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Abstract. Groundwater is the largest store of liquid fresh water, yet because it is hidden beneath the surface, it is primarily understood through models of porous-media flow. Richards' equation, a physically faithful description of variably saturated groundwater flow, remains costly for basin-scale groundwater modelling. Consequently, operational models use reduced physics, either by assuming linearity or by using approximate solutions. Here, we introduce a scalable, open-source, three-dimensional solver for variably saturated groundwater flow implemented in G-ADOPT, an automated finite-element platform built on Firedrake. The implementation combines a locally conservative discontinuous Galerkin discretisation, Newton iteration with an exact Jacobian, implicit time integration through the Irksome library, used here with Backward Euler, and vertically lumped multigrid preconditioners designed for the extreme aspect ratios of production scenarios. We verify accuracy, convergence, and strict local mass conservation on two- and three-dimensional benchmarks, and apply the solver to the Lower Murrumbidgee floodplain in New South Wales, Australia. The model covers approximately 30 000 km², contains several hundred million degrees of freedom, and uses public stratigraphic, rainfall, and water-table data. Our scaling experiments show that the effectiveness of the preconditioner is controlled primarily by the physical regime in which the model is integrated. A single-level method (e.g., block-Jacobi ILU(0)) suffices at the short time steps of an infiltration study, but at seasonal time steps, the admissible time step decreases significantly under horizontal refinement. In a near-saturated profile without specific storage, such single-level methods do not converge at any tested step size. By contrast, a vertically lumped coarse-space preconditioner takes the full three-month step on every mesh, requiring four to ten linear iterations per Newton step and reducing wall-clock time by a factor of two to four. It also integrates the near-saturated profile on every mesh. Because basin simulations use seasonal scenarios, we report, for the first time with a discontinuous Galerkin discretisation and a vertically lumped multigrid preconditioner, a seasonal, basin-scale, fully three-dimensional solution of the full Richards' equation on an unstructured, terrain-following mesh. This study demonstrates basin-scale capability rather than predictive skill for the synthetic Murrumbidgee test case, whose hydraulic parameters require calibration. The algorithmic differentiation in the underlying Firedrake and pyadjoint frameworks supports future inverse and data-assimilation studies, including the calibration needed to assess predictive skill.

1 Introduction

25 Groundwater is the largest store of liquid fresh water on the planet, accounting for close to 99 per cent of all unfrozen fresh water, supplying about half of the water withdrawn worldwide for domestic use (UNESCO, 2022) and over 40 per cent of that consumed by irrigation (Siebert et al., 2010). Access to clean water and sanitation is set out as the sixth of the United Nations Sustainable Development Goals (United Nations General Assembly, 2015), and groundwater underpins a large share of that provision, particularly across regions where surface water is scarce or unreliable (UNESCO, 2022). Yet among the major
30 components of the hydrological cycle it remains one of the most crudely represented in current global modelling frameworks (Condon et al., 2021; Fan et al., 2013). These shortcomings are becoming increasingly consequential as a changing climate alters precipitation patterns, drives sea-level rise, intensifies extraction, and elevates the risk of contaminant transport, so that the demand for predictive capability grows precisely as the systems themselves come under mounting stress (Bear, 2013; Earman and Dettinger, 2011; Kumar, 2017; Taylor et al., 2013).

35 The full subsurface state and flow field cannot be observed directly. Instead, hydraulic states are sampled at scattered boreholes, while the properties that govern flow, such as hydraulic conductivity and storativity, are known, if at all, at only a handful of locations and over a narrow range of conditions (Ojha et al., 2015). These observations cannot simply be interpolated into a coherent flow field, because water between observation points follows physical laws rather than an assumed spatial smoothness. A physical model is therefore required to simulate the evolution of pressure and water content from material properties, initial
40 conditions, and forcing. This forward problem also underlies inverse modelling: observations constrain uncertain properties and states through repeated forward solves and their sensitivities (Carrera et al., 2005; Zhou et al., 2014; Cockett et al., 2018). The present work addresses that forward component.

The movement of water through a saturated porous medium was first quantified by Darcy (1856), whose empirical law relates the specific discharge to the gradient of hydraulic head and provides, in effect, the momentum balance for slow flow through
45 soil and rock; combined with the conservation of mass it yields the classical groundwater-flow equation. In the fully saturated regime this equation is essentially linear, the hydraulic conductivity being a fixed property of the medium, and it is on this basis that mature and efficient tools such as MODFLOW (Harbaugh, 2005; Langevin et al., 2017) have been built and adopted as de facto standards. The subsurface at large, however, is not fully saturated. Above the water table lies the unsaturated, or vadose, zone, in which water and air share the pore space and the water is held under negative pressure by capillary and adsorptive
50 forces. There, the volumetric moisture content varies with the pressure head through the strongly curved soil-water retention relation, and the hydraulic conductivity ceases to be a constant, falling by many orders of magnitude as the soil dries and rising again as it wets (Van Genuchten, 1980). Because both the stored water and its capacity to move depend on the pressure head that the flow itself sets up, the governing balance becomes nonlinear, and the linear, saturated description that underlies MODFLOW-type codes no longer holds. Capturing variably saturated flow therefore requires a governing law in which storage
55 and conductivity respond to the state of the soil, which is precisely what Richards' equation provides.

First stated by Richards (1931), and more properly termed the Richardson–Richards equation in recognition of Richardson's earlier statement of the same balance (Zha et al., 2019), it expresses the conservation of water in a variably saturated medium

as

$$\frac{\partial \theta}{\partial t} = \nabla \cdot K(h) \nabla (h + z), \quad (1)$$

60 where θ is the volumetric moisture content, K the hydraulic conductivity, h the pressure head and z the elevation head representing gravity. Its nonlinearity, already implied by the dependence of θ and K on h , is accompanied by degeneracy at both saturation extremes. In dry soil $K \rightarrow 0$, so that the flux collapses and infiltration advances as a sharp front (Celia et al., 1990), whereas in fully saturated regions θ becomes constant, the time derivative vanishes and the equation turns locally elliptic. This mixed, shifting character is what makes Richards' equation notoriously hard to solve, and considerable effort has gone into
65 mitigating it, through adaptive regularisation of the constitutive relations (Févote et al., 2024) and adaptive temporal linearisation (List and Radu, 2016; Stokke et al., 2023). Resolving flow across heterogeneous and discontinuous media, where the conductivity jumps sharply between layers, remains particularly challenging (Kirkland et al., 1992; Miller et al., 1998); for a comprehensive survey of the numerical approaches we refer the reader to Farthing and Ogden (2017) and Zha et al. (2019).

Solving Richards' equation numerically, as for other Earth system processes, turns broadly on three choices, namely how
70 space is discretised, how time is advanced, and how the resulting computation is made to scale. Scalability is important since such applications are often large either because the domain itself is extensive or because the required resolution is high. Among the available spatial discretisation methods, the finite element method is the most flexible, accommodating the unstructured and deformed meshes needed to follow real topography and stratigraphy while leaving the polynomial basis and its order free to be chosen (Rivière, 2008). Within this family, a continuous Galerkin approximation, though flexible, is prone to spurious
75 oscillations where sharp infiltration fronts develop in dry soils (Ju and Kung, 1997). The discontinuous Galerkin method is more robust in this respect. Because its approximation space is defined independently on each element, it is locally conservative and tolerant of the sharp gradients and material discontinuities that the equation produces, which makes it well suited to representing wetting fronts as they advance through layered soils (Clément et al., 2020, 2021, 2023; Li et al., 2007). The temporal discretisation is equally important. Because the Richards' equation is stiff and strongly nonlinear, explicit marching
80 is impractical, and one turns instead to implicit schemes that remain stable at the time steps of interest while offering enough order to keep long simulations accurate, each step requiring the solution of a nonlinear system (Kavetski et al., 2001; Farthing and Ogden, 2017). With space and time under control, the remaining difficulty is one of scale. Most solutions of Richards' equation are still reported in one or two dimensions, or in three dimensions over domains no larger than a few metres, whereas the problems that matter for water resources concern entire basins and continents (Condon et al., 2021; Gleeson et al., 2021).
85 Reaching those sizes requires that the linear solves within each nonlinear step remain efficient as the mesh is refined. This in turn depends on multigrid preconditioners whose performance is complicated by the extreme horizontal-to-vertical aspect ratios of groundwater domains and therefore calls for anisotropy-aware strategies such as semi-coarsening (Tuminaro et al., 2016). It is largely because of these scalability constraints that many operational basin-scale models fall back on reduced physics, neglecting gravity, assuming steady state, or omitting the unsaturated zone altogether (Hughes et al., 2017; Zhou and
90 Li, 2011). The present work addresses this gap.

Here, we develop the forward component in G-ADOPT (the Geoscientific ADjoint Optimisation Platform) (Davies et al., 2022; Ghelichkhan et al., 2024; Scott et al., 2025; Gibson et al., 2026). G-ADOPT is an open-source framework for simulating geoscientific flows with adjoint-based methods, built on Firedrake (Ham et al., 2023), PETSc, and pyadjoint (Mitusch et al., 2019), and already applied to mantle and lithosphere dynamics and glacial isostatic adjustment. Within this framework, we

95 add a solver for Richards’ equation that operates across scales, from metre-scale verification problems to basin-scale three-dimensional domains. The solver combines a locally conservative discontinuous Galerkin discretisation with Newton iteration on an exact, symbolically derived Jacobian, and uses Irksome (Farrell et al., 2021) for time integration. Although Irksome supports higher-order Runge–Kutta methods, all reported results use Backward Euler. The formulation stabilises both diffusive and gravity-driven fluxes while retaining strict local mass balance, accommodates the heterogeneous and discontinuous

100 media typical of the subsurface, and employs anisotropy-aware multigrid preconditioning suited to the extreme aspect ratios of groundwater domains. We show that, in the regimes considered here, preconditioner performance is better described by a column-integrated diffusion number than by the mesh aspect ratio alone. A simple single-level method suffices at short time steps, whereas a vertically lumped coarse space is required at the seasonal time steps and for the near-saturated profiles typical of basin models. We verify accuracy, convergence, and strict mass conservation on a suite of two- and three-dimensional

105 benchmarks, and we demonstrate efficiency and practical utility at scale through weak- and strong-scaling studies and a basin-scale case study of the Lower Murrumbidgee floodplain. The remainder of the paper is organised as follows. Section 2 presents the mathematical formulation, the constitutive relations, and the DG weak form together with the time-integration strategy; Section 3 reports the verification benchmarks establishing accuracy, convergence, and strict mass conservation; and Section 4 develops the scalable preconditioning strategy and applies it through scaling studies and the Lower Murrumbidgee case study

110 to demonstrate efficiency and utility at scale. Section 5 interprets the preconditioner results, situates our modelling choices against existing groundwater solvers, and outlines the route to inverse modelling. Section 6 concludes.

2 Mathematical Formulation and Discretisation

Richards’ equation describes the flow of water through a variably saturated porous medium under the combined action of gravity and capillarity. It can be regarded as a reduction of the general two-phase (air–water) description of porous-media flow,

115 obtained by assuming that the air phase is passive. Because the mobility contrast between water and air is large, the gas phase offers negligible resistance to flow and remains everywhere at atmospheric pressure, so that its dynamics may be disregarded and only the liquid-phase mass balance needs to be retained (Farthing and Ogden, 2017; Zha et al., 2019). We further take the porous skeleton to be non-deforming and the water to be of constant density, and we neglect thermal effects and water-vapour transport.

120 Under these assumptions the volumetric liquid flux $\mathbf{q}$ is given by the Darcy–Buckingham law

$$\mathbf{q} = -\frac{k_{rl}(S_l)k}{\mu_l}(\nabla p_l + \rho_l g \mathbf{e}_z), \quad (2)$$

where k is the absolute permeability of the porous skeleton, $k_{rl}(S_l)$ is the dimensionless relative permeability of the liquid phase, S_l is the liquid saturation, μ_l and ρ_l are the liquid viscosity and density, g is the gravitational acceleration, p_l is the

liquid pore pressure, and e_z is the upward unit vector. Introducing the pressure head $h = p_l/(\rho_l g)$, which has units of length, and combining (2) with the liquid-phase mass balance $\partial\theta/\partial t + \nabla \cdot \mathbf{q} = s$ yields Richards' equation. Following Farthing and Ogden (2017) and Zha et al. (2019), it admits three equivalent forms at the continuous level. In its *mixed* form it reads

$$\frac{\partial\theta}{\partial t} - \nabla \cdot (K \nabla(h+z)) = s, \quad (3)$$

where θ is the volumetric moisture content, z is the vertical coordinate representing the elevation (gravitational) head, s is a volumetric source/sink term with units $[T^{-1}]$, and $K = k_{rl}(S_l) \rho_l g / \mu_l$ is the hydraulic conductivity $[LT^{-1}]$. Using the soil-water retention relationship $\theta = \theta(h)$ to eliminate the moisture content in favour of the head gives the *head-based* form

$$C(h) \frac{\partial h}{\partial t} = \nabla \cdot (K \nabla h) + \frac{\partial K}{\partial z} + s, \quad C(h) \equiv \frac{\partial\theta}{\partial h}, \quad (4)$$

in which $C(h)$ is the specific moisture capacity $[L^{-1}]$. Conversely, eliminating the head in favour of the moisture content casts the capillary flux as a nonlinear diffusion, $\nabla \cdot (D \nabla \theta)$ with soil-water diffusivity $D = K/C$, giving the *moisture-content* form (Zha et al., 2019). The three forms are equivalent for smooth solutions in a homogeneous medium, but they cease to be interchangeable both once discretised and across a material interface, where the retention curve changes discontinuously. There the head h stays continuous while the moisture content θ jumps, so the moisture-content form cannot represent flow across the interface and only the head-based and mixed forms remain valid.

We solve the form in (3) in this study, as it is the formulation that conserves mass and remains valid across material interfaces. That is, although the chain rule employed in deriving the head form gives the exact identity $\partial\theta/\partial t = C(h) \partial h/\partial t$ at the continuous level, this equality no longer holds once the time derivative is discretised, because $C(h)$ is strongly nonlinear. Evaluating the storage term as $C(h^{n+1} - h^n)/\Delta t$ is not equal to the true moisture increment $(\theta^{n+1} - \theta^n)/\Delta t$ unless the head change over the step is vanishingly small, so a direct discretisation of the head-based form (4) accumulates mass-balance errors. Discretising the mixed form instead retains $\partial\theta/\partial t$ as the primary storage term and preserves global mass conservation even with low-order time integration (Celia et al., 1990; Farthing and Ogden, 2017; Zha et al., 2019). This has become the standard route to a conservative scheme for Richards' equation, and it is the one we adopt.

In fully saturated regions the retention curve flattens, so that $C(h) = 0$ and the head-based form (4) degenerates into a locally elliptic problem. To represent elastic storage in the saturated zone we augment the storage coefficient with a specific-storage term, following Farthing and Ogden (2017), and define

$$C_{\text{eff}}(h) \equiv C(h) + S_s S(h), \quad S(h) = \frac{\theta(h) - \theta_r}{\theta_s - \theta_r}, \quad (5)$$

where S_s is the specific storage $[L^{-1}]$, $S(h)$ is the effective (normalised) saturation, and θ_r and θ_s are the residual and saturated moisture contents. Since we discretise the mixed form for mass conservation, this augmentation enters as an explicit elastic-storage term rather than through C , so that the equation we actually solve is

$$\frac{\partial\theta}{\partial t} + S_s S(h) \frac{\partial h}{\partial t} - \nabla \cdot (K \nabla(h+z)) = s, \quad (6)$$

in which the retention storage is kept in the conservative form $\partial\theta/\partial t$ while the elastic term $S_s S(h) \partial h/\partial t$ is the mixed-form counterpart of replacing C by C_{eff} in the head-based form.

2.1 Finite element discretisation

Let $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) denote the computational domain (the porous medium of interest), with Lipschitz boundary $\partial\Omega$ and outward unit normal $\mathbf{n}$. We partition Ω into a mesh of non-overlapping cells e and seek the pressure head in a finite-dimensional space V of piecewise polynomials of degree p defined cell by cell, using the same space for the test functions. The discrete problem is then to find $h \in V$ such that the weak residual defined below vanishes for all $v \in V$. We note that here we use a scalar head formulation that does not introduce the Darcy flux $\mathbf{q}$ as an independent unknown. Mixed formulations that solve a coupled head–flux system are possible, but are outside the scope of this work. Furthermore, the space V may be continuous, so that degrees of freedom are shared between neighbouring cells, or discontinuous, so that each cell carries its own degrees of freedom. As shown below, this choice determines which of the interior-facet terms are active. The formulation itself fixes neither the continuity nor the order of V . Through Firedrake’s automated code generation (Ham et al., 2023), the weak form derived below is assembled once and applies unchanged to continuous or discontinuous spaces of arbitrary polynomial degree, on simplicial or tensor-product cells, so that raising the order or changing the element family requires no modification of the discretisation. Here, we present results for discontinuous (DG) and continuous (CG) spaces of degree $p = 1, 2$ on quadrilateral (two-dimensional) and hexahedral (three-dimensional) cells, with the discontinuous discretisation being our primary focus for its local mass conserving property. We therefore leave the higher-order and simplicial spaces that the same formulation admits to future work. Any member of V is written as a linear combination of basis functions, $h(\mathbf{x}, t) = \sum_i h_i(t) \phi_i(\mathbf{x})$, where for a discontinuous space each ϕ_i is supported on a single cell and the coefficients $h_i(t)$ form the discrete solution vector that is advanced in time. To derive the weak form we multiply equation (3) by a test function $v \in V$ and integrate over a cell e (the elastic-storage term of equation (6) is a volume term carried unchanged alongside $\partial\theta/\partial t$, and for clarity we reinstate it only in the final residual (14)),

$$\int_e \frac{\partial\theta}{\partial t} v \, dx - \int_e \nabla \cdot (K \nabla h) v \, dx - \int_e \frac{\partial K}{\partial z} v \, dx - \int_e s v \, dx = 0. \quad (7)$$

We identify the second term as diffusion and the third as a gravity-driven (advective-like) contribution, since $\partial_z K = (\partial_h K) \partial_z h$ within a soil of uniform properties (more generally $K = K(\mathbf{x}, h)$ and the chain rule acts on the head dependence). Integrating the divergence terms by parts yields

$$\int_e \frac{\partial\theta}{\partial t} v \, dx + \int_e K \nabla h \cdot \nabla v \, dx + \int_e K \frac{\partial v}{\partial z} \, dx - \int_{\partial e} (K \nabla h \cdot \mathbf{n}_e) v \, ds - \int_{\partial e} (K \mathbf{e}_z \cdot \mathbf{n}_e) v \, ds - \int_e s v \, dx = 0, \quad (8)$$

where $\mathbf{n}_e$ is the unit normal pointing outward from cell e and $\mathbf{e}_z = \nabla z$ is the upward unit vector $((0, 1)$ in two dimensions and $(0, 0, 1)$ in three).

Before treating the facet terms, we establish the following notation. Interior facets are collected in Γ_{int} . An interior facet is shared by two cells e^+ and e^- , each carrying its own outward unit normal, $\mathbf{n}^+$ and $\mathbf{n}^-$, with $\mathbf{n}^- = -\mathbf{n}^+$. For a scalar quantity a we take the jump to be the vector $[[a]] = a^+ \mathbf{n}^+ + a^- \mathbf{n}^-$ and the average $\{\{a\}\} = \frac{1}{2}(a^+ + a^-)$, and for a vector $\mathbf{b}$ we likewise write $\{\{\mathbf{b}\}\} = \frac{1}{2}(\mathbf{b}^+ + \mathbf{b}^-)$. Defining the jump of a scalar as a vector aligned with the facet normal keeps the combinations that appear below, $\{\{\mathbf{b}\}\} \cdot [[a]]$ and $[[a]] \cdot \{\{b\}\}$, independent of which cell is labelled e^+ . Relabelling exchanges

the two sides and flips each normal, so both the average and the jump behave consistently and their products are unchanged. Summing the elemental boundary integrals in equation (8) over all cells splits them into interior and boundary contributions,

190 $\sum_e \int_{\partial e} (\cdot) ds = \int_{\Gamma_{\text{int}}} (\cdot) ds + \int_{\partial\Omega} (\cdot) ds$, each interior facet being visited once from either adjoining cell. It is at this stage that the discrete flux on each facet must be prescribed, and it is here that the continuous and discontinuous cases differ.

2.1.1 Upwind numerical flux for the gravity term.

The gravity-driven flux is $\mathbf{q}_g = K \mathbf{e}_z$. Although $\mathbf{q}_g$ points upward, gravity transports water downward, so the relevant characteristic speed is directed downward and the numerical flux must take the conductivity from the cell lying *above* each facet,

195 the upstream side for gravity-driven drainage. On an interior facet we therefore replace $\mathbf{q}_g$ by the single-valued upwind flux $\hat{\mathbf{q}}_g = K_{\text{up}} \mathbf{e}_z$, in which K_{up} is the trace of K from the upper cell; the upper trace is selected by the negative part of the normal flux,

$$q_n = \frac{1}{2} (\mathbf{q}_g \cdot \mathbf{n} - |\mathbf{q}_g \cdot \mathbf{n}|), \quad (9)$$

evaluated from either adjoining cell. Assembling the gravity boundary term of equation (8) over all cells then gives

200
$$\sum_e \int_{\partial e} (K \mathbf{e}_z \cdot \mathbf{n}_e) v ds = \int_{\Gamma_{\text{int}}} [[v]] \cdot \hat{\mathbf{q}}_g ds + \int_{\partial\Omega} (K \mathbf{e}_z \cdot \mathbf{n}) v ds, \quad (10)$$

where the vector jump $[[v]] = v^+ \mathbf{n}^+ + v^- \mathbf{n}^-$ makes the interior term independent of facet orientation. If V is continuous the interior jump vanishes. The boundary contribution is not upwinded; it is carried together with the diffusive boundary flux, since on a weakly imposed head boundary the two combine into a single treatment of the potential $h + z$, while on a prescribed-flux boundary both are replaced by the imposed flux, as detailed below.

2.1.2 SIPG fluxes for the diffusive term.

For diffusion we use the symmetric interior penalty Galerkin (SIPG) method (Rivière, 2008), which adds a consistency term, its adjoint counterpart and a penalty term. Let h_F be a characteristic face length scale (not to be confused with the pressure head). Denoting by $|e|$ the measure of a cell and by $|F|$ that of a facet F , we set $h_F^{-1} = \{\{|F|/|e|\}\} = \frac{1}{2} (|F|/|e^+| + |F|/|e^-|)$ on an interior facet shared by cells $e^\pm$, and $h_F = |e|/|F|$ on a boundary facet, where no averaging is required. Assembling the

210 diffusive boundary term of equation (8) gives

$$\sum_e \int_{\partial e} (K \nabla h \cdot \mathbf{n}_e) v ds = \int_{\Gamma_{\text{int}}} \left(\{\{K \nabla h\}\} \cdot [[v]] + \{\{K \nabla v\}\} \cdot [[h]] - \frac{\sigma_F}{h_F} \{\{K\}\} [[h]] \cdot [[v]] \right) ds + \int_{\partial\Omega} (K \nabla h \cdot \mathbf{n}) v ds. \quad (11)$$

The penalty carries the average conductivity $\{\{K\}\}$ so that it scales with the diffusive flux; this is essential for Richards' equation, where K varies over many orders of magnitude between wet and dry soil. We set $\sigma_F = C_f (p+1)^2$ for polynomial degree $p \geq 1$, where C_f is twice the number of facets of a cell and $(p+1)^2$ is the trace-inequality constant for quadrilateral and hexahedral cells (for simplices the corresponding constants are $(p+1)(p+2)/2$ and $(p+1)(p+3)/3$ in two and three

215 dimensions respectively; see Rivière, 2008). Across a sharp conductivity contrast the interface flux is carried by the consistency

term, whose normal trace is exact at a material contact, so that the penalty is not called upon to represent the flux there. If V is continuous the jumps vanish, $[[\cdot]] = 0$, while the averages reduce to the single-valued trace, $\{\{a\}\} = a$, so only the boundary term survives.

220 2.1.3 Boundary conditions.

We consider two boundary conditions, a weakly imposed head and a prescribed normal flux, treated in turn.

- (i) A *Dirichlet* head $h = h_B$ is enforced weakly, by the same SIPG construction applied on $\partial\Omega$; imposed as a boundary condition in this way it is also known as Nitsche's method (Nitsche, 1971). The consistency term carries the full flux $K\nabla(h+z)$, so that the gravity boundary contribution of equation (10) is absorbed here rather than upwinded; the adjoint-consistency and
225 penalty terms act on $h - h_B$ alone, the elevation head z being known exactly and needing no penalty:

$$-\int_{\partial\Omega} (K\nabla(h+z) \cdot \mathbf{n}) v \, ds - \int_{\partial\Omega} (K\nabla v \cdot \mathbf{n}) (h - h_B) \, ds + \int_{\partial\Omega} \frac{2\sigma_F}{h_F} K (h - h_B) v \, ds. \quad (12)$$

- These are the consistency, adjoint-consistency and penalty contributions, with signs chosen so that the penalty is positive and the bilinear form remains symmetric and coercive, and with the penalty again weighted by K . The conductivity in the consistency term is taken from the interior trace, so the diffusive and gravity boundary fluxes together form a single weak
230 enforcement of the potential $h + z$. For a boundary facet $h_F = |e|/|F|$ directly, as no averaging over two cells is required, and the factor of two in the penalty compensates for the absence of a neighbouring cell to share it. If a continuous function space is used, the Dirichlet condition may instead be imposed strongly, without SIPG.

- (ii) A prescribed normal flux $\mathbf{q} \cdot \mathbf{n} = q_T$, where $\mathbf{q} = -K\nabla(h+z)$ is the Darcy flux, replaces *both* the diffusive and the gravity boundary terms on that part of $\partial\Omega$ by the single contribution

$$235 \int_{\partial\Omega} q_T v \, ds, \quad (13)$$

an impermeable (no-flow) boundary being the special case $q_T = 0$. On head boundaries, by contrast, the gravity contribution is already carried by the full-potential consistency term of equation (12), so no separate upwinding is applied there.

2.1.4 Time discretisation.

- The spatial discretisation above leaves time continuous. With the understanding that the SIPG and upwind facet terms supplement the elemental integrals, and reinstating the elastic-storage term of equation (6), the semi-discrete problem is to find
240 $h(t) \in V$ such that, for all $v \in V$,

$$F(h; v) = \int_{\Omega} \frac{\partial \theta(h)}{\partial t} v \, dx + \int_{\Omega} S_s S(h) \frac{\partial h}{\partial t} v \, dx + \int_{\Omega} K(h) \nabla(h+z) \cdot \nabla v \, dx - \int_{\Omega} s v \, dx = 0, \quad (14)$$

a system of nonlinear ordinary differential equations for the discrete coefficients $h_i(t)$. Rather than committing the implementation to a particular marching scheme, we retain the time derivative symbolically and delegate its discretisation to Irksome

(Farrell et al., 2021), which combines the semi-discrete residual with the Butcher tableau of a chosen Runge–Kutta method to assemble a fully discrete variational problem for the stage unknowns, with time-dependent sources and boundary data evaluated at the stage times prescribed by the tableau. The entire Runge–Kutta family is thereby available behind a single interface and without any change to the spatial discretisation, from the one-step methods available here, namely Backward Euler, the implicit midpoint rule and Crank–Nicolson, through higher-order implicit collocation schemes and embedded pairs for adaptive time-stepping. The tableau alone determines where the nonlinear conductivity flux is sampled in time. Writing $h^n = h(t^n)$ for the head at time t^n and $\Delta t = t^{n+1} - t^n$, Backward Euler evaluates it at h^{n+1} , while the multi-stage members evaluate it at intermediate stage heads. All numerical results reported in this paper are obtained with Backward Euler; a systematic assessment of the higher-order and adaptive schemes lies beyond the present scope and is left for future work.

A feature central to conservation is that the time derivative in equation (14) acts on the moisture content directly, so that its discrete counterpart is formed from differences of θ and, for Backward Euler, reduces to the exact increment $(\theta(h^{n+1}) - \theta(h^n))/\Delta t$ rather than an expansion through the chain-ruled capacity $C(h) = d\theta/dh$. This property extends the mass-conservative discretisation of Celia et al. (1990) and holds for every member of the family irrespective of how the flux is integrated in time; we verify the resulting discrete mass balance in §3.2. At each time step the fully discrete problem is nonlinear in the stage heads, which for the one-step methods used here collapse to the single unknown h^{n+1} , and we solve it with Newton’s method,

$$h_{k+1}^{n+1} = h_k^{n+1} + \delta h, \quad (15)$$

where

$$J(h_k^{n+1})\delta h = -F(h_k^{n+1}), \quad (16)$$

and J is the Jacobian of F with respect to the unknown, generated symbolically through Firedrake’s automatic differentiation.

2.1.5 Solution strategy.

The Newton iteration (16) and the linear solves it entails are carried out through PETSc (Balay et al., 2026, 1997), whose composable interface exposes the full range of parallel nonlinear solvers, Krylov methods and preconditioners as runtime options, in the same spirit as the temporal and spatial flexibility described above. For the nonlinear solve we use PETSc’s line-searched Newton method with backtracking, together with the exact symbolic Jacobian assembled by Firedrake. The convergence controls are held identical across every problem in this work and are collected in Listing 1, where `snes_rtol` and `snes_atol` are the relative and absolute nonlinear-residual tolerances, `snes_stol` the smallest admissible relative Newton step, and `snes_max_it` the iteration cap. A step is accepted only once the backtracking line search has secured a sufficient decrease of the residual, which stabilises the iteration in the strongly nonlinear, near-degenerate regime.

```

1: newton_parameters = {
2:     "snes_type": "newtonls",          # Newton with line search
3:     "snes_linesearch_type": "bt",     # backtracking line search
4:     "snes_rtol": 1e-8,                # relative residual tolerance
5:     "snes_atol": 1e-12,               # absolute residual tolerance
6:     "snes_stol": 1e-8,                # minimum relative Newton step

```

```

7: "snes_max_it": 50,           # maximum Newton iterations
8: }

```

Listing 1. Nonlinear (Newton) solver options, held fixed for every problem considered in this work.

The linear system for the update δh is where the cost of a time step is ultimately incurred, and its treatment differs with dimensionality. For the two-dimensional problems the systems are small enough to be factorised exactly, and we adopt a sparse direct LU decomposition through MUMPS (Amestoy et al., 2001, 2006) as the default. Direct factorisation does not carry over to the three-dimensional, high-aspect-ratio meshes demanded by basin-scale forecasting, where its memory and communication costs become prohibitive. There we solve the systems with a preconditioned Krylov method, for which the preconditioner rather than the Krylov iteration itself determines whether the solve scales. Because no single preconditioner performs well across the range of geometries and anisotropies, we do not fix one here but compare several strategies, whose exact PETSc option sets and scaling behaviour are presented in §4. The direct configuration used for the two-dimensional benchmarks is collected in Listing 2; it appends the Newton block of Listing 1 unchanged.

```

295 1: direct_parameters = {
2:   "ksp_type": "preonly",           # no Krylov iteration; factor and back-substitute
3:   "pc_type": "lu",                # parallel sparse direct solver
4:   "pc_factor_mat_solver_type": "mumps",
5: }

```

Listing 2. Direct linear solver for the two-dimensional benchmarks: a sparse LU factorisation through MUMPS. `ksp_type preonly` disables the outer Krylov wrapper, so each solve is a single factorisation and back-substitution. The Newton options of Listing 1 are appended.

2.1.6 Mass-balance diagnostics.

We measure global and local conservation from a cellwise mass-balance residual. For a discontinuous space, this residual follows by testing equation (14) with the indicator function of each cell. The gradient terms then vanish and leave the change in cell storage and the facet fluxes. For cell e and time step n ,

$$r_e^n = \int_e [\theta(h^{n+1}) - \theta(h^n)] dx + \Delta t \int_{\partial e \setminus \partial \Omega} \hat{F} \cdot \mathbf{n}_e ds \quad (17)$$

$$+ \Delta t \int_{\partial e \cap \partial \Omega} q_T ds, \quad (18)$$

where $\hat{F}$ is the single-valued numerical flux, $\mathbf{n}_e$ is the outward cell normal, and q_T is positive outward. The numerical flux combines the interior-penalty flux of equation (11) and the upwind gravity flux of equation (10). Summing the residuals with and without a cellwise modulus gives

$$M_{\text{loc}} = \sum_{n=1}^N \sum_e |r_e^n|, \quad M_{\text{glob}} = \sum_{n=1}^N \left| \sum_e r_e^n \right|, \quad (19)$$

where N is the number of time steps. The first quantity measures cellwise balance, while the second measures balance over the domain and permits deficits and surpluses in different cells to cancel. Interior fluxes cancel in the global sum, so $M_{\text{glob}} \leq M_{\text{loc}}$.

A continuous Galerkin discretisation does not supply an elementwise conservative flux. We therefore evaluate its diagnostic with the facetwise average of the Darcy flux obtained from the head field. An auxiliary recovery problem can construct a locally

conservative flux (Hughes et al., 2000; Larson and Niklasson, 2004), but the resulting conservation belongs to the recovered
 315 flux rather than to the original continuous discretisation.

3 Verification and convergence tests

3.1 Accuracy and convergence against Tracy's exact solution

We first test whether the discretisation recovers its expected spatial and temporal accuracy. The constitutive laws of Tracy
 (2006),

$$320 \quad \theta = \theta_r + (\theta_s - \theta_r)e^{\alpha h} \quad K = K_s e^{\alpha h}, \quad (20)$$

where α has dimensions of inverse length, admit exact steady and transient solutions. We evaluate spatial convergence against
 the steady-state solution, for which $\partial h / \partial t = 0$. The two-dimensional domain is the square $\Omega = [0, L] \times [0, L]$, where $L =$
 15.24 m. We impose $h = -L$ on the left, right, and bottom boundaries and

$$h = \frac{\ln|e^{-\alpha L} + (1 - e^{-\alpha L}) \sin(\pi x / L)|}{\alpha}, \quad (21)$$

325 on the top boundary. The analogous three-dimensional solution has the top boundary condition

$$h(x, y, L) = \frac{\ln(e^{-\alpha L} + \bar{h}_0 \sin \frac{\pi x}{L} \sin \frac{\pi y}{L})}{\alpha}, \quad (22)$$

and $h = -L$ on the other five boundaries, with $\bar{h}_0 = 1 - e^{-\alpha L}$.

We obtain the steady state by integrating the transient problem with Backward Euler. The time step grows by five per cent
 from $\Delta t = 5 \times 10^4$ s to a ceiling of 2.5×10^5 s. At $t = 2.5 \times 10^6$ s, the magnitude of $\partial h / \partial t$ is approximately at machine
 330 precision. Because the target is the stationary solution, the reported errors are insensitive to this time-step sequence. The
 three-dimensional calculations use GMRES preconditioned by block-Jacobi ILU(0); §4.1 gives the solver configuration and
 alternatives.

Fig. 2 reports the L^2 error for polynomial degrees $p = 1$ and 2 in two and three dimensions. In both cases, the error decreases
 monotonically with mesh refinement. We measure each convergence rate from the log-log slope between the two finest meshes,
 335 where the discretisation is closest to its asymptotic regime. The measured slopes are consistent with the expected rate of $p + 1$.
 The coarsest meshes have not reached this regime, and their local slopes increase with refinement. At a given Δx , the higher-
 order space produces a smaller error. This comparison supports the expected spatial accuracy.

We next test temporal accuracy against Tracy's transient two-dimensional solution using the same soil parameters. Starting
 from the analytical initial condition, we integrate to $t = 10^5$ s at constant Δt on a uniform mesh of 201^2 points with a DG
 340 space of degree $p = 1$. This resolution is chosen to make the spatial error small relative to the temporal error. Fig. 3 shows that
 the L^2 error decreases linearly with Δt , consistent with the first-order accuracy of Backward Euler.

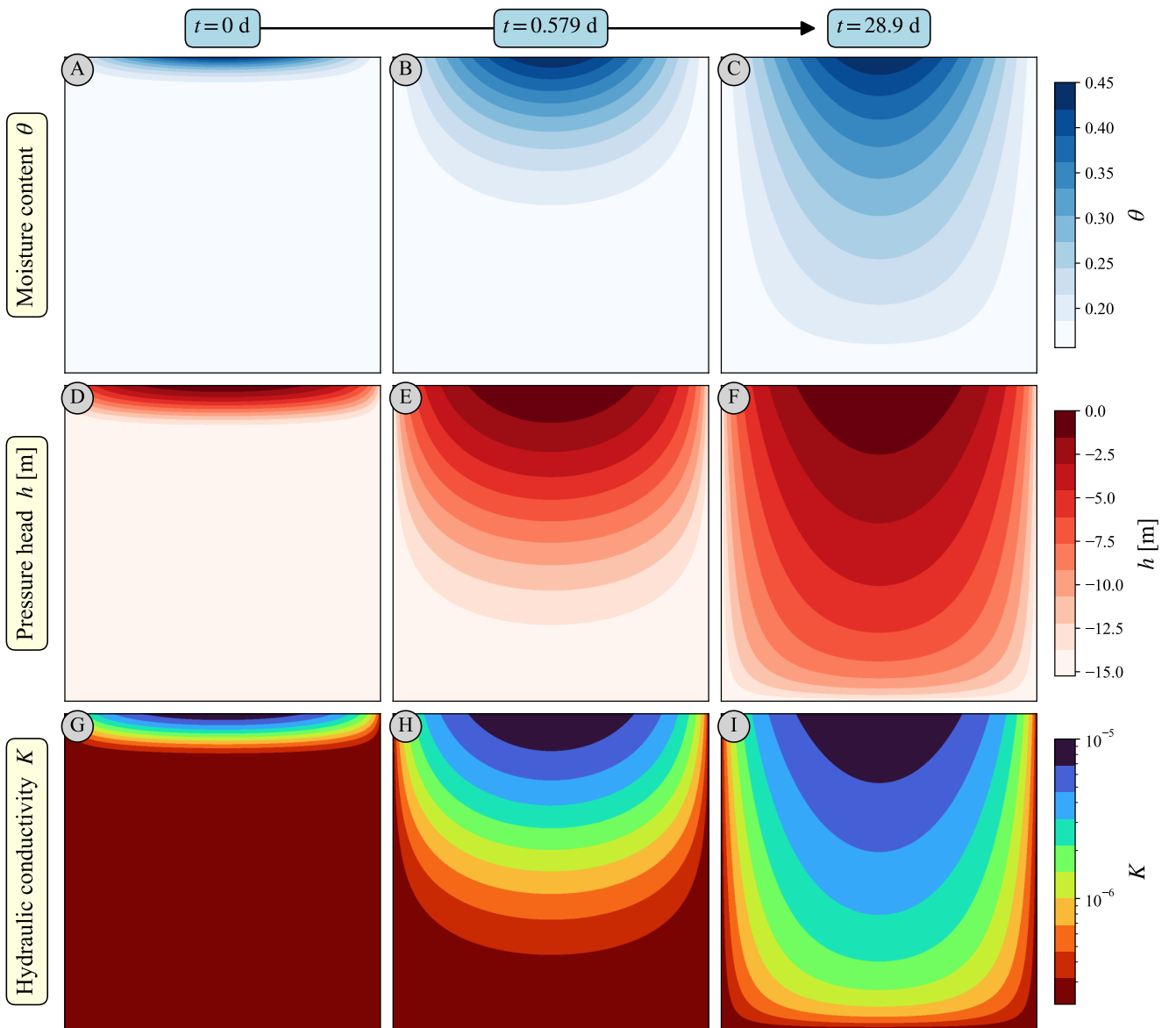


Figure 1. Evolution of pressure head towards Tracy's steady-state solution in § 3.1. The square domain has side length $L = 15.24$ m, with $\alpha = 0.25 \text{ m}^{-1}$, $\theta_r = 0.15$, $\theta_s = 0.45$, $S_s = 0$, and $K_s = 10^{-5} \text{ m/s}$. The left, right, and bottom boundaries have $h = -L$, while the top boundary follows equation (21). Panels show $t = 0$, 5×10^4 , and 2.5×10^6 s from left to right.

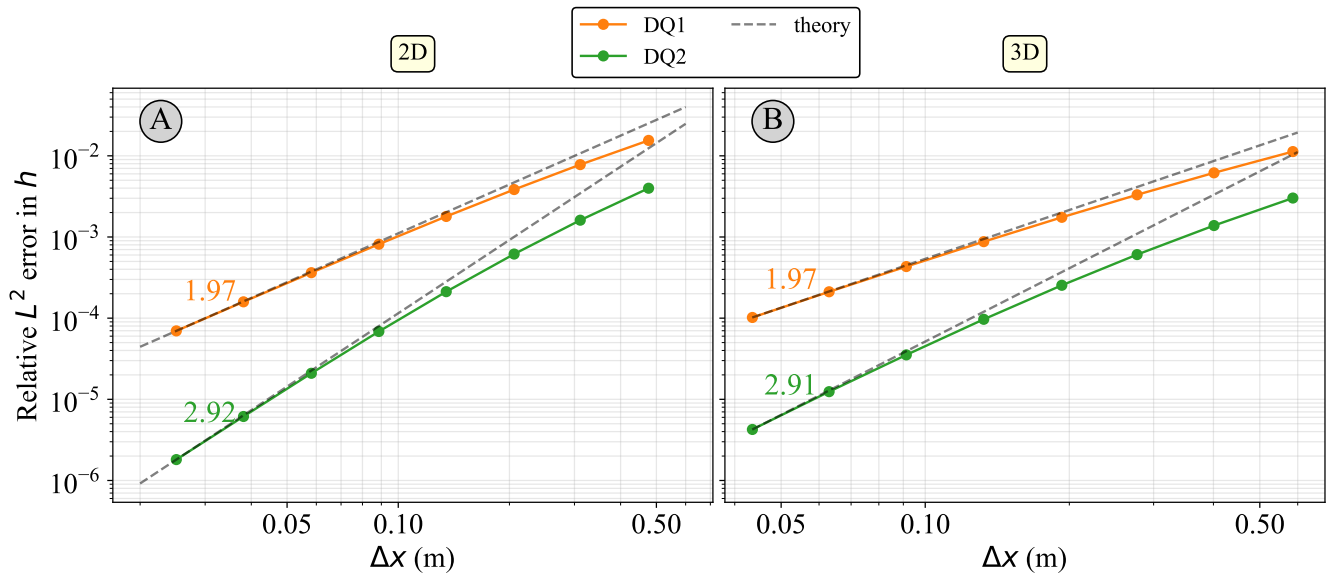


Figure 2. Convergence of the L^2 error norm against Tracy's steady-state exact solution (Tracy, 2006) in (a) two and (b) three dimensions. The parameter p is the polynomial degree. Other parameters follow Fig. 1. Dashed grey lines show theoretical rates; each annotated rate is the log-log slope between the two finest meshes.

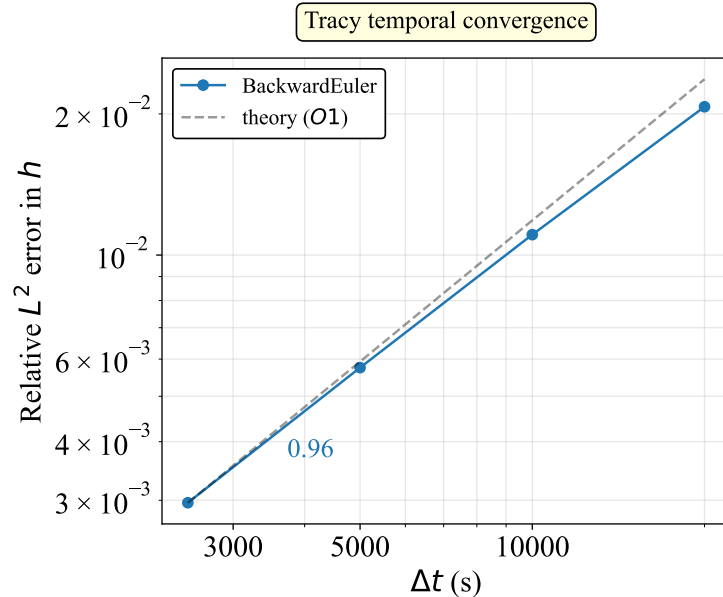


Figure 3. Temporal convergence against Tracy's two-dimensional exact solution (Tracy, 2006) for Backward Euler. Simulations end at $t = 10^5$ s and use a discontinuous space of degree $p = 1$ on a uniform mesh of 201^2 points. The plotted quantity is $\|h_h - h_{\text{exact}}\|_{L^2}$.

3.2 Global and local mass conservation

Conservation of mass is a defining requirement for hydrological models, especially for Richards' equation. It is also the reason for our use of the mixed form (3). Here we test whether global balance distinguishes the continuous and discontinuous spaces, and whether the same spaces conserve water cell by cell.

We consider infiltration into a unit square with initial condition $h(t = 0) = -1$ m. The left, right, and bottom boundaries have no flux. The top boundary has the constant infiltration flux 10^{-6} m/s, or $q_T = -10^{-6}$ m/s in the outward-positive convention of §2.1. The total volume therefore follows $\text{Vol}(t) = \text{Vol}(0) + 10^{-6}t$. We use the constitutive relations of Haverkamp et al. (1977), as adopted by Vauclin et al. (1979):

$$\theta = \theta_r + \frac{\alpha(\theta_s - \theta_r)}{\alpha + |h|^\beta}, \quad K = K_s \frac{A}{A + |h|^\gamma}. \quad (23)$$

Here α , β , γ , and A are fitting parameters. Because α is added to $|h|^\beta$ and A to $|h|^\gamma$, the two carry units of m^β and m^γ . We set $\theta_r = 0.05$, $\theta_s = 0.4$, $K_s = 10^{-5}$ m/s, $\alpha = 0.5 \text{ m}^{1.3}$, $\beta = 1.3$, $A = 0.01 \text{ m}^{1.5}$, $\gamma = 1.5$, and $S_s = 0$. Thus, moisture content carries every change in storage. We take 300 Backward Euler steps with $\Delta t = 100$ s, so 0.03 m^3 enters the domain by $t = 3 \times 10^4$ s. At that time the medium remains unsaturated and its maximum head is -0.32 m. This endpoint isolates mass conservation from the vanishing moisture capacity that enters when upper cells reach $h = 0$.

Fig. 4a shows that CG and DG give indistinguishable global imbalances. Every space returns between 4.28 and $4.36 \times 10^{-11} \text{ m}^3$ at every degree and resolution. This floor is 1.4×10^{-9} of the supplied volume and is comparable with the Newton relative tolerance $\text{snes_rtol} = 10^{-8}$. Reducing that tolerance to 10^{-10} lowers the imbalance to $4.1 \times 10^{-14} \text{ m}^3$, which confirms that the nonlinear tolerance controls the floor. Global conservation holds for both families because constants belong to V : setting $v = 1$ in equation (14) gives an exact identity between the moisture increment and the boundary supply. The global metric therefore does not distinguish CG from DG in this test.

The local imbalance in Fig. 4b separates the two families. The DG spaces remain at the nonlinear-solve floor, between 4.29 and $4.37 \times 10^{-11} \text{ m}^3$, so each cell balances water to the accuracy of the solve. The CG spaces, however, do not. On the coarsest mesh, they produce cellwise imbalances of $2.38 \times 10^{-3} \text{ m}^3$ for $p = 1$ and $1.05 \times 10^{-3} \text{ m}^3$ for $p = 2$, equal to 7.9 and 3.5 per cent of the supplied volume. Refinement reduces these values to 9.61×10^{-5} and $4.66 \times 10^{-5} \text{ m}^3$ at $\Delta x = 10^{-2} \text{ m}$, but they remain six orders of magnitude above the DG result. Increasing the degree from one to two reduces the imbalance by a factor of about two. Thus, global balance alone hides the cellwise difference between the spaces.

Changing Δt from 200 to 50 s changes M_{loc} by less than five per cent, so temporal resolution does not control this contrast.

3.3 Water-table recharge and self-convergence

The simplified recharge problem of Vauclin et al. (1979) tests the evolution of a water table under localised surface infiltration. The original benchmark uses a seepage face whose saturated extent forms part of the solution. Here we prescribe a hydrostatic head on the right boundary, which gives a Dirichlet-boundary variant suitable for a self-convergence test.

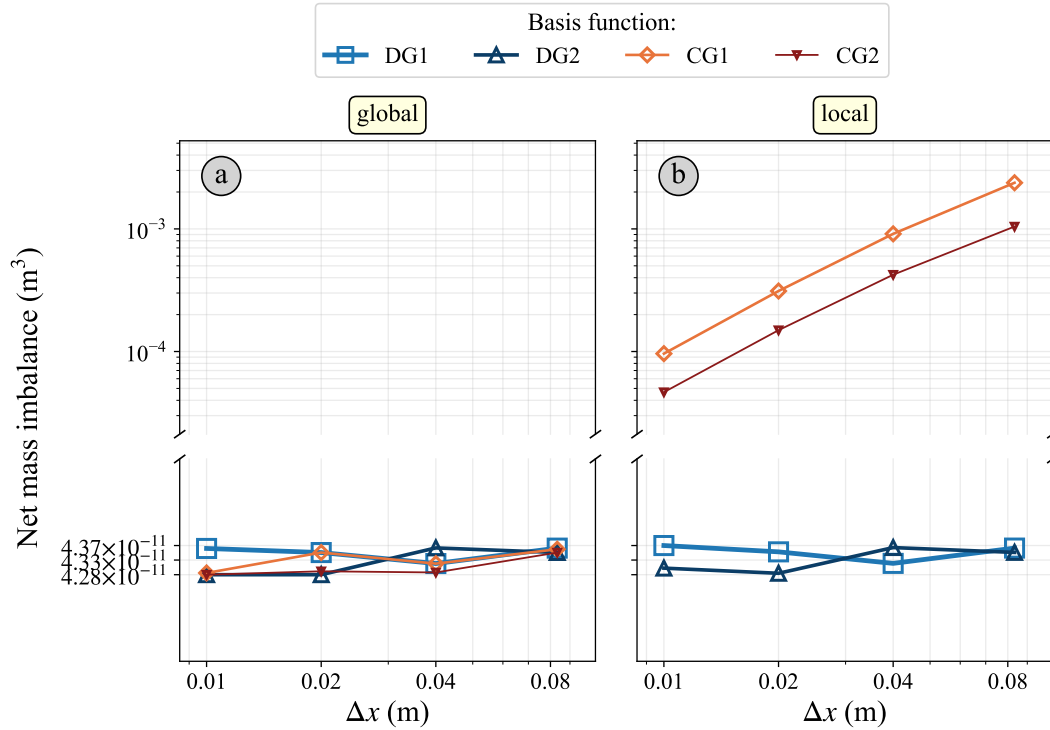


Figure 4. Mass conservation for continuous (CG) and discontinuous (DG) spaces of degrees $p = 1$ and 2. Panels show (a) global imbalance and (b) local imbalance from equation (19) against mesh spacing Δx , using Backward Euler with $\Delta t = 100$ s. The broken axis separates the CG local imbalance from the nonlinear-tolerance floor. Scatter within the lower band reflects termination of the Newton iteration and does not rank the spaces.

The rectangular domain is 3 m wide and 2 m deep, with initial water-table height 0.65 m and $h(x, z, t = 0) = 0.65 - z$ m. The left and bottom boundaries are impermeable. We impose 4.11×10^{-5} m/s of infiltration over $x \leq 0.5$ m on the top boundary and no flux elsewhere. The right boundary retains the hydrostatic head $h = 0.65 - z$ m. Equation (23) supplies the hydraulic properties, with $\alpha = 6.340 \times 10^{-2} \text{ m}^{2.90}$, $\beta = 2.90$, $A = 2.99 \times 10^{-4} \text{ m}^5$, $\gamma = 5$, $K_s = 9.722 \times 10^{-5} \text{ m/s}$, $\theta_r = 0$, and $\theta_s = 0.30$.

Fig. 5 shows the resulting moisture evolution. The localised wet region first descends beneath the recharge boundary and then spreads laterally towards the fixed-head boundary. The streamlines show the same transition from predominantly downward infiltration to lateral groundwater flow. This figure provides a qualitative check of the recharge response; the self-convergence test below quantifies numerical accuracy.

No exact solution exists for this boundary-condition variant. We therefore measure self-convergence against a reference solution with $p = 2$ on a 240×160 -cell mesh. The comparison uses $p = 1$ and 2 on coarser meshes with spacings from $\Delta x = 0.25$ to 0.025 m. Every coarse mesh nests in the reference mesh, so representation on the reference space adds no interpolation error. All runs use the same time step.

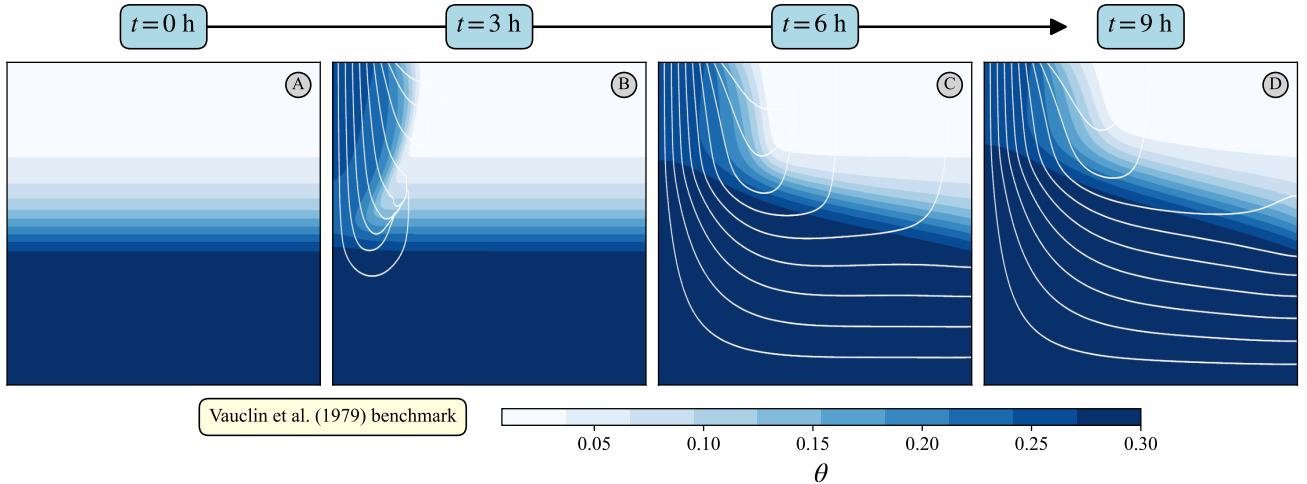


Figure 5. Moisture content for the Dirichlet-boundary variant of the Vauclin recharge problem (Vauclin et al., 1979) at $t = 0, 3, 6$, and 9 hours from left to right. White curves show instantaneous streamlines. Hydraulic properties follow equation (23) and the parameters in §3.3.

385 Fig. 6 shows that the relative L^2 error decreases monotonically under refinement for both polynomial degrees. The slopes
between the two finest available meshes are consistent with the expected order of $p + 1$. On coarser meshes, the local slopes
remain below the asymptotic values. For $p = 2$, Newton iteration fails at the common time step on the two coarsest meshes, so
the curve begins at $\Delta x = 0.125$ m. Moreover, the $p = 2$ reference mesh is only twice as fine as the finest comparison mesh.
Its discretisation error therefore biases the finest fitted slope downward, so this self-convergence result does not independently
390 establish the asymptotic order.

4 Preconditioner design and performance

4.1 Preconditioner design and comparison protocol

The direct factorisation used for the two-dimensional benchmarks does not extend to the three-dimensional meshes and proces-
sor counts required for basin-scale simulation. We therefore use a preconditioned Krylov method, whose performance depends
395 primarily on the preconditioner. Preconditioning strategies for Richards' equation have been examined before by Herbst et al.
(2008), who combined a Galerkin finite-element discretisation with Picard iteration and a parallel conjugate-gradient solver
and showed that the three-dimensional problem can be solved efficiently in parallel. Here we compare eight preconditioners in
three families, namely two single-level baselines, two algebraic multigrids, and four geometric multigrids. Table 1 summarises
their defining differences. Appendix A gives the complete PETSc option dictionaries.

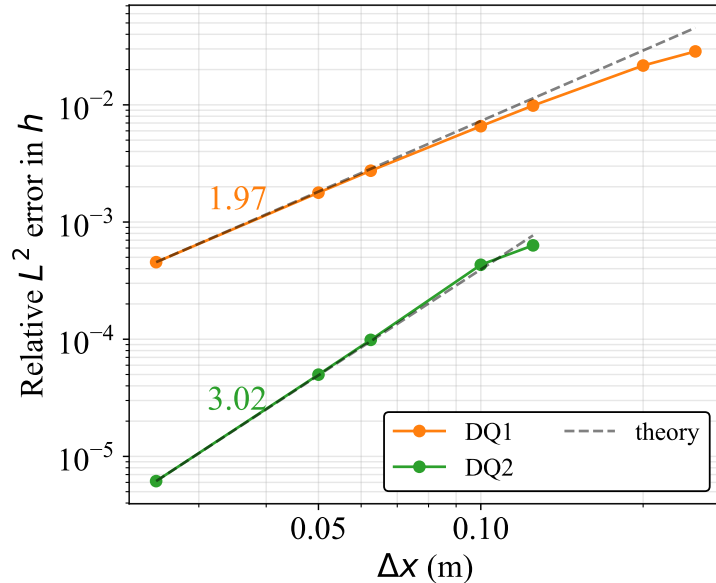


Figure 6. Self-convergence of the relative L^2 error in pressure head for the Vauclin problem at $t = 8$ hours. Numerical reference values use $p = 2$ on a 240×160 -cell mesh. Dashed lines show the theoretical rates $O(\Delta x^{p+1})$; each annotated rate is the log-log slope between the two finest available meshes. Other parameters follow Fig. 5.

Table 1. Preconditioners used in the performance experiments. The coarse-space column states the principal correction beyond the fine-level smoother.

Method	Family	Coarse space	Distinguishing component
BJac-SOR	single level	none	rank-local symmetric SOR
BJac-ILU	single level	none	rank-local ILU(0)
BoomerAMG	algebraic MG	algebraic hierarchy	anisotropy-tuned coarsening
PETSc GAMG	algebraic MG	aggregate hierarchy	smoothed aggregation
GMG-H	geometric MG	horizontally coarsened 3-D	block-ILU smoother
VLumping	geometric MG	vertically constant 2-D	direct coarse solve
VLumping-linesmooth	geometric MG	vertically constant 2-D	column-LU smoother
VLumping-HMG	geometric MG	vertically constant 2-D	iterative coarse solve

400 The two single-level baselines use no coarse-grid correction. BJac-SOR applies symmetric SOR to each rank-local block, while BJac-ILU applies rank-local ILU(0). Because the extruded mesh is partitioned into vertical columns, the ILU(0) blocks act as column smoothers and address the stiff vertical coupling without a coarse space.

Algebraic multigrid constructs its hierarchy from the fine-grid matrix and requires no mesh information. For DG systems, however, the enlarged stencil and block structure increase setup cost, while an algebraic coarse space need not preserve the
 405 layered geometry. We tune Hypr’s BoomerAMG for this anisotropy. We raise the strength-of-connection threshold from its isotropic default of 0.25 to 0.5 so coarsening follows the strong vertical couplings. HMIS coarsening and `ext+1` interpolation address the anisotropic stencil, while aggressive coarsening on the first two levels limits hierarchy complexity.

Geometric multigrid instead coarsens a prescribed mesh hierarchy and preserves the element structure between levels. GMG-H uses horizontal semi-coarsening: the two-dimensional base mesh changes between levels, but each level retains the same
 410 vertical layer count. A Chebyshev sweep over rank-local ILU(0) blocks smooths each level.

GMG-H retains the full vertical discretisation on every level. VLumping instead uses a vertically constant coarse space, which eliminates the vertical dimension and leaves a two-dimensional coarse problem. This idea originates in ocean modelling, where Kramer et al. (2010) developed a vertically lumped preconditioner for the Poisson equation on small-aspect-ratio unstructured meshes. It was motivated by the observation of Marshall et al. (1997) that the corresponding three-dimensional
 415 elliptic operator is dominated by vertical anisotropy. A related semi-coarsening approach was developed by Tuminaro et al. (2016) for extruded ice-sheet meshes. Richards’ equation adds a changing operator to this construction. Because $K(h)$ depends on the solution, each Newton iteration updates the Jacobian and rebuilds the Galerkin coarse operator $A_c = P^T A P$, where P is the prolongation operator that injects a 2D lumped solution vector into a vertically constant 3D solution. The setup cost therefore enters once per Newton step.

420 The three VLumping variants isolate the fine smoother applied to the full 3D problem and the coarse solver for the lumped 2D problem. VLumping uses two rank-local ILU(0) sweeps as the fine smoother and a direct MUMPS coarse solve. VLumping-linesmooth retains that coarse solve but uses one exact LU solve per vertical column. VLumping-HMG retains the line smoother and solves the coarse problem with geometric multigrid. Its coarse operator is the same Galerkin operator $A_c = P^T A P$, and what changes is the mesh that carries the coarse space. VLumping builds that space on the extruded mesh itself, with a single
 425 degree of freedom in the vertical, while VLumping-HMG builds it on the two-dimensional base mesh from which the fine mesh was extruded. The two spaces hold the same unknowns in the same parallel layout, so P is unchanged, and only the base mesh belongs to a hierarchy of coarser meshes. The coarse solve can therefore descend that hierarchy, with the operator on each further level again formed by Galerkin projection.

In every vertically lumped configuration reported here, the fine-level Chebyshev smoother is replaced by Richardson it-
 430 eration, and the coarse operator is left unchanged. PETSc’s Chebyshev smoother estimates the extreme eigenvalues of the preconditioned fine-level operator with a short GMRES run after every Jacobian update. The Richardson smoother needs only a damping factor, which we derive from the same kind of estimate. We make that estimate at the start of a run and repeat it only when the norm of the Jacobian diagonal changes by more than a factor of two, when the linear iteration count of a Newton step grows by more than half, or after 200 Newton steps, so that most Newton steps reuse the previous damping factor. We refer to

Table 2. Van Genuchten parameters for the heterogeneous infiltration benchmark.

Soil	θ_r	θ_s	α (m ⁻¹)	n	K_s (m s ⁻¹)
Loamy sand	0.035	0.401	11.5	1.474	1.69×10^{-5}
Sand	0.020	0.417	13.8	1.592	5.82×10^{-5}

435 this scheme as runtime damping. For VLumping-HMG, this substitution applies only to the fine level; the base-mesh hierarchy retains its Chebyshev smoothers. The performance sections below test whether this change helps in both near-isotropic and anisotropic regimes.

GMG-H and the VLumping methods use FGMRES, while BJac-SOR, BJac-ILU, BoomerAMG, and PETSc GAMG use GMRES. For a fixed preconditioner, GMRES and FGMRES generate the same iterates, so this choice does not alter their
440 comparison.

All solvers use right preconditioning and stop when the true residual $\|b - Ax\|$ has been sufficiently reduced according to a specified relative tolerance (`ksp_rtol` in the options configurations detailed in the Appendix). This common norm avoids a bias from preconditioner-dependent residuals. We set `ksp_rtol`= 10^{-4} for inexact Newton, which reduces the true linear residual by four orders of magnitude without oversolving each update.

445 4.2 Near-isotropic regime: heterogeneous infiltration

Vertical lumping targets the strong vertical anisotropy of basin meshes. We first test whether it provides an advantage without that anisotropy, using the three-dimensional heterogeneous infiltration benchmark of Cockett et al. (2018). Sharp wetting fronts and large parameter contrasts test the nonlinear solver, while cells with an aspect ratio close to 1:1 keep the linear system near-isotropic.

450 The domain is $2 \times 2 \times 2.6$ m, with $h = -0.1$ m at the top, $h = -0.3$ m at the bottom, and no flux through the sides. The initial condition is $h(t=0) = 0.2e^{5(z-2.6)} - 0.3$ m. Water retention and conductivity follow the van Genuchten model (Van Genuchten, 1980),

$$\theta = \theta_r + \frac{\theta_s - \theta_r}{(1 + |\alpha|h|^n)^m}, \quad K = K_s \frac{[1 - (\alpha|h|)^{n-1}(1 + (\alpha|h|)^n)^{-m}]^2}{(1 + (\alpha|h|)^n)^{m/2}}, \quad (24)$$

with $m = 1 - 1/n$ and $S_s = 0$. An indicator field assigns each element to loamy sand or sand with the properties in Table 2.

455 Fig. 7A shows the resulting soil structure. The moisture fields in Fig. 7C–E show faster infiltration through the more conductive sand inclusions than through the loamy sand. The experiment therefore isolates nonlinear heterogeneity without the strong vertical anisotropy targeted by the vertically lumped methods.

We test weak scaling by increasing the problem size in proportion to the node count. All runs use Gadi `normal` nodes, each with 104 Intel Sapphire Rapids cores and 500 GB of memory. A degree-one DG mesh supplies approximately 1.8×10^7
460 degrees of freedom per node, or 1.7×10^5 unknowns per core. The node count increases from one to eight, and the global problem grows from 1.8×10^7 to 1.4×10^8 degrees of freedom. Each run takes 30 fixed time steps of $\Delta t = 300$ s.

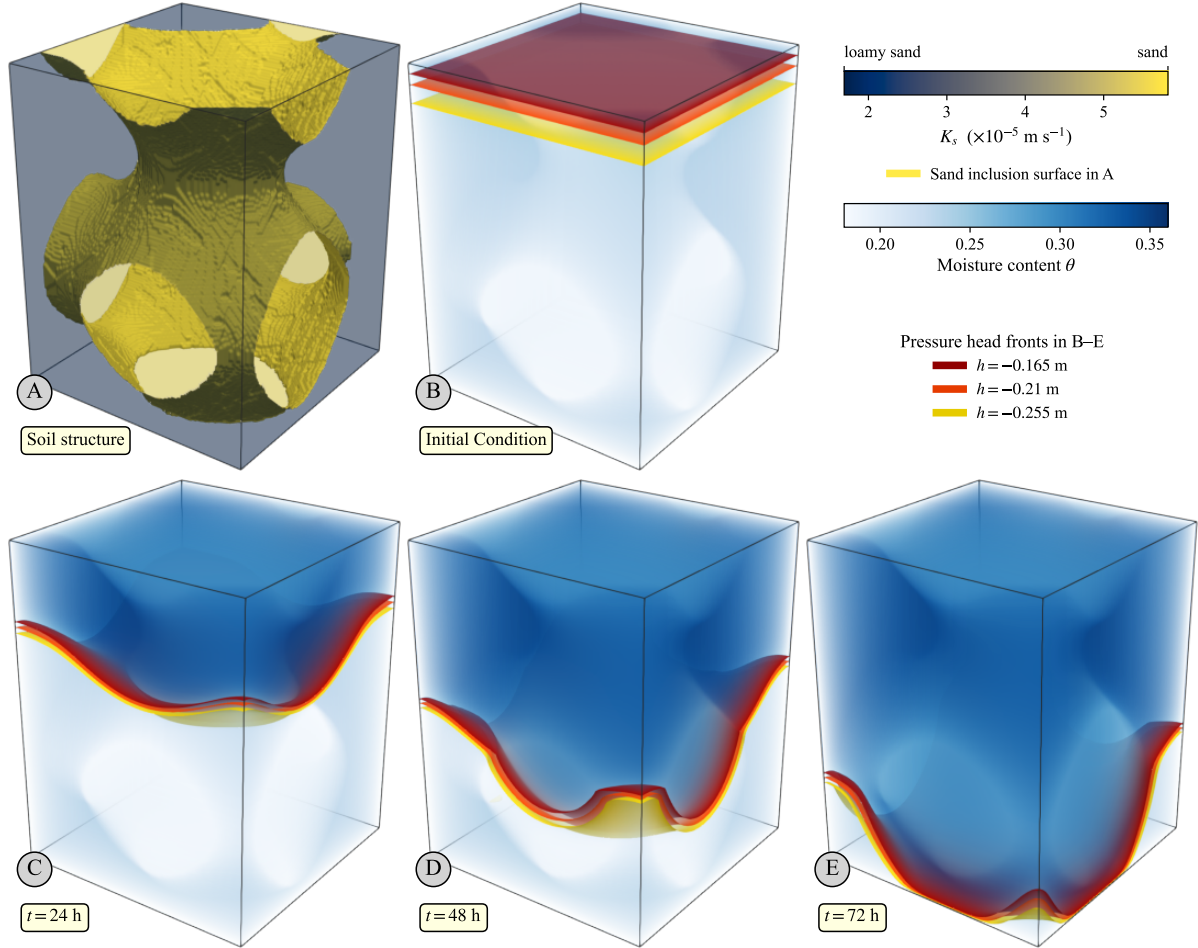


Figure 7. (A) Sand inclusions within the loamy-sand matrix, with saturated conductivity K_s shown by the colourbar. (B) Initial moisture content. (C–E) Moisture content at $t = 24, 48,$ and 72 hours, with pressure-head isosurfaces $h = -0.165, -0.21,$ and -0.255 m . The hydraulic properties follow equation (24) and Table 2.

An effective weak-scaling preconditioner maintains its iteration count and time per Newton step as the global problem grows. We use these two metrics because iteration count measures algorithmic scaling, while wall time also includes setup and communication, and the costs per iteration can be significantly different between methods. Every method drawn in Fig. 8
465 completes the 30 time steps in 91 Newton iterations at all node counts. Normalisation per Newton step therefore isolates the linear-solver cost without changing the shapes of the curves.

BJac-ILU is the fastest solver at every scale, despite having the highest iteration count. Its count rises from 60 to 126 iterations per Newton step (Fig. 8B), while its time per Newton step rises from 5.3 to 9.6 s (Fig. 8A). BJac-SOR performs less well: its count rises from 85 to 154 and its time per Newton step from 8.0 to 13.4 s before the Krylov solve diverges at eight
470 nodes. Thus, the cheaper BJac-ILU iterations outweigh their larger number in this near-isotropic regime.

Among the multilevel methods, GMG-H holds its iteration count nearly constant, from 19 to 20, while BoomerAMG rises from 21 to 29. Their time per Newton step still increases, from 8.7 to 11.3 s for BoomerAMG and from 8.6 to 17.0 s for GMG-H. Stable iteration counts therefore do not yield stable wall time because each multilevel iteration carries greater setup, smoothing, and coarse-solve costs than a BJac-ILU iteration.

475 On this near-isotropic mesh, the iteration counts of the vertically lumped methods increase with mesh size. VLumping rises from 21 to 41 iterations per Newton step and from 6.9 to 15.0 s per Newton step. VLumping-linesmooth and VLumping-HMG carry nearly twice as many iterations, rising from 45.6 to 97.5 and from 45.9 to 97.6, respectively. Their counts remain within one per cent at every scale, although their coarse solvers differ. This similarity indicates that the line smoother, rather than the coarse solver, controls the iteration increase. Their times per Newton step rise from 7.8 to 21.4 s and from 7.8 to 19.8 s,
480 respectively.

The vertically lumped methods target error associated with strong vertical anisotropy, which is absent from this experiment. Their increasing iteration counts are consistent with a horizontal coarse space that does not capture all error modes on near-cubic cells, although we do not diagnose those modes directly. BJac-ILU instead applies inexpensive rank-local ILU(0) blocks on the column partition. The resulting larger iteration count still costs less than the multilevel alternatives. The basin
485 experiments below test whether this advantage persists under strong vertical anisotropy and longer time steps.

Runtime-damped Richardson smoothing also gives no advantage in this regime. Relative to the Chebyshev configuration, it requires about one-fifth more iterations and increases wall time by 5–12 per cent. Thus, any benefit from runtime damping is specific to the anisotropic basin problems tested below.

We also test PETSc’s smoothed-aggregation algebraic multigrid. Its time per Newton step rises from roughly 27 to 47 s
490 between two and eight nodes, and its peak memory is two to three times that of the other methods. We therefore retain the anisotropy-tuned BoomerAMG as the algebraic-multigrid representative for the remaining comparisons. Tables 3 and 4 report all outcomes.

For the retained solvers, peak resident memory per process lies between approximately 0.7 and 1.6 GB and does not grow with node count (Table 3). Memory did not constrain these runs.

Table 3. Solver robustness across the six scaling experiments. Each cell reports the outcome of one run at the indicated node count. • converged and completed the run at the requested timestep; ◦ completed the run, but only at a smaller timestep than was requested, so its admissible step was set by the preconditioner; D diverged; M failed on an out-of-memory error; T did not reach the final time within the wall-clock limit; a dash marks a configuration that was not attempted. The final column is the peak resident memory per process over the successful runs in that row. The seasonal blocks report the presets compared in §4.4.2, where VLumping-HMG appears only for the strong-scaling decompositions.

Solver	Nodes						Peak mem
	1	2	4	8	16	32	(GB)
<i>Cockett 3D — isotropic box, cell AR $\approx 1:1$</i>							
BJac-SOR	•	•	•	D	—	—	0.9
BJac-ILU	•	•	•	•	—	—	1.0
GAMG	•	•	•	•	—	—	3.1
BoomerAMG	•	•	•	•	—	—	1.3
GMG-H	•	•	•	•	—	—	1.6
VLumping	•	•	•	•	—	—	1.3
VLumping-linesmooth	•	•	•	•	—	—	1.3
VLumping-HMG	•	•	•	•	—	—	1.2
<i>Lower Murrumbidgee — horizontal weak scaling (300 layers; Δx 1775$\rightarrow$620 m, AR 1000:1$\rightarrow$350:1)</i>							
BJac-SOR	D	D	D	D	—	—	—
BJac-ILU	•	•	•	•	—	—	1.3
GAMG	M	M	M	M	—	—	—
BoomerAMG	D	D	D	D	—	—	—
GMG-H	•	•	•	•	—	—	1.7
VLumping	•	•	•	•	—	—	1.7
VLumping-linesmooth	•	•	•	•	—	—	1.7
VLumping-HMG	•	•	•	•	—	—	1.7
<i>Lower Murrumbidgee — vertical weak scaling ($\Delta x = 1775$ m; 150$\rightarrow$1200 layers, AR 500:1$\rightarrow$4000:1)</i>							
BJac-SOR	D	D	D	D	—	—	—
BJac-ILU	•	•	•	•	—	—	1.0
GAMG	M	M	M	M	—	—	—
BoomerAMG	D	D	D	D	—	—	—
GMG-H	•	•	•	•	—	—	1.2
VLumping	•	•	•	•	—	—	1.2
VLumping-linesmooth	•	•	•	•	—	—	1.2
VLumping-HMG	•	•	•	•	—	—	1.2
<i>Lower Murrumbidgee — graded stress test (as horizontal, three-month ceiling, water table +5 m, $\theta_s - \theta_r$ /3)</i>							
BJac-ILU	•	•	◦	◦	—	—	1.3
GMG-H	•	•	•	•	—	—	1.7
VLumping	•	•	•	•	—	—	1.7
VLumping-linesmooth	•	•	•	•	—	—	1.8
<i>Lower Murrumbidgee — saturated stress test (water table +10 m, $\theta_s - \theta_r$ /10)</i>							
BJac-ILU	D	D	D	D	—	—	—
GMG-H	T	◦	•	•	—	—	1.7
VLumping	◦	•	•	•	—	—	1.7
VLumping-linesmooth	◦	•	•	•	—	—	1.7
<i>Lower Murrumbidgee — strong scaling (fixed 3.2×10^8 DOF, $\Delta x = 620$ m, 300 layers)</i>							
BJac-ILU	M	•	•	•	•	•	2.9
VLumping	M	•	•	•	M	M	4.2
VLumping-HMG	—	•	•	•	•	•	4.3

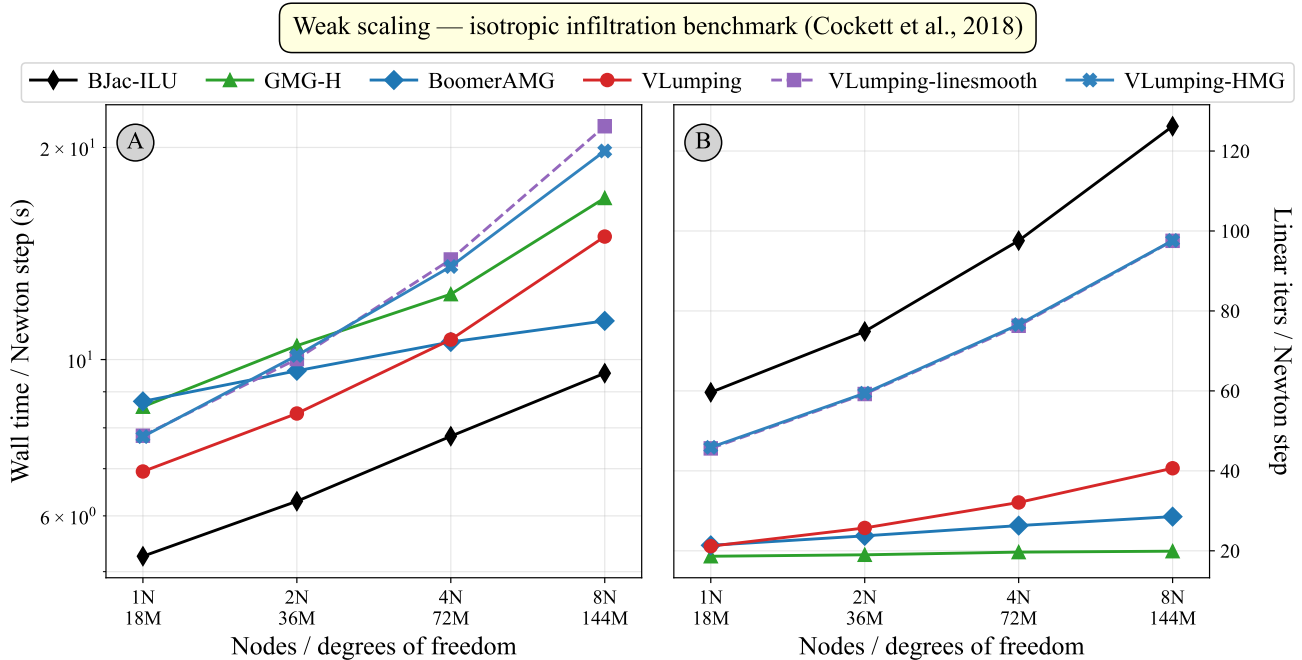


Figure 8. Weak scaling for the near-isotropic infiltration benchmark. Each 104-core node carries approximately 1.8×10^7 degrees of freedom, and each run takes 30 time steps with $\Delta t = 300$ s. Panels show (A) wall time and (B) Krylov iterations per Newton step. The dashed VLumping-linesmooth curve separates it from the nearly coincident VLumping-HMG curve. Table 3 reports BJac-SOR, smoothed aggregation, and peak memory.

4.3 Lower Murrumbidgee basin: model setup

We next construct a basin-scale model of the Lower Murrumbidgee floodplain in south-western New South Wales, Australia. This case tests computational feasibility on a field-derived geometry; it does not test predictive hydrological skill. The hydraulic parameters remain uncalibrated, and the recharge is a simplified first estimate. The region in Fig. 9 spans approximately 280 km from east to west and 130 km from north to south, with an area of about 3×10^4 km². It lies west of the Great Dividing Range, where elevation and the regional groundwater flow direction generally decrease from east to west. Public topographic, stratigraphic, rainfall, and water-table data define the model. The terrain-following mesh resolves the aquifers in thin vertical layers, so the resulting elements test the preconditioners under strong vertical anisotropy.

The mesh and its hydraulic structure are assembled from public Australian data sources with OMEGA (Ghelichkhan, 2026c), which builds terrain-following G-ADOPT meshes from open data. A Geoscience Australia SRTM digital elevation model, retrieved with `ausdem` (Ghelichkhan, 2026a), defines the surface. Boreholes from the Bureau of Meteorology National Groundwater Information System, retrieved with `austrata` (Ghelichkhan, 2026b), define the subsurface stratigraphy. We build the mesh in two stages. First, an unstructured triangulation with target spacing Δx follows the irregular basin boundary. Its interior

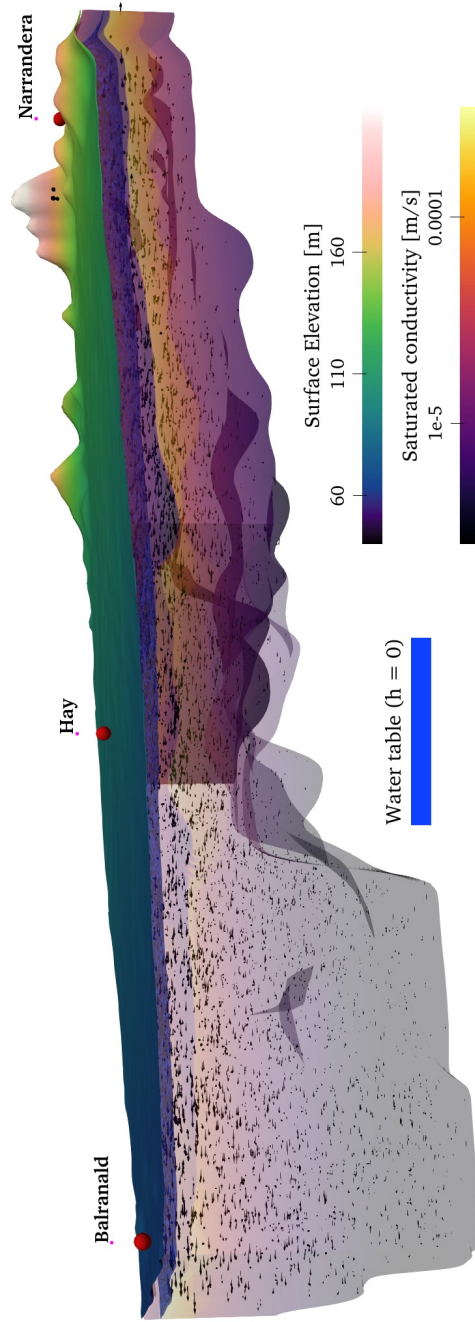


Figure 9. Lower Murrumbidgee model with the vertical axis exaggerated. Input fields comprise surface elevation and the saturated-conductivity structure of the three aquifers. Output fields comprise the modelled water table and Darcy flux. Balranald, Hay, and Narrandera provide geographic reference.

Table 4. Performance of the preconditioners that complete each experiment, evaluated at the eight-node point (the largest common weak-scaling case). For each run we report the linear (Krylov) iterations per Newton step, the wall-clock time per Newton step, the share of the solve time spent forming the preconditioner (Setup) and applying it (Apply), and the peak resident memory per process. The iteration count measures the conditioning the preconditioner achieves, while the “Setup” and “Apply” shares show where each preset spends its solve, the vertically lumped presets carrying most of theirs in applying the preconditioner rather than in forming it. The “Apply” share covers both the fine smoother and the coarse solve, whose relative sizes are given for the basin in §4.4.1. The last block repeats the measurement in the graded stress test of §4.4.2 on the same mesh, where the ranking of the first basin block inverts. Presets that diverge, exhaust memory, or exceed the wall-clock limit at this scale carry no entry here and are recorded in Table 3. The saturated stress test has no block, because BJac-ILU completes no time step there. The basin blocks include the two direct-coarse lumped presets recommended in §4.4, which are configured identically apart from the fine-level smoother.

Solver	Iterations per Newton	Wall time per Newton (s)	Setup Time (%)	Apply Time (%)	Peak mem (GB)
<i>Cockett 3D — isotropic box</i>					
BJac-ILU	126.2	9.56	1.2	32.5	1.0
GAMG	23.2	47.13	77.7	16.1	2.4
BoomerAMG	28.5	11.35	14.3	62.0	1.3
GMG-H	19.9	16.95	19.8	48.8	0.7
VLumping	40.6	14.95	4.9	74.9	1.3
VLumping-linesmooth	97.5	21.44	2.6	72.8	1.3
VLumping-HMG	97.6	19.77	1.6	71.2	1.2
<i>Lower Murrumbidgee — horizontal weak scaling, short-step regime</i>					
BJac-ILU	34.7	5.20	3.2	31.1	1.3
GMG-H	16.8	11.89	17.4	65.1	1.7
VLumping	11.3	7.87	17.6	63.0	1.7
VLumping-linesmooth	14.8	6.98	17.0	55.4	1.7
<i>Lower Murrumbidgee — vertical weak scaling, short-step regime</i>					
BJac-ILU	13.3	2.15	6.1	25.7	0.9
GMG-H	16.9	8.64	18.8	62.0	1.0
VLumping	4.9	3.16	13.8	52.4	1.0
VLumping-linesmooth	6.6	2.90	10.6	45.5	1.0
<i>Lower Murrumbidgee — graded stress test</i>					
BJac-ILU	108.3	13.46	1.1	35.4	1.3
GMG-H	27.4	19.49	9.8	75.6	1.7
VLumping	8.1	6.11	17.6	59.9	1.7
VLumping-linesmooth	10.2	5.43	17.4	52.3	1.8

resolution remains approximately uniform. Second, Firedrake’s `ExtrudedMesh` extends this base between the surface and bedrock with a fixed number of terrain-following layers. This is the sigma-coordinate construction familiar from ocean mod-
510 elling. Because the layer count is fixed, the layer thickness scales with the local depth to bedrock, which ranges from about 120 to 420 m across the basin, so that with 300 layers the thinnest layers are about 0.4 m and the thickest about 1.4 m thick. A fixed layer count does not place resolution where the solution requires it, and where the depth to bedrock changes rapidly it produces thin and strongly tilted cells. In ocean models such tilted layers are a known source of error in the horizontal pressure gradient, which there is evaluated as the difference of two large terms in the transformed coordinate. Our finite-element integrals are
515 evaluated on the deformed cells in physical coordinates, so this particular error does not arise, although accuracy still degrades on thin, strongly tilted cells. In the Lower Murrumbidgee the layer interfaces tilt by less than half a degree, so the cells remain close to right prisms, and we leave depth-adaptive layering to basins with steeper relief. The horizontal spacing Δx and the maximum vertical spacing Δz can be selected independently. We use polynomial degree $p = 1$ in both directions. The digital elevation model provides a common reference surface for the mesh and geological boundaries. Each borehole boundary is
520 recorded as depth below this surface, so a node at elevation z has depth $d = \text{DEM}(x, y) - z$. This common datum prevents a mismatch between the surface and borehole data. Because borehole picks are sparse and clustered, we use Gaussian-kernel regression to construct smooth depth surfaces. Nearby conflicting logs then average rather than produce spurious near-vertical facets.

The Lower Murrumbidgee comprises three aquifers, from top to bottom the Shepparton formation, the Calivil formation,
525 and the Upper Renmark group, underlain by impermeable basement. Interbedded sand and clay, coarse sand, and fine sand dominate these aquifers, respectively. We therefore set the surface values $K_{s,0} = 2.5 \times 10^{-5}$, 10^{-3} , and 5×10^{-4} m/s, while acknowledging that these uncalibrated values remain uncertain. The unsaturated hydraulic behaviour of all three formations follows the Haverkamp relations in equation (23) (Haverkamp et al., 1977) with common parameters $\alpha = 0.44 \text{ m}^{1.2924}$, $\beta = 1.2924$, $A = 0.0104 \text{ m}^{1.5722}$, $\gamma = 1.5722$, $\theta_r = 0.025$, and a surface value $\theta_{s,0} = 0.40$, so the formations differ only in $K_{s,0}$.
530 We set $S_s = 0$ in every basin calculation. The formation boundaries are smoothed over about ten metres with a hyperbolic-tangent indicator, which avoids a discontinuous K_s at a borehole-derived surface that is itself uncertain. Saturated conductivity and porosity decrease with depth d in metres according to (Chen et al., 2020a; Shahbazi et al., 2020)

$$K_s(d) = K_{s,0} \left(1 - \frac{d}{58 + 1.02d} \right)^3 \quad \text{and} \quad \theta_s(d) = \frac{\theta_{s,0}}{(1 + 7.1 \times 10^{-5} d)^{5.989}}. \quad (25)$$

We initialise the pressure head from Bureau of Meteorology water-table data, retrieved with `austrata`, using the hydrostatic
535 profile $h(t = 0) = d - d_{\text{wt}}$, where d_{wt} is the measured depth to the water table. The lateral boundaries retain their initial water-table height. This simplified condition does not represent time-dependent regional inflow.

At the surface, we impose recharge equal to 15 per cent of rainfall. This fraction is an uncalibrated forcing choice. The basement has no flux. Although the source term in equation (3) can represent groundwater extraction, the reported calculations omit that sink and isolate the response to the imposed recharge.

540 The output fields in Fig. 9 come from a 30-day proof-of-concept calculation on a mesh with $\Delta x = 1.5$ km and 150 extruded layers. The calculation starts from the water-table initial condition and uses the rainfall forcing described above without extrac-

tion. It completes without artificial regularisation or manual intervention. This result demonstrates numerical feasibility for the field-derived geometry and imposed heterogeneity. It does not assess hydrological accuracy because the hydraulic parameters and recharge remain uncalibrated.

545 4.4 Preconditioner performance on the basin mesh

The basin mesh tests the methods under stronger vertical anisotropy than the near-isotropic infiltration problem. BJac-SOR and BoomerAMG diverge at every tested scale, while PETSc’s smoothed-aggregation multigrid exhausts memory (Table 3). GMG-H and the vertically lumped methods retain a horizontal coarse correction and remain viable. But their performance depends on the integration regime. We therefore compare short steps on an unsaturated profile with seasonal steps on wetter profiles. Each weak-scaling configuration holds a fixed number of degrees of freedom per 104-core Sapphire Rapids node. The horizontal experiment retains 300 layers while Δx decreases from 1775 to 620 m. It tests whether a horizontal coarse correction becomes more important under horizontal refinement. The vertical experiment retains $\Delta x = 1775$ m while the layer count increases from 150 to 1200. It tests sensitivity to increasing vertical anisotropy.

4.4.1 Short-step unsaturated regime

555 We first test whether the horizontal coarse correction is necessary for a 30-day infiltration calculation on an unsaturated profile. The adaptive time step starts at $\Delta t = 60$ s, increases by a factor of 1.5 after a successful solve, and has a 12-hour ceiling. In this regime, BJac-ILU has the lowest wall time at every scale (Fig. 10). Under horizontal refinement, the BJac-ILU time per Newton step increases from 2.65 to 5.20 s and its iteration count from 14.2 to 34.7. The vertically lumped methods require fewer iterations: 5.0 to 11.3 for VLumping and 6.9 to 14.8 for VLumping-linesmooth. At eight nodes, VLumping-linesmooth is the faster lumped variant (6.98 s per Newton step, compared with 7.87 s for VLumping), but neither is faster than BJac-ILU. 560 GMG-H also converges, but takes 7.69 to 11.89 s per Newton step. Vertical refinement favours the lumped coarse spaces. Increasing the layer count from 150 to 1200 changes the VLumping iteration count only from 4.5 to 4.9, and the VLumping-linesmooth count from 6.3 to 6.6. BJac-ILU nevertheless remains the fastest method, increasing from 1.26 to 2.15 s per Newton step despite using about 2.7 times as many iterations as VLumping. Thus, for short steps on this unsaturated profile, the cheaper 565 BJac-ILU iterations outweigh the lower iteration counts of the lumped methods.

Timing profiles explain the wall-time contrast (Table 4). In the eight-node horizontal case, BJac-ILU spends 3.2 per cent of the solve time forming the preconditioner and 31.1 per cent applying it. The lumped method instead spends 63.0 per cent applying its preconditioner. Within that application, the coarse solve accounts for 20 per cent and the fine block-ILU smoother for 71 per cent. Rebuilding the coarse operator adds another 17.6 per cent of the total solve time. These profiles show that the 570 fine smoother, rather than the coarse solve, dominates the lumped application. The BJac-ILU result is consistent with effective vertical coupling through its column-aligned rank-local blocks (§5.1). Unlike the near-isotropic case, runtime damping reduces basin wall time by 11–24 per cent at comparable iteration counts. VLumping-linesmooth is also as fast as or faster than VLumping in every basin experiment. These contrasts show that both choices benefit the target anisotropic regime rather than

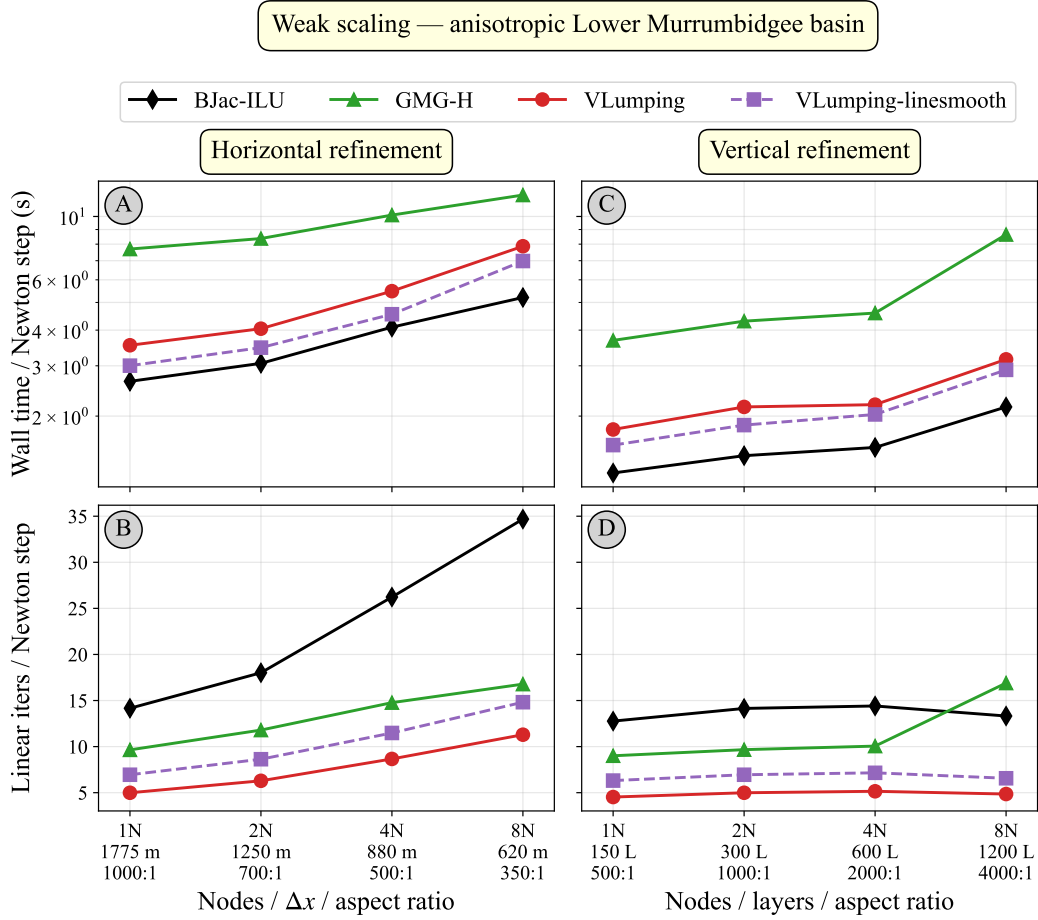


Figure 10. Weak scaling in the short-step unsaturated regime. Panels (A,B) refine Δx from 1775 to 620 m at 300 layers. Panels (C,D) increase the layer count from 150 to 1200 at $\Delta x = 1775$ m. The top row shows wall time per Newton step, and the bottom row shows Krylov iterations per Newton step. The integration starts at $\Delta t = 60$ s and has a 12-hour ceiling.

the method in general. These results apply to relatively short time steps on an unsaturated profile. The seasonal experiment tests whether they hold for the longer steps required in a basin simulation.

4.4.2 Seasonal regimes

We next test whether the short-step ranking persists when longer time steps and wetter profiles strengthen horizontal coupling. This is a controlled stress test, not a reconstruction of the Lower Murrumbidgee. We repeat the horizontal weak-scaling experiment on the same four meshes and raise the initial water table 5 m above the measured one. We also raise the residual water content so that $\theta_s - \theta_r$ decreases by a factor of three, while θ_s , the Haverkamp shape parameters, and $K(h)$ remain unchanged. Because the Haverkamp moisture capacity $C(h) = d\theta/dh$ is proportional to $\theta_s - \theta_r$, this change reduces the moisture capacity

Table 5. Horizontal weak scaling in the graded stress test of §4.4.2, mesh by mesh. The four meshes are those of Fig. 10A,B, so the cell aspect ratio is unchanged from the short-step regime. The graded stress test also raises the initial water table by 5 m and reduces $\theta_s - \theta_r$ by a factor of three. D_{col} is the median over the basin columns of the column-integrated diffusion number of Eq. (27), evaluated at the imposed three-month step. For each solver the table reports the largest timestep the run sustained, the number of failed steps the adaptive ramp absorbed, and the wall-clock time per simulated year, which is the comparable cost measure because the runs terminate between 468 and 545 simulated days. A dagger marks a step limited by the imposed three-month ceiling rather than by the solver, so that entry is a lower bound on what the preconditioner would sustain. BJac-ILU’s admissible step is the only one that falls below the ceiling, and it does so on the two finest meshes, where D_{col} approaches and passes unity. GMG-H is omitted for width, and it holds the full step at all four meshes, at 1.08, 1.15, 1.49, 1.82 hours per simulated year, between three and six times the cost of VLumping, the ratio falling as the mesh is refined.

Mesh			BJac-ILU			VLumping			VLumping-linesmooth		
Δx	DOF	D_{col}	Δt_{max}	failed	wall	Δt_{max}	failed	wall	Δt_{max}	failed	wall
(m)			(d)	steps	(h/yr)	(d)	steps	(h/yr)	(d)	steps	(h/yr)
1775	4.0×10^7	0.14	93 [†]	0	0.35	93 [†]	0	0.18	93 [†]	0	0.15
1250	8.0×10^7	0.29	93 [†]	1	0.68	93 [†]	0	0.22	93 [†]	1	0.22
880	1.6×10^8	0.59	70	4	1.12	93 [†]	1	0.36	93 [†]	1	0.30
620	3.2×10^8	1.19	39	10	1.31	93 [†]	2	0.55	93 [†]	2	0.52

by a factor of three throughout the unsaturated range. The resulting soil has a lower specific yield, which reduces the column storativity and increases the column-integrated diffusion number discussed in §5.1.

The adaptive time step starts at 60 s, increases by a factor of 1.5 after a successful solve, and has a 93-day ceiling. We refer to this experiment as the graded stress test, and to the wetter experiment below as the saturated stress test. A failed solve halves the time step. We integrate for approximately 463 days. Individual runs end between 468 and 545 days because each stops after its first step beyond the target time. We therefore compare wall time per simulated year. Importantly, the largest sustained time step is a result rather than a fixed run parameter. The solver ranking reverses in this seasonal regime (Fig. 11 and Table 5). Both vertically lumped methods with a direct coarse solve reach the imposed three-month ceiling at every scale. VLumping requires 4.1 to 8.1 linear iterations per Newton step, and VLumping-linesmooth requires 5.3 to 10.2. Each has at most two failed steps over an entire run. BJac-ILU requires 46.7 to 108.3 iterations per Newton step. Its largest sustained time step decreases from three months at 1775 and 1250 m to 70 days at 880 m and 39 days at 620 m. The lumped methods are also faster: their wall times per simulated year are 1.9 to 3.7 times lower than those of BJac-ILU. GMG-H converges but requires three to six times more wall time than VLumping because its coarse correction remains three-dimensional.

The saturated stress test uses a wetter profile (Fig. 12). The initial water table is raised 10 m above the measured one, and the residual water content is raised so that $\theta_s - \theta_r$, and hence $C(h)$, decrease by a factor of ten. The saturated water content, Haverkamp shape parameters, and $K(h)$ remain unchanged. The time step grows by a factor of two after a successful solve rather than 1.5. BJac-ILU completes no time step at any scale: each Newton solve reaches the 200-iteration limit before the solution advances.

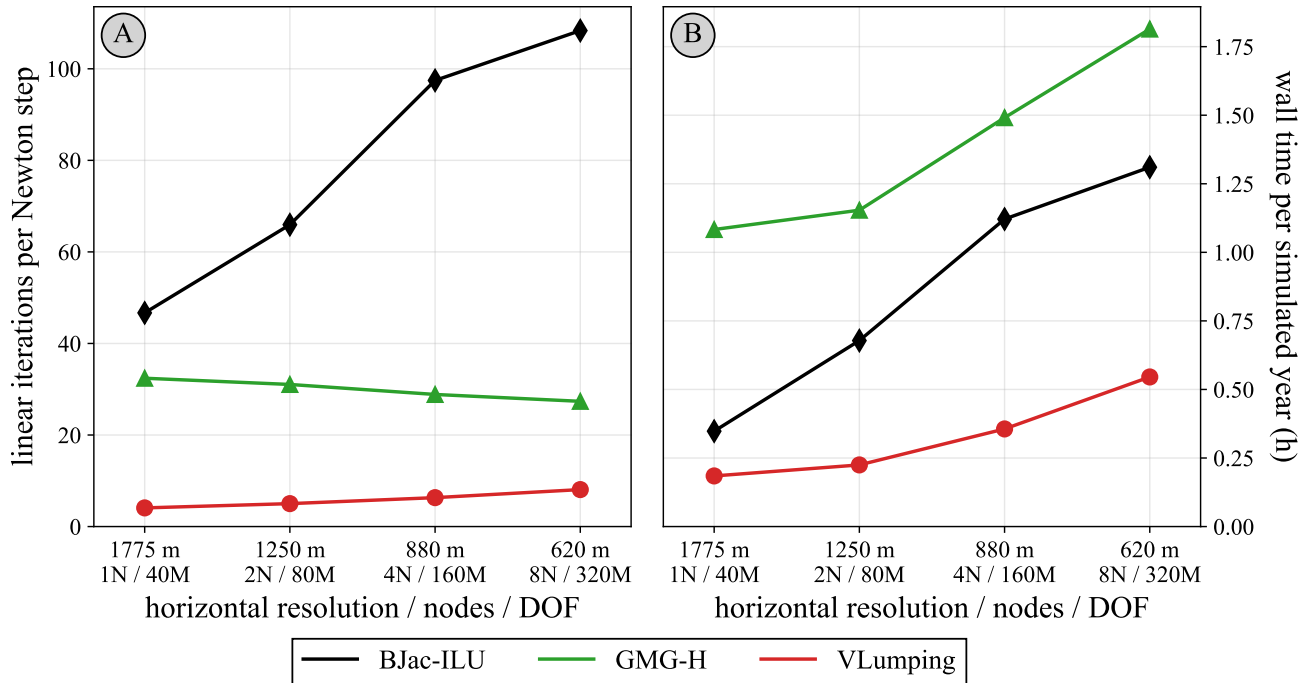


Figure 11. Horizontal weak scaling in the seasonal regime on the four meshes of Fig. 10A,B. The initial water table is raised 5 m above the measured one, and $\theta_s - \theta_r$, and hence $C(h)$, are reduced by a factor of three while θ_s and $K(h)$ remain unchanged. The adaptive time step has a 93-day ceiling. Panels show (A) Krylov iterations per Newton step and (B) wall time per simulated year. Table 5 reports the largest sustained time step; entries at the imposed ceiling are lower bounds on the admissible step.

VLumping solves every scale in 3.8 to 8.2 iterations per Newton step, and VLumping-linesmooth requires 5.1 to 10.8 (Table 3). They reach the three-month step on every mesh except the coarsest. That exception is a nonlinear-solver failure, not a loss of linear convergence: the coarsest mesh has the lowest linear iteration count. GMG-H completes the runs at 1250 m and finer, but exceeds the six-hour limit at 1775 m and requires about 40 iterations per Newton step where it completes. The contrast with BJac-ILU shows that a horizontal coarse correction is necessary in this tested regime.

Fig. 13 reports the cost decomposition for the graded stress test, in which every method advances the solution. VLumping is the least expensive at 27–66 s per nonlinear solve, followed by BJac-ILU at 54–135 s and GMG-H at 174–230 s. GMG-H applies its three-dimensional coarse correction in each of 27–32 Krylov iterations per Newton step, which dominates its total cost. VLumping requires only 4–8 iterations, although preconditioner application still dominates its cost. Within that application, the coarse solve increases from 3 to 21 per cent under horizontal refinement, while the fine block-ILU smoother decreases from 86 to 70 per cent. Thus, the fine smoother remains more expensive than the coarse solve. BJac-ILU again spends more time in Krylov work than in preconditioning, but now requires 47–108 iterations per Newton step. The cost of a BJac-ILU solve falls between 880 and 620 m. This apparent decrease does not indicate improved efficiency. Its admissible time

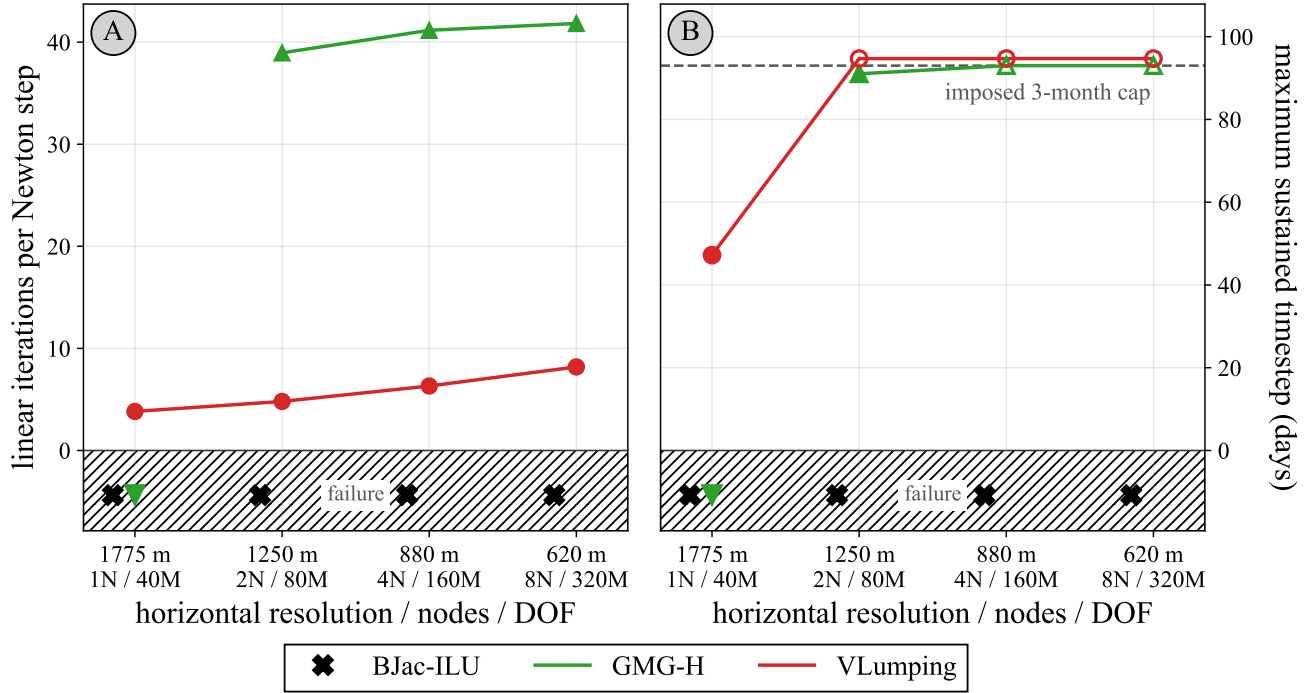


Figure 12. Weak scaling for the saturated stress test. The initial water table is raised 10 m above the measured one, and $\theta_s - \theta_r$, and hence $C(h)$, are reduced by a factor of ten while θ_s and $K(h)$ remain unchanged. The time step grows by a factor of two. Panels show (A) Krylov iterations per Newton step and (B) the largest sustained time step. The hatched band denotes runs that produce no result. Open symbols denote the imposed three-month ceiling and thus a lower bound. No wall-time comparison is possible because BJac-ILU does not complete a single time step.

step has fallen from 70 to 39 days, so each individual solve is an easier problem and more of them are required per simulated year. Wall time per simulated year therefore provides the relevant comparison.

615 We finally test the strong scaling of the vertically lumped methods on the largest basin mesh: $\Delta x = 620$ m, 300 layers, and about 3.2×10^8 degrees of freedom (Fig. 14). These runs use the 30-day short-step configuration of §4.4.1, with the measured water table and a 12-hour ceiling on the time step. The mesh does not fit on one node. VLumping reaches eight nodes, where it takes 23.8 s per time step, but its MUMPS coarse factorisation stops with an out-of-memory error at 16 and 32 nodes. The error most likely reflects the workspace that MUMPS estimates for the factorisation, not a lack of physical memory, because

620 the eight-node run peaks at 1.7 GB per process against about 4.8 GB available per core. We did not test a larger workspace setting. VLumping-HMG avoids this factorisation and reaches 16 nodes at 17.0 s per time step, with the same iteration count as at eight nodes. At 32 nodes, however, communication in its fixed two-dimensional coarse solve increases the time to 109.9 s per step. VLumping-HMG therefore runs at the processor counts where the direct coarse factorisation fails, reaching 16 and 32 nodes where VLumping stops at eight, but its iterative coarse solve becomes a communication bottleneck.

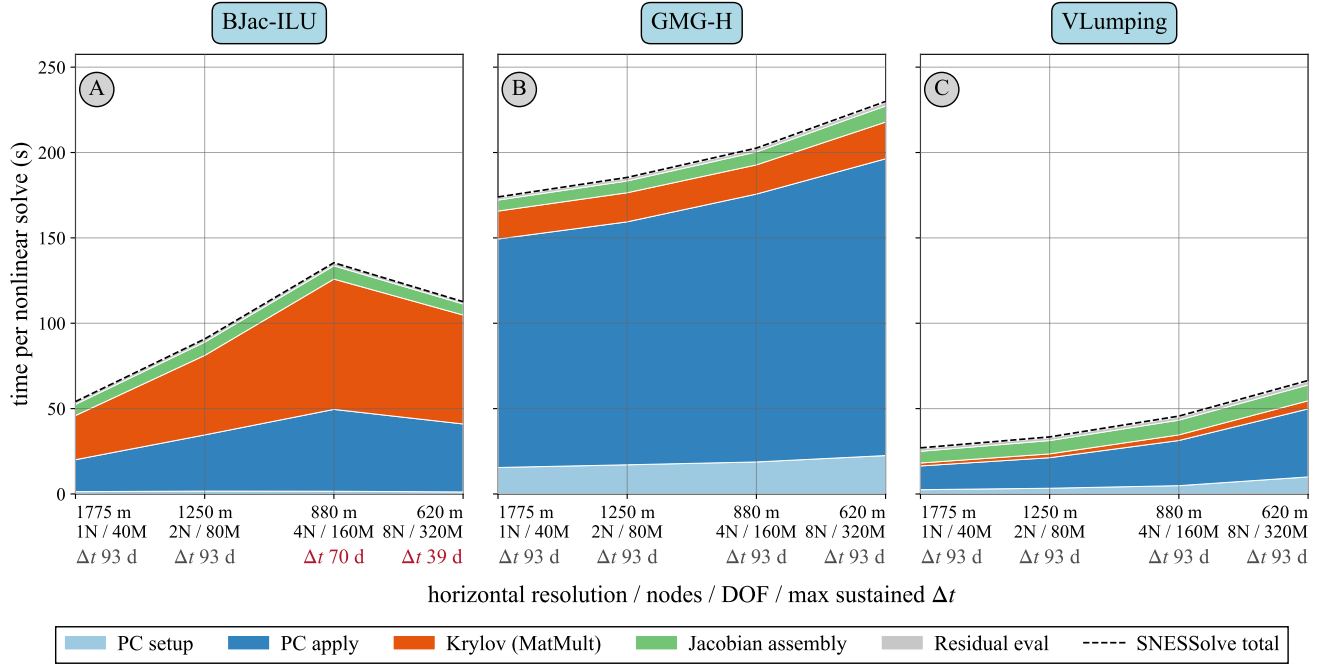


Figure 13. Cost per nonlinear solve in the graded stress test. The timing profile separates preconditioner setup and application, Krylov work, Jacobian assembly, and residual evaluation. The dashed curve gives the total `SNESolve` time, the time spent in PETSc’s Newton solver including the calls to Firedrake that assemble the Jacobian and the residual at every iteration. One solve is one attempted time step, and averages include failed attempts. Labels below the axis give the largest sustained time step; red labels mark loss of the three-month ceiling.

625 The basin experiments show that the appropriate solver depends on the temporal integration regime. At short time steps on the unsaturated profile, BJac-ILU has the lowest wall time despite its larger iteration count. At seasonal time steps and on wetter profiles, its admissible step decreases with horizontal refinement and can fail to advance the solution. Conversely, the direct-coarse lumped methods maintain low iteration counts, retain the imposed three-month ceiling, and have the lowest seasonal wall time. Across these weak-scaling experiments, peak memory remains between 1 and 2 GB per process. We
630 therefore adopt VLumping and VLumping-linesmooth as the default methods for the basin configurations examined here. VLumping-HMG runs at processor counts where the direct coarse factorisation fails, at the cost of an iterative coarse solve. Taken together, the results show that the physical integration regime, rather than the mesh aspect ratio alone, determines the effective preconditioner.

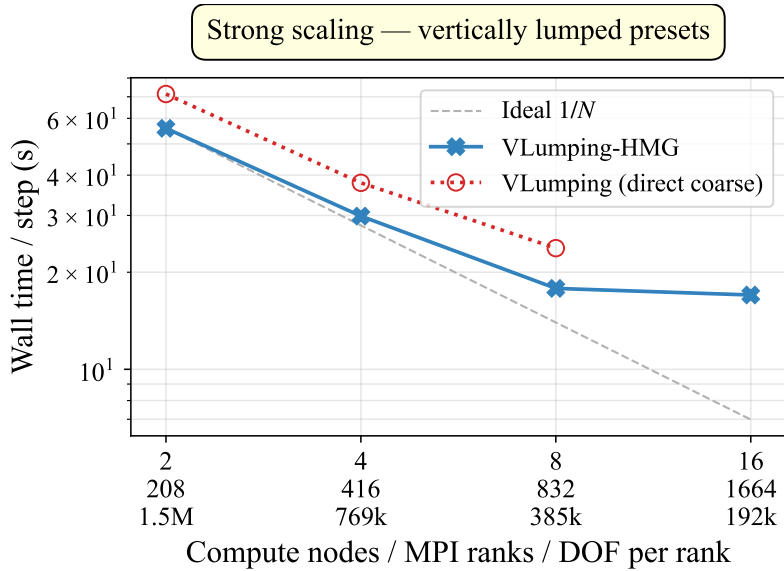


Figure 14. Strong scaling of the vertically lumped methods on the largest basin mesh, with $\Delta x = 620$ m, 300 layers, and approximately 3.2×10^8 degrees of freedom. Tick labels give node count, MPI ranks, and degrees of freedom per rank. The one-node case exceeds available memory. The VLumping curve ends at eight nodes because the MUMPS coarse factorisation fails at larger decompositions.

5 Discussion

5.1 Preconditioning

Preconditioning must address the strong vertical coupling created by the thin geometry of a basin. For this reason, we draw on preconditioners developed for ocean and ice-sheet models, where a shallow domain with strong vertical structure creates the same numerical problem. The vertically lumped preconditioners used in §4 come from that literature. Marshall et al. (1997) used the vertical dominance of a shallow elliptic operator to motivate a domain decomposition that assigns complete vertical columns to individual processing units. The same column structure explains a feature of our baseline results in §4.2. The extruded mesh is partitioned by columns, so the rank-local ILU(0) in BJac-ILU acts as a line smoother along each vertical column. BJac-ILU therefore resolves much of the stiff vertical coupling without a coarse-grid correction. The baseline can remain competitive in wall time even while its iteration count deteriorates. The same block smoother also provides the fine-level solver in every multilevel preset reported here.

Later work extended the same idea through semi-coarsening for extruded ice-sheet meshes (Tuminaro et al., 2016) and the Firedrake-based unstructured ocean model of Scott et al. (2023). Basin-scale groundwater flow has the same thin-sheet geometry as a shallow ocean, so these methods provide a natural starting point for the present problem.

Richards' equation is nonlinear, so the conductivity changes as Newton's method updates the solution. The linear operator and the Galerkin coarse operator must therefore also change at every Newton iteration. This is the main difference from the

650 ocean application, where the Galerkin coarse operator is assembled once during setup. Rebuilding the coarse operator turns a one-off setup cost into a cost paid at every Newton iteration of every time step. We also tested a coarse operator reused across several Newton iterations to avoid that cost, but it did not perform well enough to include, and we do not report those runs.

The recommended presets also avoid a second recurring cost. A Chebyshev smoother requires a new eigenvalue estimate after every Jacobian update, whereas the Richardson smoother used here repeats its estimate only when the operator has
 655 changed appreciably (§4.1). On the near-isotropic benchmark of §4.2, this substitution gives no advantage, the coarse correction contributes little, and BJac-ILU is faster than every multilevel method. On the basin mesh of §4.4, runtime damping reduces wall time by 11–24 per cent, and the coarse correction becomes necessary once the time step reaches seasonal length, which justifies the cost of rebuilding the coarse operator.

At short time steps, BJac-ILU remains effective on the basin problem. The storage term, $(S_s S + C) \delta h / \Delta t$, represents
 660 the linearised change in water stored in the porous medium over a time step. At short time steps, the storage term is large relative to the flow operator. BJac-ILU also resolves the strong vertical coupling through the vertical-column line smoothing described above. A horizontal coarse correction therefore adds little at short time steps. This trade-off changes as the time step increases. The storage contribution decreases with $1/\Delta t$, so the anisotropic flow operator becomes more important. The horizontal modes then emerge as the difficult part of the linear system. Vertical line smoothing does not remove those horizontal
 665 modes. A horizontal coarse space is needed to represent and correct the slowly varying horizontal error.

Our results also show that a small aspect ratio does not, by itself, make the linear system ill-conditioned. For an anisotropic elliptic operator on a domain with horizontal extent L and vertical extent Δz , the horizontal and vertical eigenvalues differ by a ratio of $\approx (L/\Delta z)^2$. For the related thin-domain pressure problem Kramer et al. (2010) showed that a Neumann, or rigid-lid, boundary condition gives a condition number that scales as ε^{-2} with aspect ratio ε . A Dirichlet boundary condition instead
 670 keeps the operator well conditioned, independently of ε . The boundary condition therefore determines whether the small aspect ratio produces an ill-conditioned linear system.

The Lower Murrumbidgee problem is not purely Neumann because the lateral boundaries are held at fixed head. These Dirichlet boundaries constrain the horizontal null mode, while the storage term controls how the conditioning changes with the time step. To relate this argument to the results in §4, we measure the horizontal coupling that remains after BJac-ILU controls
 675 the vertical error. Integrating the linearised equation over a column of depth b gives the transmissivity and storativity

$$T = \int_0^b K \, dz, \quad S_{\text{col}} = \int_0^b (S_s S + C) \, dz. \quad (26)$$

The column-integrated diffusion number

$$D_{\text{col}} = \frac{\Delta t T}{S_{\text{col}} \Delta x^2}, \quad (27)$$

with horizontal mesh spacing Δx , measures the horizontal coupling over one time step. When $D_{\text{col}} \ll 1$, storage dominates
 680 and neighbouring columns remain weakly coupled, so BJac-ILU is sufficient. As D_{col} approaches unity, the column-constant modes couple across the basin and require a horizontal coarse correction. The vertically lumped preconditioners assemble and solve this depth-integrated horizontal problem.

Storage is not the only contribution that damps these column-constant modes. At the land surface the recharge is imposed as a total flux, $K \nabla(h + z) \cdot \mathbf{n} = q_T$, and because K depends on h the linearisation of this condition for the Newton update δh gives

$$\frac{\partial K}{\partial h} \delta h \nabla(h + z) \cdot \mathbf{n} + K \nabla \delta h \cdot \mathbf{n} = 0, \quad (28)$$

which is a Robin condition for δh . Where $\partial K / \partial h > 0$, that is, where the surface is unsaturated, this boundary term damps the column-constant modes even in the absence of storage, with a strength set by the ratio $(\partial K / \partial h) \nabla(h + z) \cdot \mathbf{n} / (K \Delta z)$. D_{col} therefore describes the conditioning only where this contribution is small, and elsewhere it overstates the horizontal coupling.

The seasonal results follow this interpretation. In Table 5, D_{col} increases from 0.14 to 1.19 under horizontal refinement, while the largest time step sustained by BJac-ILU falls from the imposed three-month ceiling to 39 days. We note that the available values do not determine a universal threshold, but they show the expected loss of BJac-ILU performance as horizontal coupling becomes stronger.

In the saturated stress test, raising the water table by 10 m brings approximately 0.45 per cent of the columns to saturation at the surface. In these fully saturated columns, the specific moisture capacity $C = d\theta/dh$ vanishes because the saturated moisture content does not change with pressure head. Since $S_s = 0$, the column storativity S_{col} is also zero. The conductivity equals K_s for $h \geq 0$, so $\partial K / \partial h = 0$ and the surface Robin term vanishes as well. Neither storage nor the boundary condition then damps a head perturbation that is constant along these columns, whatever the time step. BJac-ILU consequently fails at the initial 60 s step and again after the controller reduces the step to 1.9 s. The vertically lumped coarse space contains these column-constant modes and removes them, which is consistent with the lumped presets completing the runs. This result depends on $S_s = 0$. A nonzero specific storage would restore a storage term in the saturated columns, and BJac-ILU would then be expected to converge at sufficiently short time steps.

These results explain why the preferred preconditioner depends on the physical regime. At short time steps on an unsaturated profile, BJac-ILU combines cheap iterations with sufficient control of the vertical error and is the fastest solver. At seasonal time steps and near saturation, the vertically lumped preconditioners maintain low iteration counts and remain both faster and more reliable. The column-integrated diffusion number describes the linear problem only. On the coarsest saturated mesh, the lumped methods have their lowest linear iteration count but do not reach the full three-month step, which points instead to a limit in the nonlinear solve. We therefore recommend the two direct-coarse vertically lumped presets as the G-ADOPT defaults for basin-scale Richards problems. BJac-ILU remains available for short-step simulations of unsaturated profiles.

Other Richards and groundwater solvers address the same anisotropy with semicoarsening multigrid applied directly to the three-dimensional operator. Jones and Woodward (2001) combine an inexact Newton–Krylov method with semicoarsening for heterogeneous, variably saturated flow, following methods developed for anisotropic elliptic operators (Schaffer, 1998). Related studies examine boundary-condition sensitivity (Ashby and Falgout, 1996) and algebraic multigrid (Herbst et al., 2008). Our method instead collapses the vertical dimension and solves a smaller two-dimensional coarse problem. Both approaches require the preconditioner to account for the anisotropy of the operator.

5.2 Scope and comparison with existing groundwater solvers

A common practice in the Earth sciences for benchmarking numerical results is the intercomparison of outputs between codes. For groundwater simulations, we note that such a comparison, for accuracy of numerical methods, is only meaningful when the underlying equations are the same, otherwise what is being demonstrated is difference in physics. For example, while MODFLOW remains the operational standard in groundwater modelling for good reasons of maturity, documentation, and efficiency (Harbaugh, 2005; Langevin et al., 2017), it is not, in general, a solver for the full Richards' equation. Its core model solves saturated groundwater flow by a control-volume finite-difference method, while unsaturated flow is represented through an optional vadose-zone addition. That package follows the one-dimensional vertical kinematic-wave approach of Niswonger et al. (2006), using volumetric water content as the state variable together with a Brooks–Corey conductivity relation. As a result, capillary diffusion is absent from the unsaturated flux, lateral unsaturated flow is excluded, and the vadose-zone state is not solved as part of the same nonlinear system as the saturated heads.

Among the codes that do solve the full Richards' equation, several offer broader physical capability than our present solver. ParFlow (Kuffour et al., 2020; Maxwell et al., 2015), for example, couples three-dimensional Richards flow to overland flow and land-surface processes, while HydroGeoSphere (Brunner and Simmons, 2012), CATHY (Camporese et al., 2010), PFLO-TRAN (Hammond et al., 2014), DuMu^x (Koch et al., 2021), OpenGeoSys (Kolditz et al., 2012), and Amanzi/ATS (Coon et al., 2016) all embed Richards flow within broader surface–subsurface or multiphysics frameworks. Moreover, several have been demonstrated at continental scale, ParFlow across the contiguous United States (Maxwell et al., 2015) and HydroGeoSphere over Canada (Chen et al., 2020b), using cell-centred finite-difference and control-volume finite-element discretisations, respectively. Other established tools, including the OpenFOAM-based solver of Orgogozo et al. (2014), HYDRUS (Šimůnek et al., 2016), and TOUGH3 (Jung et al., 2017), likewise provide mature capability across a range of scales and applications. We therefore claim neither broader process coverage nor greater raw extent. Nonetheless, the contribution of the present work lies in the flexibility of the numerical framework: our solver combines locally conservative spatial discretisation with interchangeable function spaces, temporal schemes, and mesh types within a single implementation. Expressing the residual in the Unified Form Language (UFL) also allows `pyadjoint` (Mitusch et al., 2019) to derive adjoint equations and gradients by algorithmic differentiation, providing the basis for formal inversion (§5.3). The solver is also released as open-source software, with the workflows that are open to reproduce, inspect, and extend any results.

The benchmarks in §3 and §4.2 provide an indirect check of the scheme against other codes. The Tracy problem has an exact solution and therefore leaves no uncertainty in the reference. The Vauclin and Cockett problems use the published geometries and hydraulic parameters of Vauclin et al. (1979) and Cockett et al. (2018), and our solutions agree qualitatively with the reported results (Clement et al., 1994; Clément et al., 2023; Shen and Phanikumar, 2010; Xu et al., 2012; Cockett et al., 2018). For the Vauclin problem, the self-convergence test in §3.3 further shows that the solution converges under mesh refinement. What we have not attempted is a controlled head-to-head against a single named code on a common mesh at a common tolerance, of the kind carried out for integrated surface–subsurface models in the IH-MIP intercomparisons (Maxwell et al., 2014; Kollet et al., 2017). Such a comparison is harder to interpret than it first appears, because codes differ in the order of

750 their discretisations and in how well they tolerate anisotropic elements, so that a difference in cost at a common mesh may only reflect a difference in accuracy. A useful comparison would therefore match accuracy, not mesh or tolerance, and we leave it to future work.

In the present study, the hydraulic conductivity K is treated as a scalar. This is a choice made for the benchmarks and scaling experiments reported here, rather than a limitation of the formulation. Within the unified form-language approach
755 used throughout this paper, extending K to a tensor field is straightforward. Following that logic, the main limitation of our present work is the calibration of K . In the Murrumbidgee case, for example, the saturated conductivity assigned to each aquifer is likewise a single scalar value per formation, as the composition of each formation is known more securely than its conductivity. Our results show that spatially varying fields already pass naturally through the G-ADOPT data pipeline. Terrain is taken from a national elevation model through `ausdem` (Ghelichkhan, 2026a), stratigraphy and the initial water table from
760 Bureau of Meteorology boreholes through `austrata` (Ghelichkhan, 2026b), and the mesh is assembled from these inputs by OMEGA (Ghelichkhan, 2026c). Better conductivity estimates, whether scalar or tensor-valued, could therefore be inserted without changing the solver structure. This is the practical motivation for the inverse-modelling discussion in §5.3.

5.3 Towards inverse modelling

The parameters that matter most to groundwater prediction are often those that are known least well. Saturated conductivity may
765 vary over orders of magnitude and is usually constrained by a small number of pumping tests, while the retention parameters of the soil-curve relations are commonly transferred from soil classes rather than measured in situ. In the Lower Murrumbidgee case, recharge is represented by a basin-wide average even though it varies with land use, irrigation, and season, and the initial water table is interpolated from scattered bore observations. In practice, such quantities are often calibrated either by derivative-free search or by simplifying the physics until the parameter space becomes tractable. Both approaches become difficult as the
770 number of unknowns grows, which is why calibration is often restricted to a few zonal constant coefficients, even when the underlying fields are spatially distributed. Gradient-based inversion offers an alternative, and it has already been shown to be feasible for Richards’ equation. Using a cell-centred finite-volume discretisation of the mixed form, Cockett et al. (2018) computed implicit adjoint sensitivities and carried out a three-dimensional inversion. Our approach differs from theirs in two respects. First, the adjoint equations and the gradient are derived automatically from the same discrete residual that defines the
775 forward model, so no separate adjoint discretisation has to be derived or maintained. Second, the forward model is designed for problems several orders of magnitude larger.

In our case, every term of the residual (14) is written symbolically, including the interior-penalty terms, the upwind gravity flux, the constitutive relations, and the time discretisation. This allows `pyadjoint` (Mitusch et al., 2019) to derive the adjoint equation and the gradient of an arbitrary misfit functional by algorithmic differentiation of the same residual statement, without
780 a separate adjoint derivation or a second code path to maintain. The natural inversion targets are the saturated conductivity of each formation, the Haverkamp or van Genuchten parameters, the recharge field, and the initial water table. Of these, the last is the most immediate, because the Bureau of Meteorology water-table observations used to initialise the Lower Murrumbidgee model in §4.3 could also serve as a misfit target. By default, `pyadjoint` solves the adjoint equations with the same solver

configuration as the forward problem, so the preconditioners developed here apply to the adjoint without further implementa-
785 tion. Because the upwind gravity flux makes the Jacobian non-symmetric, the adjoint operator differs from the forward one,
and we have not yet tested the performance of these preconditioners on it. The gradient is therefore already available within
the present framework, and the next step is to put it to use.

6 Conclusion

We have presented a scalable three-dimensional solver for Richards' equation, built around a locally conservative discontinu-
ous Galerkin discretisation, a symbolic residual in Firedrake, Newton iteration with an exact Jacobian, implicit time integration
790 through Irksome, and anisotropy-aware multigrid preconditioning. Tests against analytical benchmarks in two and three dimen-
sions produced spatial slopes consistent with the expected $p + 1$ rate and first-order temporal accuracy for Backward Euler. The
mass-conservation tests also clarified an important point. The mixed formulation preserved the domain-integrated water bal-
ance for both continuous and discontinuous spaces, but only the discontinuous discretisation preserved water cell by cell. This
795 local property matters for basin-scale flows, where the quantities of interest are water balances over parts of the aquifer, such
as a formation, an irrigation district, or a management zone.

The scaling experiments showed that the choice of the most effective preconditioner depends on the regime of the model, in
particular the balance between transmissivity and storativity. On the near-isotropic infiltration problem, block-Jacobi ILU(0)
was the fastest solver in wall time despite carrying the highest iteration count, because each of its iterations is cheap. The
800 multilevel methods held a flatter iteration count, but each iteration cost more, so none matched block-Jacobi's wall time. The
same held on the anisotropic basin meshes while the time step stayed short. At the seasonal time steps a basin model requires,
the ranking inverted on the same meshes. BJac-ILU's admissible time step was significantly limited as the horizontal mesh was
refined, falling from three months to thirty-nine days across an eightfold growth in problem size. On a profile driven closer
to saturation it could not advance the solution at all at any step size we allowed it. The vertically lumped presets instead held
805 four to ten linear iterations per Newton step in the graded stress test and up to eleven near saturation. They admitted the full
three-month step at every scale except the coarsest mesh of the saturated stress test, and in the graded stress test they ran 1.9 to
3.7 times faster than BJac-ILU. Their memory per process remained stable up to several hundred million degrees of freedom.
The Discussion identifies the quantity that separates the two regimes, the column-integrated diffusion number, and shows that
it increases through unity over the meshes on which BJac-ILU loses the three-month step. We therefore recommend the two
810 direct-coarse vertically lumped presets as defaults for the basin configurations examined here; block-Jacobi ILU(0) remains
the faster choice whenever a model genuinely operates at short steps on an unsaturated profile. In the Lower Murrumbidgee
application, these methods made it possible to integrate a terrain-following basin model built from public Australian datasets
without manual solver intervention. This case study demonstrates computational feasibility rather than predictive skill.

The solver is released within G-ADOPT, the Geoscientific ADjoint Optimisation Platform (Davies et al., 2022; Ghelichkhan
815 et al., 2024; Scott et al., 2025; Gibson et al., 2026), an open-source, adjoint-capable platform built on Firedrake, PETSc, and
pyadjoint (Ham et al., 2023; Mitusch et al., 2019; Balay et al., 2026), including all the benchmark problems presented in this

work. Because the residual is written symbolically, adjoint equations and gradients of misfit functionals can be obtained by algorithmic differentiation. The present work is therefore not only a forward solver for basin-scale variably saturated flow, but also a practical basis for future inverse modelling with the full Richards' equation.

820 Appendix A: Solver configurations

The following listings give the PETSc options that define the preconditioners in §4.1. Shared convergence criteria and the Newton options of Listing 1 are omitted.

```

825 1: sor_parameters = {
2:   "ksp_type": "gmres",
3:   "pc_type": "sor",
4: }
5:
830 6: bjacobi_parameters = {
7:   "ksp_type": "gmres",
8:   "pc_type": "bjacobi",
9:   "sub_pc_type": "ilu",
10:  # Zero fill limits memory while retaining coupling within each column block.
835 11:  "sub_pc_factor_levels": 0,
12: }
```

Listing A1. Single-level baselines. BJac-SOR uses rank-local symmetric SOR. BJac-ILU uses rank-local ILU(0), which acts on complete vertical columns under the mesh partition.

```

840 1: boomeramg_parameters = {
2:   "pc_type": "hypre",
3:   "pc_hypre_type": "boomeramg",
4:   # A higher threshold excludes weak horizontal couplings from coarsening.
5:   "pc_hypre_boomeramg_strong_threshold": 0.5,
6:   "pc_hypre_boomeramg_coarsen_type": "HMIS",
845 7:   "pc_hypre_boomeramg_interp_type": "ext+i",
8:   # Aggressive early coarsening controls the complexity of the DG hierarchy.
9:   "pc_hypre_boomeramg_agg_n1": 2,
10:  "pc_hypre_boomeramg_truncfactor": 0.3,
850 11:  "pc_hypre_boomeramg_P_max": 4,
12: }
```

Listing A2. Hypre BoomerAMG with parameters selected for the vertical anisotropy of the extruded DG systems.

```

855 1: gmh_parameters = {
2:   "ksp_type": "fgmres",
3:   "pc_type": "gmh",
4:   # One block-ILU sweep smooths the vertical columns on each level.
5:   "mg_levels_ksp_type": "chebyshev",
6:   "mg_levels_ksp_max_it": 1,
7:   "mg_levels_pc_type": "bjacobi",
860 8:   "mg_levels_sub_pc_type": "ilu",
9:   # The coarse mesh remains three-dimensional after horizontal coarsening.
10:  "mg_coarse_ksp_type": "gmres",
11:  "mg_coarse_pc_type": "bjacobi",
865 12:  "mg_coarse_sub_pc_type": "ilu",
13: }
```

Listing A3. Geometric multigrid with horizontal-only coarsening (GMG-H). The hierarchy retains the full vertical discretisation on every level.

```

870 1: vlumping_parameters = {
2:   "ksp_type": "fgmres",
3:   "pc_type": "python",
4:   "pc_python_type": "gadopt.VerticallyLumpedPC",
5:   # Two inexact fine sweeps balance smoothing cost against coarse correction.
6:   "lumped_mg_levels_ksp_type": "chebyshev",
7:   "lumped_mg_levels_ksp_max_it": 2,
875 8:   "lumped_mg_levels_pc_type": "bjacobi",
9:   "lumped_mg_levels_sub_pc_type": "ilu",
10:  "lumped_mg_levels_sub_pc_factor_levels": 0,
11:  # A direct solve removes error represented on the collapsed coarse space.
12:  "lumped_mg_coarse_ksp_type": "preonly",
13:  "lumped_mg_coarse_pc_type": "lu",
880 14:  "lumped_mg_coarse_pc_factor_mat_solver_type": "mumps",
15: }
```

Listing A4. Vertical lumping (VLumping). A Galerkin projection forms the vertically constant coarse space, which MUMPS solves directly.

```

885 1: vlumping_linesmooth_parameters = {
2:   # One exact column solve replaces the two inexact block sweeps.
```

```

3:     "lumped_mg_levels_ksp_max_it": 1,
4:     "lumped_mg_levels_pc_type": "python",
5:     "lumped_mg_levels_pc_python_type": "firedrake.ASMLinesmoothPC",
6:     "lumped_mg_levels_pc_linesmooth_codims": "0",
7:     "lumped_mg_levels_pc_linesmooth_sub_sub_pc_type": "lu",
8: }

```

Listing A5. VLumping-linesmooth options that replace the inexact fine smoother in Listing A4. The direct coarse solve remains unchanged.

```

1: vlumping_hmg_parameters = {
2:     "pc_python_type": "gadopt.VerticallyLumpedHMGPC",
3:     # The iterative coarse solve avoids a monolithic distributed factorisation.
4:     "lumped_mg_coarse_pc_type": "mg",
5:     "lumped_mg_coarse_mg_levels_ksp_type": "chebyshev",
6:     "lumped_mg_coarse_mg_levels_pc_type": "bjacobi",
7:     "lumped_mg_coarse_mg_coarse_pc_type": "lu",
8: }

```

Listing A6. VLumping-HMG options that replace the direct coarse solve with geometric multigrid on the two-dimensional base mesh. The line smoother follows Listing A5.

```

1: recommended_vlumping_overrides = {
2:     # Runtime damping avoids repeated spectral estimates after Jacobian updates.
3:     "vlumping_omega_auto": True,
4: }

```

Listing A7. Runtime-damping override used by all reported vertically lumped configurations.

Code and data availability. The G-ADOPT finite-element framework used in this study is open-source and available at <https://github.com/g-adopt/g-adopt> (archived at <https://doi.org/10.5281/zenodo.19688884>; Gibson et al., 2026). The Richards-equation module and the vertically lumped preconditioners described here are maintained on a development branch of that repository and can be inspected through the associated open pull requests. The exact solution of the Tracy problem and the set-ups of the Vauclin and Cockett benchmarks are provided by the `gwassess` package at <https://github.com/g-adopt/gwassess>. The driver scripts, solver-parameter presets, parsed run records, and plotting routines that reproduce every figure and number reported in this paper are collected in a dedicated repository at <https://github.com/sghelichkhani/morrow2026> (archived at <https://doi.org/10.5281/zenodo.22910968>; Morrow et al., 2026). The parallel-scaling simulations were carried out on the Gadi supercomputer at the Australian National Computational Infrastructure (NCI) under project `xd2`.

Author contributions. LM and SG implemented the solver and ran the simulations. SG conceived and designed the study. SCK developed the preconditioning methodology and reviewed the implementation. All authors contributed to the writing of the manuscript.

Competing interests. The authors declare that they have no competing interests.

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925 manuscript for internal consistency and to fact-check the reported values against the underlying simulation records. The authors verified all resulting changes and take full responsibility for the content of this paper.

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References

- Amestoy, P. R., Duff, I. S., L'Excellent, J.-Y., and Koster, J.: A Fully Asynchronous Multifrontal Solver Using Distributed Dynamic Scheduling, *SIAM Journal on Matrix Analysis and Applications*, 23, 15–41, <https://doi.org/10.1137/S0895479899358194>, 2001.
- Amestoy, P. R., Guermouche, A., L'Excellent, J.-Y., and Pralet, S.: Hybrid Scheduling for the Parallel Solution of Linear Systems, *Parallel Computing*, 32, 136–156, <https://doi.org/10.1016/j.parco.2005.07.004>, 2006.
- Ashby, S. F. and Falgout, R. D.: A parallel multigrid preconditioned conjugate gradient algorithm for groundwater flow simulations, *Nuclear Science and Engineering*, 124, 145–159, <https://doi.org/10.13182/NSE96-A24230>, 1996.
- Balay, S., Gropp, W. D., McInnes, L. C., and Smith, B. F.: Efficient Management of Parallelism in Object-Oriented Numerical Software Libraries, in: *Modern Software Tools for Scientific Computing*, edited by Arge, E., Bruaset, A. M., and Langtangen, H. P., pp. 163–202, Birkhäuser Boston, https://doi.org/10.1007/978-1-4612-1986-6_8, 1997.
- Balay, S., Abhyankar, S., Adams, M. F., Benson, S., Brown, J., Brune, P., Buschelman, K., Constantinescu, E., Dalcin, L., Dener, A., Eijkhout, V., Faibussowitsch, J., Gropp, W. D., Hapla, V., Isaac, T., Jolivet, P., Karpeev, D., Kaushik, D., Knepley, M. G., Kong, F., Kruger, S., May, D. A., McInnes, L. C., Mills, R. T., Mitchell, L., Munson, T., Roman, J. E., Rupp, K., Sanan, P., Sarich, J., Smith, B. F., Suh, H., Zampini, S., Zhang, H., Zhang, H., and Zhang, J.: PETSc/TAO Users Manual, Tech. Rep. ANL-21/39 - Revision 3.25, Argonne National Laboratory, <https://doi.org/10.2172/3025790>, 2026.
- Bear, J.: *Dynamics of Fluids in Porous Media*, Dover Publications, Mineola, NY, ISBN 978-0-486-65675-5, reprint of the 1972 American Elsevier edition, 2013.
- Brunner, P. and Simmons, C. T.: HydroGeoSphere: A fully integrated, physically based hydrological model, *Groundwater*, 50, 170–176, <https://doi.org/10.1111/j.1745-6584.2011.00882.x>, 2012.
- Camporese, M., Paniconi, C., Putti, M., and Orlandini, S.: Surface-subsurface flow modeling with path-based runoff routing, boundary condition-based coupling, and assimilation of multisource observation data, *Water Resources Research*, 46, W02512, <https://doi.org/10.1029/2008WR007536>, 2010.
- Carrera, J., Alcolea, A., Medina, A., Hidalgo, J., and Slooten, L. J.: Inverse problem in hydrogeology, *Hydrogeology Journal*, 13, 206–222, <https://doi.org/10.1007/s10040-004-0404-7>, 2005.
- Celia, M. A., Bouloutas, E. T., and Zarba, R. L.: A general mass-conservative numerical solution for the unsaturated flow equation, *Water Resources Research*, 26, 1483–1496, <https://doi.org/10.1029/WR026i007p01483>, 1990.
- Chen, J., Kuang, X., and Zheng, C.: An empirical porosity–depth model for Earth’s crust, *Hydrogeology Journal*, 28, 2331–2339, <https://doi.org/10.1007/s10040-020-02214-x>, 2020a.
- Chen, J., Sudicky, E. A., Davison, J. H., Frey, S. K., Park, Y.-J., Hwang, H.-T., Erler, A. R., Berg, S. J., Callaghan, M. V., Miller, K., Ross, M., and Peltier, W. R.: Towards a climate-driven simulation of coupled surface-subsurface hydrology at the continental scale: a Canadian example, *Canadian Water Resources Journal*, 45, 11–27, <https://doi.org/10.1080/07011784.2019.1671235>, 2020b.
- Clément, J.-B., Golay, F., Ersoy, M., and Sous, D.: Adaptive discontinuous Galerkin method for Richards equation, in: *Topical problems of fluid mechanics 2020*, pp. 27–34, Institute of Thermomechanics, AS CR, v.v.i., <https://doi.org/10.14311/tpfm.2020.004>, 2020.
- Clément, J.-B., Golay, F., Ersoy, M., and Sous, D.: An adaptive strategy for discontinuous Galerkin simulations of Richards’ equation: Application to multi-materials dam wetting, *Advances in Water Resources*, 151, 103897, <https://doi.org/10.1016/j.advwatres.2021.103897>, 2021.

- Clément, J.-B., Sous, D., Bouchette, F., Golay, F., and Ersoy, M.: A Richards' equation-based model for wave-resolving simulation of variably-saturated beach groundwater flow dynamics, *Journal of Hydrology*, 619, 129–344, <https://doi.org/10.1016/j.jhydrol.2023.129344>, 2023.
- 970 Clement, T., Wise, W. R., and Molz, F. J.: A physically based, two-dimensional, finite-difference algorithm for modeling variably saturated flow, *Journal of Hydrology*, 161, 71–90, [https://doi.org/10.1016/0022-1694\(94\)90121-X](https://doi.org/10.1016/0022-1694(94)90121-X), 1994.
- Cockett, R., Heagy, L. J., and Haber, E.: Efficient 3D inversions using the Richards equation, *Computers & Geosciences*, 116, 91–102, <https://doi.org/10.1016/j.cageo.2018.04.006>, 2018.
- Condon, L. E., Kollet, S., Bierkens, M. F., Fogg, G. E., Maxwell, R. M., Hill, M. C., Fransen, H.-J. H., Verhoef, A., Van Loon, A. F., Sulis, M., et al.: Global groundwater modeling and monitoring: Opportunities and challenges, *Water Resources Research*, 57, e2020WR029500, <https://doi.org/10.1029/2020WR029500>, 2021.
- 975 Coon, E. T., Moulton, J. D., and Painter, S. L.: Managing complexity in simulations of land surface and near-surface processes, *Environmental Modelling & Software*, 78, 134–149, <https://doi.org/10.1016/j.envsoft.2015.12.017>, 2016.
- Darcy, H.: *Les Fontaines Publiques de la Ville de Dijon*, Victor Dalmont, Paris, <https://gallica.bnf.fr/ark:/12148/bpt6k624312>, 1856.
- 980 Davies, D. R., Kramer, S. C., Ghelichkhan, S., and Gibson, A.: Towards automatic finite-element methods for geodynamics via Firedrake, *Geoscientific Model Development*, 15, 5127–5166, <https://doi.org/10.5194/gmd-15-5127-2022>, 2022.
- Earman, S. and Dettinger, M.: Potential impacts of climate change on groundwater resources—a global review, *Journal of Water and Climate Change*, 2, 213–229, <https://doi.org/10.2166/wcc.2011.034>, 2011.
- Fan, Y., Li, H., and Miguez-Macho, G.: Global patterns of groundwater table depth, *Science*, 339, 940–943, <https://doi.org/10.1126/science.1229881>, 2013.
- 985 Farrell, P. E., Kirby, R. C., and Marchena-Menéndez, J.: Irksome: Automating Runge–Kutta time-stepping for finite element methods, *ACM Transactions on Mathematical Software*, 47, 1–26, <https://doi.org/10.1145/3466168>, 2021.
- Farthing, M. W. and Ogden, F. L.: Numerical solution of Richards' equation: A review of advances and challenges, *Soil Science Society of America Journal*, 81, 1257–1269, <https://doi.org/10.2136/sssaj2017.02.0058>, 2017.
- 990 Févotte, F., Rappaport, A., and Vohralík, M.: Adaptive regularization for the Richards equation, *Computational Geosciences*, 28, 1371–1388, <https://doi.org/10.1007/s10596-024-10309-7>, 2024.
- Ghelichkhan, S.: ausdem: Australian DEM data from Geoscience Australia as xarray arrays, <https://doi.org/10.5281/zenodo.20755954>, 2026a.
- Ghelichkhan, S.: austrata: access to Geoscience Australia and Bureau of Meteorology NGIS groundwater data, <https://doi.org/10.5281/zenodo.20756406>, 2026b.
- 995 Ghelichkhan, S.: OMEGA: Optimised Modelling of Environmental Groundwater in Australia, mesh generation for G-ADOPT, <https://doi.org/10.5281/zenodo.23083131>, 2026c.
- Ghelichkhan, S., Gibson, A., Davies, D. R., Kramer, S. C., and Ham, D. A.: Automatic adjoint-based inversion schemes for geodynamics: reconstructing the evolution of Earth's mantle in space and time, *Geoscientific Model Development*, 17, 5057–5086, <https://doi.org/10.5194/gmd-17-5057-2024>, 2024.
- 1000 Gibson, A., Davies, R., Kramer, S., Ghelichkhan, S., Turner, R., Duvernay, T., and Scott, W.: G-ADOPT: the Geoscientific ADjoint Optimisation Platform, Zenodo, software version 2026.4.0, <https://doi.org/10.5281/zenodo.19688884>, 2026.

- Gleeson, T., Wagener, T., Döll, P., Zipper, S. C., West, C., Wada, Y., Taylor, R., Scanlon, B., Rosolem, R., Rahman, S., et al.: GMD perspective: The quest to improve the evaluation of groundwater representation in continental-to global-scale models, *Geoscientific Model Development*, 14, 7545–7571, <https://doi.org/10.5194/gmd-14-7545-2021>, 2021.
- Ham, D. A., Kelly, P. H. J., Mitchell, L., Cotter, C. J., Kirby, R. C., Sagiya, K., Bouziani, N., Vorderwuelbecke, S., Gregory, T. J., Betteridge, J., Shaper, D. R., Nixon-Hill, R. W., Ward, C. J., Farrell, P. E., Brubeck, P. D., Marsden, I., Gibson, T. H., Homolya, M., Sun, T., McRae, A. T. T., Luporini, F., Gregory, A., Lange, M., Funke, S. W., Rathgeber, F., Bercea, G.-T., and Markall, G. R.: *Firedrake User Manual*, Imperial College London and University of Oxford and Baylor University and University of Washington, first edition edn., <https://doi.org/10.25561/104839>, 2023.
- Hammond, G., Lichtner, P., and Mills, R.: Evaluating the performance of parallel subsurface simulators: An illustrative example with PFLO-TRAN, *Water Resources Research*, 50, 208–228, <https://doi.org/10.1002/2012WR013483>, 2014.
- Harbaugh, A. W.: MODFLOW-2005, the U.S. Geological Survey modular ground-water model: the Ground-Water Flow Process, Tech. Rep. Techniques and Methods 6-A16, U.S. Geological Survey, Reston, VA, <https://doi.org/10.3133/tm6A16>, 2005.
- Haverkamp, R., Vauclin, M., Touma, J., Wierenga, P. J., and Vachaud, G.: A comparison of numerical simulation models for one-dimensional infiltration, *Soil Science Society of America Journal*, 41, 285–294, <https://doi.org/10.2136/sssaj1977.03615995004100020024x>, 1977.
- Herbst, M., Gottschalk, S., Reißel, M., Hardelauf, H., Kasteel, R., Javaux, M., Vanderborght, J., and Vereecken, H.: On preconditioning for a parallel solution of the Richards equation, *Computers & Geosciences*, 34, 1958–1963, <https://doi.org/10.1016/j.cageo.2008.02.020>, 2008.
- Hughes, J. D., Langevin, C. D., and Banta, E. R.: Documentation for the MODFLOW 6 framework, Tech. Rep. Techniques and Methods 6-A57, U.S. Geological Survey, Reston, VA, <https://doi.org/10.3133/tm6A57>, 2017.
- Hughes, T. J. R., Engel, G., Mazzei, L., and Larson, M. G.: The Continuous Galerkin Method Is Locally Conservative, *Journal of Computational Physics*, 163, 467–488, <https://doi.org/10.1006/jcph.2000.6577>, 2000.
- Jones, J. E. and Woodward, C. S.: Newton–Krylov-multigrid solvers for large-scale, highly heterogeneous, variably saturated flow problems, *Advances in Water Resources*, 24, 763–774, [https://doi.org/10.1016/S0309-1708\(00\)00075-0](https://doi.org/10.1016/S0309-1708(00)00075-0), 2001.
- Ju, S.-H. and Kung, K.-J.: Mass types, element orders and solution schemes for the Richards equation, *Computers & Geosciences*, 23, 175–187, [https://doi.org/10.1016/S0098-3004\(97\)85440-4](https://doi.org/10.1016/S0098-3004(97)85440-4), 1997.
- Jung, Y., Pau, G. S. H., Finsterle, S., and Pollyea, R. M.: TOUGH3: A new efficient version of the TOUGH suite of multiphase flow and transport simulators, *Computers & Geosciences*, 108, 2–7, <https://doi.org/10.1016/j.cageo.2016.09.009>, 2017.
- Kavetski, D., Binning, P., and Sloan, S.: Adaptive time stepping and error control in a mass conservative numerical solution of the mixed form of Richards equation, *Advances in Water Resources*, 24, 595–605, [https://doi.org/10.1016/S0309-1708\(00\)00076-2](https://doi.org/10.1016/S0309-1708(00)00076-2), 2001.
- Kirkland, M. R., Hills, R., and Wierenga, P.: Algorithms for solving Richards’ equation for variably saturated soils, *Water Resources Research*, 28, 2049–2058, <https://doi.org/10.1029/92WR00802>, 1992.
- Koch, T., Gläser, D., Weishaupt, K., Ackermann, S., Beck, M., Becker, B., Burbulla, S., Class, H., Coltman, E., Fetzer, T., et al.: DuMux 3 – an open-source simulator for solving flow and transport problems in porous media with a focus on model coupling, *Computers & Mathematics with Applications*, 81, 423–443, <https://doi.org/10.1016/j.camwa.2020.02.012>, 2021.
- Kolditz, O., Bauer, S., Bilke, L., Böttcher, N., Delfs, J., Fischer, T., Görke, U., Kalbacher, T., Kosakowski, G., McDermott, C., et al.: OpenGeoSys: an open-source initiative for numerical simulation of thermo-hydro-mechanical/chemical (THM/C) processes in porous media, *Environmental Earth Sciences*, 67, 589–599, <https://doi.org/10.1007/s12665-012-1546-x>, 2012.

- Kollet, S., Sulis, M., Maxwell, R. M., Paniconi, C., Putti, M., Bertoldi, G., Coon, E. T., Cordano, E., Endrizzi, S., Kikinzon, E., et al.: The integrated hydrologic model intercomparison project, IH-MIP2: A second set of benchmark results to diagnose integrated hydrology and feedbacks, *Water Resources Research*, 53, 867–890, <https://doi.org/10.1002/2016WR019191>, 2017.
- Kramer, S., Cotter, C., and Pain, C.: Solving the Poisson equation on small aspect ratio domains using unstructured meshes, *Ocean Modelling*, 35, 253–263, <https://doi.org/10.1016/j.ocemod.2010.08.001>, 2010.
- Kuffour, B. N., Engdahl, N. B., Woodward, C. S., Condon, L. E., Kollet, S., and Maxwell, R. M.: Simulating coupled surface–subsurface flows with ParFlow v3.5.0: capabilities, applications, and ongoing development of an open-source, massively parallel, integrated hydrologic model, *Geoscientific Model Development*, 13, 1373–1397, <https://doi.org/10.5194/gmd-13-1373-2020>, 2020.
- Kumar, C.: Impact of climate change on groundwater resources, in: *Natural resources management: Concepts, methodologies, tools, and applications*, pp. 1094–1120, IGI Global Scientific Publishing, <https://doi.org/10.4018/978-1-5225-0803-8.ch052>, 2017.
- Langevin, C. D., Hughes, J. D., Banta, E. R., Niswonger, R. G., Panday, S., and Provost, A. M.: Documentation for the MODFLOW 6 Groundwater Flow Model, Tech. Rep. Techniques and Methods 6-A55, U.S. Geological Survey, Reston, VA, <https://doi.org/10.3133/tm6A55>, 2017.
- Larson, M. G. and Niklasson, A. J.: A conservative flux for the continuous Galerkin method based on discontinuous enrichment, *Calcolo*, 41, 65–76, <https://doi.org/10.1007/s10092-004-0084-7>, 2004.
- Li, H., Farthing, M., Dawson, C., and Miller, C.: Local discontinuous Galerkin approximations to Richards’ equation, *Advances in Water Resources*, 30, 555–575, <https://doi.org/10.1016/j.advwatres.2006.04.011>, 2007.
- List, F. and Radu, F. A.: A study on iterative methods for solving Richards’ equation, *Computational Geosciences*, 20, 341–353, <https://doi.org/10.1007/s10596-016-9566-3>, 2016.
- Marshall, J., Adcroft, A., Hill, C., Perelman, L., and Heisey, C.: A finite-volume, incompressible Navier–Stokes model for studies of the ocean on parallel computers, *Journal of Geophysical Research: Oceans*, 102, 5753–5766, <https://doi.org/10.1029/96JC02775>, 1997.
- Maxwell, R., Condon, L., and Kollet, S.: A high-resolution simulation of groundwater and surface water over most of the continental US with the integrated hydrologic model ParFlow v3, *Geoscientific Model Development*, 8, 923–937, <https://doi.org/10.5194/gmd-8-923-2015>, 2015.
- Maxwell, R. M., Putti, M., Meyerhoff, S., Delfs, J.-O., Ferguson, I. M., Ivanov, V., Kim, J., Kolditz, O., Kollet, S. J., Kumar, M., et al.: Surface–subsurface model intercomparison: A first set of benchmark results to diagnose integrated hydrology and feedbacks, *Water Resources Research*, 50, 1531–1549, <https://doi.org/10.1002/2013WR013725>, 2014.
- Miller, C. T., Williams, G. A., Kelley, C. T., and Tocci, M. D.: Robust solution of Richards’ equation for nonuniform porous media, *Water Resources Research*, 34, 2599–2610, <https://doi.org/10.1029/98WR01673>, 1998.
- Mitusch, S., Funke, S., and Dokken, J.: dolfin-adjoint 2018.1: automated adjoints for FEniCS and Firedrake, *Journal of Open Source Software*, 4, 1292, <https://doi.org/10.21105/joss.01292>, 2019.
- Morrow, L., Ghelichkhan, S., and Kramer, S. C.: morrow2026: reproducibility repository for “Scalable and locally mass-conservative simulation of basin-scale variably saturated groundwater flow”, <https://doi.org/10.5281/zenodo.22910968>, 2026.
- Niswonger, R. G., Prudic, D. E., and Regan, R. S.: Documentation of the Unsaturated-Zone Flow (UZFI) Package for modeling unsaturated flow between the land surface and the water table with MODFLOW-2005, Tech. Rep. Techniques and Methods 6-A19, U.S. Geological Survey, Reston, VA, <https://doi.org/10.3133/tm6A19>, 2006.

- 1075 Nitsche, J.: Über ein Variationsprinzip zur Lösung von Dirichlet-Problemen bei Verwendung von Teilräumen, die keinen Randbedingungen unterworfen sind, *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, 36, 9–15, <https://doi.org/10.1007/BF02995904>, 1971.
- Ojha, R., Ramadas, M., and Govindaraju, R. S.: Current and future challenges in groundwater. I: Modeling and management of resources, *Journal of Hydrologic Engineering*, 20, A4014 007, [https://doi.org/10.1061/\(ASCE\)HE.1943-5584.0000928](https://doi.org/10.1061/(ASCE)HE.1943-5584.0000928), 2015.
- 1080 Orgogozo, L., Renon, N., Soulaire, C., Hénou, F., Tomer, S., Labat, D., Pokrovsky, O., Sekhar, M., Ababou, R., and Quintard, M.: An open source massively parallel solver for Richards equation: Mechanistic modelling of water fluxes at the watershed scale, *Computer Physics Communications*, 185, 3358–3371, <https://doi.org/10.1016/j.cpc.2014.08.004>, 2014.
- Richards, L. A.: Capillary conduction of liquids through porous mediums, *Physics*, 1, 318–333, <https://doi.org/10.1063/1.1745010>, 1931.
- Rivière, B.: Discontinuous Galerkin methods for solving elliptic and parabolic equations: theory and implementation, *Society for Industrial and Applied Mathematics*, Philadelphia, PA, <https://doi.org/10.1137/1.9780898717440>, 2008.
- 1085 Schaffer, S.: A semi-coarsening multigrid method for elliptic partial differential equations with highly discontinuous and anisotropic coefficients, *SIAM Journal on Scientific Computing*, 20, 228–242, <https://doi.org/10.1137/S1064827595281587>, 1998.
- Scott, W., Hoggard, M., Duvernay, T., Ghelichkhan, S., Gibson, A., Roberts, D., Kramer, S. C., and Davies, D. R.: Automated forward and adjoint modelling of viscoelastic deformation of the solid Earth, *EGUsphere*, 2025, 1–43, <https://doi.org/10.5194/egusphere-2025-4168>, 2025.
- 1090 Scott, W. I., Kramer, S. C., Holland, P. R., Nicholls, K. W., Siegert, M. J., and Piggott, M. D.: Towards a fully unstructured ocean model for ice shelf cavity environments: Model development and verification using the Firedrake finite element framework, *Ocean Modelling*, 182, 102 178, <https://doi.org/10.1016/j.ocemod.2023.102178>, 2023.
- Shahbazi, A., Saeidi, A., and Chesnaux, R.: A review of existing methods used to evaluate the hydraulic conductivity of a fractured rock mass, *Engineering Geology*, 265, 105 438, <https://doi.org/10.1016/j.enggeo.2019.105438>, 2020.
- 1095 Shen, C. and Phanikumar, M. S.: A process-based, distributed hydrologic model based on a large-scale method for surface–subsurface coupling, *Advances in Water Resources*, 33, 1524–1541, <https://doi.org/10.1016/j.advwatres.2010.09.002>, 2010.
- Siebert, S., Burke, J., Faures, J. M., Frenken, K., Hoogeveen, J., Döll, P., and Portmann, F. T.: Groundwater use for irrigation — a global inventory, *Hydrology and Earth System Sciences*, 14, 1863–1880, <https://doi.org/10.5194/hess-14-1863-2010>, 2010.
- 1100 Šimůnek, J., van Genuchten, M. T., and Šejna, M.: Recent developments and applications of the HYDRUS computer software packages, *Vadose Zone Journal*, 15, 1–25, <https://doi.org/10.2136/vzj2016.04.0033>, 2016.
- Stokke, J. S., Mitra, K., Storvik, E., Both, J. W., and Radu, F. A.: An adaptive solution strategy for Richards’ equation, *Computers & Mathematics with Applications*, 152, 155–167, <https://doi.org/10.1016/j.camwa.2023.10.020>, 2023.
- Taylor, R. G., Scanlon, B., Döll, P., Rodell, M., Van Beek, R., Wada, Y., Longueuevergne, L., Leblanc, M., Famiglietti, J. S., Edmunds, M., et al.: Ground water and climate change, *Nature Climate Change*, 3, 322–329, <https://doi.org/10.1038/nclimate1744>, 2013.
- 1105 Tracy, F. T.: Clean two-and three-dimensional analytical solutions of Richards’ equation for testing numerical solvers, *Water Resources Research*, 42, <https://doi.org/10.1029/2005WR004638>, 2006.
- Tuminaro, R., Perego, M., Tezaur, I., Salinger, A., and Price, S.: A matrix dependent/algebraic multigrid approach for extruded meshes with applications to ice sheet modeling, *SIAM Journal on Scientific Computing*, 38, C504–C532, <https://doi.org/10.1137/15M1040839>, 2016.
- 1110 UNESCO: The United Nations World Water Development Report 2022 — Groundwater: Making the Invisible Visible, UNESCO, Paris, ISBN 978-92-3-100507-7, <https://doi.org/10.18356/9789210015363>, published by UNESCO on behalf of UN-Water, 2022.

- United Nations General Assembly: Transforming our World: The 2030 Agenda for Sustainable Development, Resolution A/RES/70/1, adopted 25 September 2015, <https://sdgs.un.org/2030agenda>, sustainable Development Goal 6: Ensure availability and sustainable management of water and sanitation for all, 2015.
- 1115 Van Genuchten, M. T.: A Closed-form Equation for Predicting the Hydraulic Conductivity of Unsaturated Soils, *Soil Science Society of America Journal*, 44, 892–898, <https://doi.org/10.2136/sssaj1980.03615995004400050002x>, 1980.
- Vauclin, M., Khanji, D., and Vachaud, G.: Experimental and numerical study of a transient, two-dimensional unsaturated-saturated water table recharge problem, *Water Resources Research*, 15, 1089–1101, <https://doi.org/10.1029/WR015i005p01089>, 1979.
- Xu, X., Huang, G., Zhan, H., Qu, Z., and Huang, Q.: Integration of SWAP and MODFLOW-2000 for modeling groundwater dynamics in shallow water table areas, *Journal of Hydrology*, 412–413, 170–181, <https://doi.org/10.1016/j.jhydrol.2011.07.002>, 2012.
- 1120 Zha, Y., Yang, J., Zeng, J., Tso, C.-H. M., Zeng, W., and Shi, L.: Review of numerical solution of Richardson–Richards equation for variably saturated flow in soils, *WIREs Water*, 6, e1364, <https://doi.org/10.1002/wat2.1364>, 2019.
- Zhou, H., Gómez-Hernández, J. J., and Li, L.: Inverse methods in hydrogeology: Evolution and recent trends, *Advances in Water Resources*, 63, 22–37, <https://doi.org/10.1016/j.advwatres.2013.10.014>, 2014.
- 1125 Zhou, Y. and Li, W.: A review of regional groundwater flow modeling, *Geoscience Frontiers*, 2, 205–214, <https://doi.org/10.1016/j.gsf.2011.03.003>, 2011.