL of EAN32 at 30 m decreases from 20.43 min at sea level to 17.71, 15.71, and 14.19 min at 1000, 2000, and 3000 m, and at 18 m it decreases from 82.77 to 55.91 min, because the lower surface pressure lowers every surfacing limit. The surface interval computed by the library agreed with the independent integration to 2.4×10^{-15} bar, and the decay of the CNS clock agreed exactly with the half-time law.

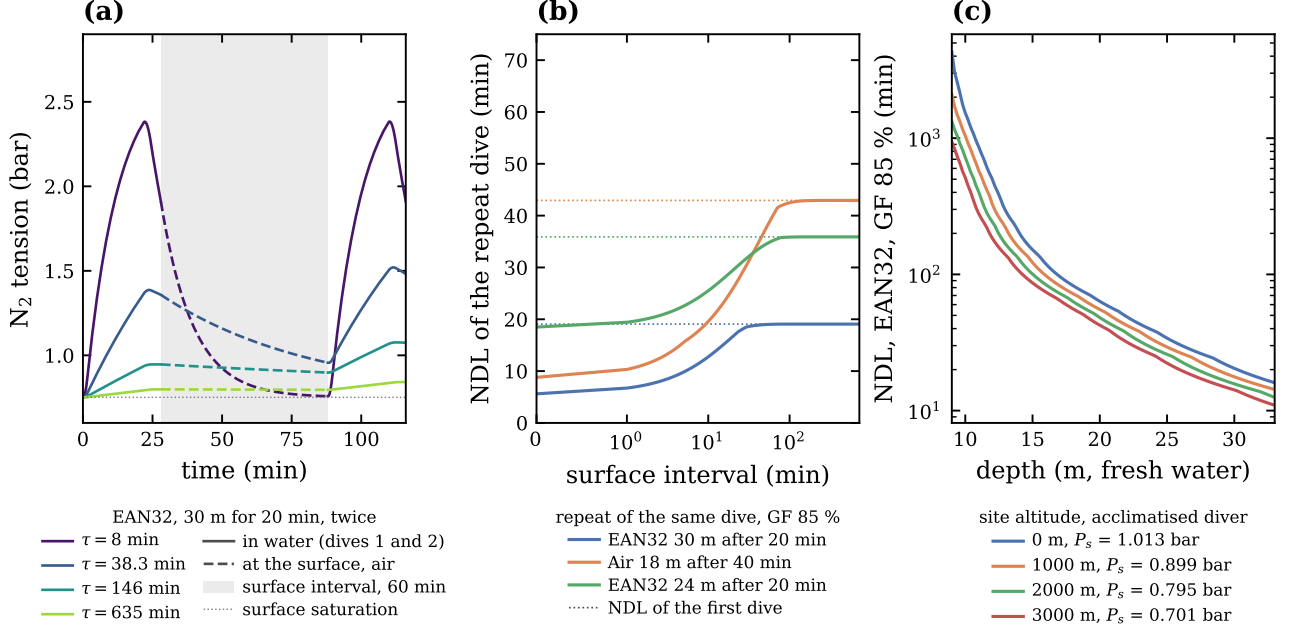


Figure 13: Repetitive dives and altitude. (a) Nitrogen tensions of four compartments through two repetitions of dive P1 separated by a 60 min surface interval. (b) NDL of a repeat dive against the length of the surface interval for three first-dive profiles at $\gamma_{hi} = 0.85$, with the NDL of the first dive. (c) NDL of EAN32 against depth in fresh water at four site altitudes, for an acclimatized diver.

The verification is summarized in Table 5. All 79 independent checks across the thirteen analyses passed. Separately, the library test suite of 113 tests passed under a virtual display.

Table 5: Independent verification of the library. Each check compares a library result with a computation written without reusing the library mathematics, or compares an interface with the library.

Analysis	Independent reference	Checks passed
GUI	Values read from the interface vs. library calls	15 / 15
CLI	Parsed output vs. library calls	17 / 17
Depth limits	Hand formulas, pre-image identity	2 / 2
Equivalent depths	Analytic offset and residual, tensions along P4	3 / 3
Tissue kinetics	RK4, 60-digit arithmetic, splitting	4 / 4
NDL	Bisection on the surfacing condition	1 / 1
GFs	One-second RK4 ceiling audit	6 / 6
Dive plans	Ceiling against depth on every trace	4 / 4
Oxygen exposure	Adaptive quadrature, symmetry, additivity, table	15 / 15
Real gas	CoolProp at 400 points, inverse round trip, gas budget	3 / 3
Blending	Simulated fills with the table and with CoolProp	3 / 3
Uncertainty bounds	Monte Carlo sampling, corner values, interval product	4 / 4
Repetitive dives	RK4 surface interval, CNS decay	2 / 2
Total		79 / 79

4 Discussion

The central property of the library is that a dive plan composed of segments of fixed gas and linearly varying depth admits closed-form solutions for every quantity that the plan requires. The tissue equation (16) is linear with a linear forcing on each segment, the oxygen doses (33) and (35) are integrals of known functions of a linearly varying partial pressure, and the gas use (37) is an integral of a linear pressure. The numerical error of a plan is therefore confined to floating-point rounding, which the comparisons with RK4 integration, 60-digit arithmetic, and adaptive quadrature bound at the level of 10^{-13} or below. The same closed forms make the plan exactly divisible. Splitting a segment, a property that a time-stepping integrator satisfies only approximately, leaves the result unchanged to within one unit in the last place, and this semigroup property is what allows the planner to simulate trial ascents on copies of the tissue state without accumulating error. A practical consequence concerns planners that subdivide an ascent into steps of a few seconds in order to track the ceiling continuously. Such planners operate in the regime $k_i t \ll 1$, in which the textbook Schreiner expression (21) cancels catastrophically. For a one-second step and the 5-min compartment, $k_i t \approx 2.3 \times 10^{-3}$, at which the condition number $2/(k_i t)^2$ is close to 4×10^5 and the textbook form retains only about ten of the sixteen significant digits of the uptake. The loss is far below the uncertainty of any decompression model and is unlikely to affect a single plan, but it would accumulate over many short steps, and it illustrates in a physically meaningful setting why algebraically equivalent expressions are not numerically equivalent [16, 24]. The rearranged form (22) removes the loss at no additional cost, and the condition-number argument that motivates it applies to any exponential relaxation driven by a ramp.

The derivation separates the parts of the calculation that follow from physical law from those that rest on empirical models, and this separation determines what the verification can and cannot establish. Hydrostatics, Dalton’s law, perfusion-limited transport, the equation of state, and the mole balance of blending are consequences of conservation laws and thermodynamics under stated assumptions, and their assumptions can be quantified. The dynamic pressure of currents, the compressibility of water, and the non-ideality of the breathed gas each affect the results at the level of 10^{-3} or less. The M-values (24), the GFs (25), the oxygen-limit table, and the OTU exponent are instead fitted or conventional, and the transport model itself omits the formation and growth of bubbles, which is the process that decompression sickness involves [41]. Agreement between the library and its independent re-implementation therefore establishes that the stated models are computed correctly. It does not establish that the models describe the physiological response of a diver, which is a question of validation rather than verification. Bubble nucleation models, which postulate stabilized gas micronuclei whose growth on decompression sets the tolerable pressure reduction, and probabilistic models of decompression outcome fitted to large sets of recorded dives address that second question from different starting points [46, 44], and they lie outside the scope of the library. The assumptions of the compartment model also fail in identifiable situations. When gas switches bring a different inert gas into contact with a tissue, diffusion between neighboring tissues can produce local supersaturation that a set of independent, perfusion-limited compartments cannot represent, as analyses of inner-ear decompression sickness after gas switches have shown [12]. Such cases fall outside nitrox diving, in which nitrogen is the only inert gas, but they mark the boundary of the first assumption used in reducing (14).

Several of the results bear directly on common teaching material. The rounded convention of 1 bar at the surface and 10 m per bar overestimates the MOD by 0.15 to 0.42 m relative to sea water. The difference is small, but it always lies in the non-conservative direction, and it places the oxygen decompression stop at 6 m, where the oxygen partial pressure in sea water slightly exceeds the 1.6 bar limit. The classical EAD is likewise always shallower than the alveolar-consistent value, by 6 to 15 cm for common mixtures. The offset is negligible in practice, but the alveolar-consistent definition carries a stronger statement than the classical one, namely that an entire nitrox profile is equivalent to an air profile in every compartment and at every instant. That statement follows from nothing more than the linearity and uniqueness of (16), and the analysis confirms it to the last bit. It also clarifies the status of the classical EAD, which is exact for a model without water vapor and approximate, with the bounded error (32), for the model actually used. The structure of the NDL corrects a frequent simplification, that the fastest compartment always limits a short dive. The limit is set by the competition between the rate constant k_i and the surfacing tolerance $M_i(P_s)$, which both decrease with the half-time, so that the controlling compartment shifts across all sixteen as the depth and the mixture change, and the most tolerant fast compartment may never reach its limit at all on nitrox. The near-equality of the times t_i of neighboring compartments close to the minimum suggests, further, that statements about which compartment controls a dive are sensitive to small changes in the inputs, whereas the NDL itself, as the minimum of a set of smooth functions, varies continuously.

The GF results show that the TTS is governed mainly by γ_{hi} , through the shallow stops, while γ_{lo} mainly sets the depth of the first stop. This division of roles follows from (28), since γ_{lo} acts only near the first stop, where it determines how early the ascent is halted, while γ_{hi} acts near the surface, where the slow compartments must off-gas against a small pressure difference and where most of the stop time accumulates. The planner keeps the realized supersaturation below the planned GF at every second of the ascent, which the independent audit confirms. Two behaviors of the present implementation deserve mention because a user may meet them. When the ascent from a decompression dive needs no wait above the safety-stop depth, as for dive P2 at (1.00, 1.00), the safety stop at 5 m is inserted before the first mandatory stop, and the switch to oxygen occurs at that depth, where the oxygen partial pressure is 1.52 bar. The schedule remains within the ceiling and the oxygen limit.

When off-gassing during the ascent removes every mandatory stop, as for dive P1, the reported first-stop depth retains the value computed from the γ_{10} ceiling at the end of the bottom phase, so that the attribute indicates where decompression would have begun rather than a stop that the plan contains. Dive P1 also illustrates that the NDL defined by an instantaneous ascent is a conservative criterion, since the off-gassing that occurs during a real ascent can clear a ceiling that the instantaneous criterion would place below the surface.

The two oxygen indices differ in the way they weight an exposure. The CNS clock weights most heavily the exposures near 1.6 bar, where the single-exposure limit falls steeply, whereas the OTU accrues at a dose rate that rises almost linearly with the oxygen partial pressure above 0.5 bar, so that it is dominated by long exposures at moderate partial pressure. The comparison of dives P2 and P3 shows that the bottom gas, rather than the decompression gas, may dominate the dose, so that the choice of a richer bottom mixture to shorten decompression is constrained by the oxygen clock as well as by the MOD. Both indices are linear functionals of the exposure history, which is what makes them additive over segments and symmetric under time reversal of a ramp, but the physiological response is neither additive nor deterministic. The tolerance to oxygen varies widely between and within individuals [11], and statistical models of toxicity treat the dose-response relation probabilistically [1]. The indices computed by the library are therefore best read as planning budgets with the same status as the GFs.

The real-gas results quantify a correction that diver training rarely treats. The usable gas of a full air cylinder is 2.8 to 9.8% smaller between 200 and 300 bar than planning with PV_c suggests, and a hot fill loses several bar more on cooling than the ideal-gas law predicts, both because Z exceeds unity for air at fill pressures [31]. Oxygen, whose compressibility stays below unity, behaves in the opposite sense, and the combination makes the textbook blending formula systematically rich. The resulting bias, about 0.3 percentage points for EAN32 at 200 bar and up to 1.1 points for EAN40 at 232 bar, or 1.2 points for the ideal-gas solution, lies within or close to the tolerance of common oxygen analyzers, so that the practice of analyzing every fill remains necessary, but it is systematic rather than random and grows with the target fraction and the fill pressure, so that a blender who calibrates a procedure against analyzer readings is in effect correcting for the real-gas behavior empirically. The Peng-Robinson equation underestimates Z by about 2 to 4% for these mixtures [34], which is comparable to the correction it is meant to supply. It serves in the library as a transparent illustration of a cubic equation of state, while the reference table, which reproduces the multiparameter model to about 2×10^{-6} , carries the quantitative results [30, 4]. The gas-budget results show that the real-gas correction shifts the time at which a reserve is reached by close to a minute on a long dive, an amount that matters mainly when a plan is already marginal.

The guaranteed bounds make the effect of measurement error explicit. An analyzer accurate to ± 1 percentage point and a gauge accurate to 0.5 m together reduce the guaranteed MOD of EAN32 by 1.8 m and widen the NDL at 30 m to about $\pm 10\%$ of its nominal value. These effects are several times larger than the difference between the sea-water and rounded conventions and about twenty times larger than the EAD offset of EAN32, which places the many small corrections examined above in perspective. The monotonicity arguments (49) and (50) make the bounds exact without sampling, and the Monte Carlo results indicate that they are also tight. For teaching, the bounds connect a familiar instrument specification to a planning consequence through a single derivative, which may help to explain why analyzer accuracy and gauge calibration are emphasized in nitrox training.

The library also has a pedagogical structure that follows from the derivation. The chain from the Cauchy momentum equation to the MOD, from the advection-diffusion equation to the NDL, and from the Helmholtz free energy to a blending instruction each passes through a small number of explicit assumptions, and each assumption can be relaxed or tested in isolation. The Learn tab of the GUI and the `--why` flag of the CLI expose that chain with the user's own numbers, so that a student can trace a planning result back to its governing equation and see which of its ingredients are physical law and which are convention. Because every closed form is verified against an independent numerical method, the same material can also serve to introduce numerical verification itself, including convergence orders, reference arithmetic at high precision, adaptive quadrature, and the distinction between truncation and rounding error.

The analysis has limitations that should be stated plainly. The library treats oxygen and nitrogen only, with argon counted as nitrogen, and excludes helium mixtures and closed-circuit rebreathers. The ZH-L16C model with GFs is one decompression model among several, and the perfusion-limited compartment omits diffusion limitation within tissues, gas exchange between neighboring tissues, and the carbon dioxide term of the alveolar gas equation [29, 14]. The single-exposure oxygen limits and their extension below 0.6 bar, the 90 min surface half-time of the CNS clock, and the absence of surface recovery for the OTU are planning conventions rather than results of a model with a documented physiological basis. The water density is held constant, the SAC is normalized to 1.01325 bar irrespective of altitude, the ventilation is assumed independent of depth and workload, fills are assumed isothermal at the settled temperature, and the reference equation of state is tabulated only between -20 and 80 °C and up to 400 bar. The guaranteed bounds are exact in real arithmetic but are not rounded outward, so they carry rounding errors of order 10^{-15} [33]. The interface audits compared the CLI and the GUI with the library at representative rather than exhaustive inputs. Finally, all cases studied are constructed planning problems, and the library has not been compared with logged dives or with the output of commercial dive computers.

5 Conclusions

This study presented **antasena**, an open-source Python library for nitrox diving calculations, together with a CLI and a GUI that display the derivation of every result with the user’s inputs substituted. The working equations were derived from the momentum balance of water at rest, the advection-diffusion equation for dissolved nitrogen in perfused tissue, and the Helmholtz free energy of a gas mixture, with each reducing assumption stated where it enters and quantified where possible. Because a dive plan is a sequence of segments of fixed gas and linearly varying depth, the library evaluates tissue tensions, oxygen doses, and gas consumption in closed form without time stepping, and an independent re-implementation based on numerical integration, high-precision arithmetic, adaptive quadrature, a reference equation of state, and simulated fills reproduced its results to the limits of double precision, including the values printed by the CLI and displayed by the GUI. The analysis showed that the textbook form of the Schreiner solution loses accuracy to cancellation on short segments while a rearranged form does not, that the alveolar-consistent EAD makes a nitrox profile equivalent to an air profile in every compartment whereas the classical definition is slightly shallower, that the compartment controlling the NDL shifts across the whole set of compartments with depth and mixture, that ideal-gas planning overstates the usable gas of a full cylinder and makes the textbook blending formula systematically rich, and that measurement uncertainty outweighs most of the conventions that distinguish one planning method from another. These findings describe the adopted models and their numerical implementation rather than the physiological risk to any diver. Future work may extend the library to helium mixtures and closed-circuit rebreathers, add alternative decompression models for comparison, and compare its schedules with logged dives, with the library, the supplementary scripts, and their outputs released openly so that the present results can be reproduced and extended.

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Data Availability

The **antasena** library source code is available on GitHub at <https://github.com/sandyherho/antasena>, from the Python Package Index at <https://pypi.org/project/antasena/>, and as an archived release at <https://doi.org/10.5281/zenodo.23410341>, with documentation at <https://antasena.readthedocs.io>. The supplementary analysis scripts that reproduce every figure, table, and verification result of this study, together with their outputs, are available on GitHub at https://github.com/sandyherho/suppl_antasena and as an archived release at <https://doi.org/10.5281/zenodo.23546931>. The library and the supplementary scripts are released under the MIT license.

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