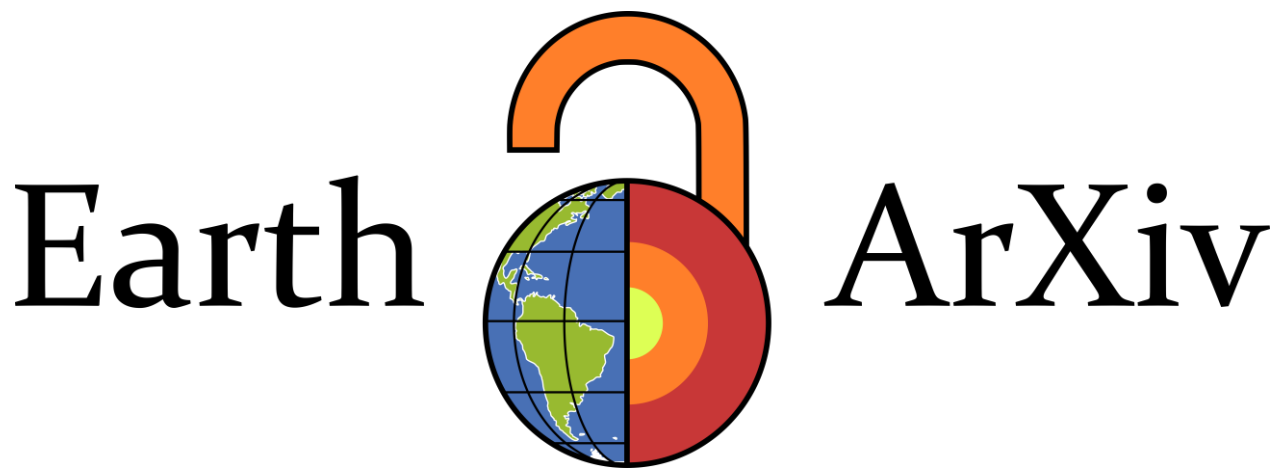




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Structure Function Sampling Uncertainty: A Gaussian-Increment

Reference Framework

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Submitted to the Journal of Atmospheric and Oceanic Technology 26 October 2026.

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7 ABSTRACT: Structure functions (SFs) are widely used to characterize scale-dependent variability in geophysical fields, but their sampling uncertainty is difficult to quantify because pairwise
8 increments are generally dependent. Dependence arises from both shared observations and spatial
9 covariance of the underlying field, so pair counts alone do not determine uncertainty. The
10 sampling variance is governed by covariance among increment powers, termed structure-function
11 contributions (SFCs). For the second-order SF, a Gaussian-increment closure expresses SFC covariance
12 in terms of the SF itself, allowing fractional sampling variance to be calculated from a
13 specified second-order model and sampling geometry. For regular one-dimensional sampling with
14 a finite integral SFC correlation scale, the well-resolved, long-record limit gives approximately
15 one effective degree of freedom per correlation scale sampled. The dependence of this scale on
16 separation yields simple uncertainty scalings for finite-scale and scale-free models. In particular,
17 for spectra near k^{-2} , the fractional standard error scales as the inverse square root of the number
18 of separation-length intervals sampled. Sampling variance can be used to construct approximate
19 confidence intervals, while cross-scale covariance allows uncertainty to be propagated to derived
20 quantities such as scaling exponents. Departures from the Gaussian reference enter through an
21 additional fourth-order cumulant contribution that is more difficult to quantify. The Gaussian
22 result thus provides a tractable baseline that explicitly accounts for sampling geometry and spatial
23 dependence.
24

25 SIGNIFICANCE STATEMENT: Structure functions are widely used to characterize spatial vari-
26 ability and infer turbulence properties, yet their sampling uncertainty is difficult to quantify because
27 the underlying increments are strongly correlated. We develop a framework that relates sampling
28 uncertainty to spatial covariance and sampling geometry, using Gaussian increments as a reference.
29 The framework quantifies uncertainty and its correlation across separation scales, providing a basis
30 for assessing the reliability of structure-function estimates and quantities derived from them.

31 1. Introduction

32 Structure functions (SFs) characterize the scale dependence of increments in a field observed in
33 space or time. Second-order structure functions (SF2) are widely used to identify scaling regimes
34 and to estimate turbulence dissipation rates (e.g., Thomson 2012; Wiles et al. 2006; Balwada et al.
35 2016), while third-order structure functions (SF3) can diagnose scale-dependent energy transfer
36 (e.g., Balwada et al. 2016; Qiu et al. 2022). Interpreting these estimates requires quantifying their
37 sampling uncertainty.

38 Historically, SF uncertainty has been treated inconsistently in the oceanographic literature. In
39 some studies, no quantitative uncertainty is reported for the SFs themselves (Balwada et al. 2016;
40 Mensa et al. 2018). Other studies provide error bars or confidence intervals using relatively brief
41 statistical prescriptions, such as standard errors, bootstrap or jackknife resampling, or variability
42 among repeated realizations, with only limited discussion of the details or underlying assumptions
43 (McCaffrey et al. 2015; Qiu et al. 2022; Hypolite et al. 2026). Such treatments can give the
44 impression that SF uncertainty is a well-established and trivial calculation. In contrast, Nichols-
45 Pagel et al. (2008) argue that a generally applicable statistical treatment of SF uncertainty does not
46 exist.

47 Geostatistics widely uses empirical semivariograms, which are equivalent to half SF2. Semi-
48 variogram sampling uncertainty has been investigated more systematically. Under Gaussian field
49 assumptions, Bogaert and Russo (1999) and Pardo-Igúzquiza and Dowd (2001) derived expres-
50 sions for variogram covariance and investigated its implications for sampling design and parameter
51 estimation. Ortiz and Deutsch (2002) examined pointwise variogram uncertainty, while Marchant
52 and Lark (2004) evaluated the performance of uncertainty estimators.

As in many statistical inference problems, quantifying the uncertainty of an SF estimate is more difficult than calculating the SF itself. The difficulty arises because the increments entering a SF estimate are generally dependent. Pairs may share observations, and spatial correlation in the underlying field can couple increments even when their endpoints are distinct. Raising increments to a power further changes their covariance. Sampling uncertainty therefore depends on both the field statistics and the sampling geometry, and cannot be inferred from pair counts alone.

A further complication is that the SF errors are strongly correlated across separations (Nichols-Pagel et al. 2008) and therefore need not appear as point-to-point scatter on a SF plot. Structure function estimates can remain visually smooth and plausible even when they depart substantially from the true population structure function. Visual smoothness and small residuals about a fitted power law can thus give a misleading impression of precision.

Here, we derive the sampling variance of second-order structure-function estimates assuming jointly Gaussian increments. The formulation accommodates arbitrary sampling geometries and shows how uncertainty depends on the covariance structure of the underlying field. For regularly sampled one-dimensional records, we derive asymptotic uncertainty scalings for finite-scale and scale-free models. We also develop approximate confidence intervals and propagate uncertainty to derived quantities. Finally, we discuss the limitations of the Gaussian-increment assumption, including uncertainty contributions that cannot be determined from second-order statistics alone.

2. Sampling uncertainty of structure functions

a. Structure-function estimator and sampling variance

Let $X(\mathbf{x})$ denote a scalar field, such as a velocity component or tracer, observed at locations $\mathbf{x}$. Here, $\mathbf{x}$ may represent a coordinate along a one-dimensional transect or a position in a two-dimensional spatial domain.

For a separation vector $\mathbf{r}$ and structure-function order p , define

$$\delta X(\mathbf{x}, \mathbf{r}) = X(\mathbf{x} + \mathbf{r}) - X(\mathbf{x}), \quad Y(\mathbf{x}, \mathbf{r}) = [\delta X(\mathbf{x}, \mathbf{r})]^p. \quad (1)$$

Here, $Y(\mathbf{x}, \mathbf{r})$ is a structure-function pair contribution (SFC). For odd orders, signed structure functions require a specified increment orientation; absolute-moment structure functions instead

79 use $Y_i = |\delta X_i|^p$. These conventions coincide for even integer orders. The covariance formulation
 80 below applies to either choice; the explicit results developed here concern $p = 2$. For a statistically
 81 homogeneous field, the population SF is defined as

$$D(r) = E[Y(\mathbf{x}, r)], \quad (2)$$

82 where the dependence on order p is left implicit here and below. If the statistics are isotropic, or
 83 if directions are averaged explicitly, the SF depends only on the separation magnitude $r = |\mathbf{r}|$, and
 84 we write $D(r)$. The empirical estimate of (2) is typically obtained by binning observation pairs
 85 according to their separation magnitude. For the N_r pairs in a bin centered on r , with endpoints
 86 $\mathbf{x}_{i_1}$ and $\mathbf{x}_{i_2}$, define

$$Y_i = [X(\mathbf{x}_{i_2}) - X(\mathbf{x}_{i_1})]^p, \quad ||\mathbf{x}_{i_2} - \mathbf{x}_{i_1}| - r| < \delta r. \quad (3)$$

87 The dependence of the SFC Y_i on the separation bin r is left implicit for notational simplicity. The
 88 empirical SF is then the mean of the SFCs,

$$\hat{D}(r) = \frac{1}{N_r} \sum_{i=1}^{N_r} Y_i. \quad (4)$$

89 We assume separation bins are sufficiently narrow that variation of the population SF $D(r)$ within
 90 each bin is negligible.

91 Our objective is to quantify the sampling uncertainty of $\hat{D}(r)$ for a finite set of observations with
 92 potentially correlated SFCs and arbitrary sampling geometry. For fixed sampling geometry and
 93 pair selection, the exact sampling variance is

$$V(r) \equiv \text{Var}[\hat{D}(r)] = \frac{1}{N_r^2} \sum_{i=1}^{N_r} \sum_{j=1}^{N_r} C_{ij}, \quad (5)$$

94 where

$$C_{ij} \equiv \text{Cov}[Y_i, Y_j] \quad (6)$$

95 is the SFC covariance, with implicit dependence on the separation bin r . The diagonal terms,
 96 $C_{ii} = \text{Var}(Y_i) \simeq C_0$, describe the variability of individual SFCs, assumed to have approximately

97 equal variance C_0 . The off-diagonal terms account for their statistical dependence. If the SFCs were
 98 uncorrelated, (5) would reduce to the familiar variance-of-the-mean expression, $V(r) \simeq C_0/N_r$. In
 99 general, however, the off-diagonal contributions cannot be neglected.

100 Dependence among the SFCs arises through several mechanisms. First, different increments
 101 may share one or more observations, introducing dependence through the construction of the
 102 increments themselves. Second, spatial correlation in the underlying field couples increments
 103 even when their endpoints are distinct. Finally, because $Y_i = (\delta X_i)^p$ is a nonlinear function of the
 104 increment, its covariance depends on higher-order joint statistics of the field. For the second-order
 105 SF, $C_{ij} = \text{Cov}[(\delta X_i)^2, (\delta X_j)^2]$ involves *fourth-order* statistics of the increments; for a third-order
 106 SF, the corresponding covariance involves sixth-order statistics, and so on. Estimation of these
 107 higher-order quantities is increasingly sensitive to finite sampling, distribution tails, intermittency,
 108 and rare coherent events.

109 Although (5) is exact, its practical evaluation remains challenging because the SFC covariance
 110 matrix must generally be inferred from finite observations. Estimating SF uncertainty can therefore
 111 be substantially more difficult than estimating the SF itself.

112 The effect of the SFC dependence can also be interpreted through an effective sample size, N_{eff} ,
 113 defined by

$$V(r) = \frac{C_0}{N_{\text{eff}}}, \quad N_{\text{eff}} = \frac{N_r^2 C_0}{\sum_{i,j} C_{ij}}. \quad (7)$$

114 For uncorrelated contributions, $N_{\text{eff}} = N_r$, recovering the familiar variance-of-the-mean result.
 115 More generally, N_{eff} summarizes the cumulative effect of covariance among SFCs. Importantly, it
 116 characterizes the covariance of SFCs of a particular order at a particular separation. It should not
 117 be confused with the number of statistically independent increments or observations in the record.

118 ¹ Although this representation provides an intuitive interpretation of the covariance sum (5), it
 119 does not simplify its estimation.

120 A substantial simplification is possible for regularly sampled, stationary one-dimensional records,
 121 where the SFC covariance can be expressed as a function of spatial lag alone.

¹This distinction is easily overlooked. For example, Camussi et al. (1996) estimated SF uncertainty using an effective number of independent velocity increments inferred from their correlation length. However, the sampling variance of an SF is governed by the covariance of the powered increments, which generally has a different correlation structure and depends on the SF order.

122 *b. Regularly sampled 1D case*

123 Consider N observations on a regular one-dimensional grid with unit spacing, $x_i = i$. At integer
 124 separation r , define $Y_i = [X(x_{i+r}) - X(x_i)]^p$ for $i = 1, \dots, N_r$, where $N_r = N - r$. If the SFC
 125 covariance is stationary, then $C_{ij} = C_r(|j - i|)$. Counting the contributions at each index lag
 126 reduces (5) to

$$V(r) = \frac{1}{N_r} \left[C_r(0) + 2 \sum_{k=1}^{N_r-1} \left(1 - \frac{k}{N_r} \right) C_r(k) \right], \quad (8)$$

127 where $C_r(k) = \text{Cov}(Y_i, Y_{i+k})$ is the SFC covariance at index lag k and separation r .

128 The covariance sum in (8) can be viewed as a discrete sampling of the covariance of an underlying
 129 spatially continuous SFC process. The cumulative effect of dependence in this process can be
 130 characterized by an integral SFC correlation scale,

$$\lambda_r = 2 \int_0^\infty \frac{C_r(s)}{C_r(0)} ds, \quad (9)$$

131 provided that the covariance is integrable. Here, s is the continuous starting-point lag, with all
 132 lengths expressed in grid intervals. The scale λ_r depends on the SF order and separation r and
 133 generally differs from the correlation scales of the underlying field and its increments.

134 If the sampling resolves the SFC covariance and the record is long compared with the lags
 135 contributing appreciably to its integral, the sum in (8) can be replaced by an integral, giving

$$V(r) \simeq C_r(0) \frac{\lambda_r}{N_r}, \quad N_{\text{eff}}(r) \simeq \frac{N_r}{\lambda_r}. \quad (10)$$

136 Thus, in this limit, the effective sample size is approximately the number of integral SFC correlation
 137 scales contained in the record.

138 Approximation (10) will be useful below, but its validity depends on both the SFC covariance
 139 and the sampling. The finite sum (8) must be retained when the covariance is poorly resolved,
 140 appreciable dependence extends to lags comparable to the record length, or the covariance is not
 141 absolutely integrable.

142 *c. Identifiability problem and estimation strategy*

143 "Sampling uncertainty" refers to the variability of $\widehat{D}(r)$ across hypothetical realizations of the
144 same statistical process observed with the same sampling geometry. An ensemble of independent
145 realizations with identical statistics would allow $V(r)$ to be estimated directly from the variability
146 of $\widehat{D}(r)$. However, such ensembles are rarely available in observations. Even with repeated
147 transects or experiments, we cannot be sure that the underlying statistics remain unchanged or that
148 the realizations are independent.

149 A single record does not, by itself, determine the ensemble sampling variance. Processes
150 consistent with the observations can differ substantially in their dependence structure or in the
151 probabilities of rare events, and thus have different values of $V(r)$. Estimating sampling uncertainty
152 therefore requires additional assumptions linking the observed variability to variability across
153 realizations. Under suitable assumptions, estimation becomes possible, but the resulting estimate
154 remains sensitive to features of the process that the record poorly constrains.

155 One approach is to specify a simplified statistical model that expresses the ensemble covariances
156 C_{ij} in terms of quantities that are easier to estimate from the record. Another is to estimate the ag-
157 gregate covariance contribution from variability among portions of the record while assuming that
158 those portions adequately represent variability across realizations and that unresolved dependence
159 is negligible. Neither approach removes the identifiability limit: the first supplies the missing
160 information through a statistical model, while the second assumes that variability not adequately
161 represented in the record can be neglected. Here, we develop the model-based approach for the
162 second-order structure function.

163 **3. Gaussian-increment reference model**

164 A useful closure is obtained by assuming that the increments of X are jointly Gaussian with
165 zero mean. This assumption does not restrict the field itself or its increments to be independent
166 or spatially uncorrelated; their covariance may remain arbitrary. Although we cannot generally
167 assume oceanic turbulence to be Gaussian, particularly in the presence of intermittency and coherent
168 structures, this remains a practical simplification. Non-Gaussian effects are addressed separately
169 below.

For the second-order structure function (SF2), the SFCs are squared increments. Isserlis' theorem (Isserlis 1918) then gives

$$C_{ij} \equiv \text{Cov} [(\delta X_i)^2, (\delta X_j)^2] = 2 (\text{Cov} [\delta X_i, \delta X_j])^2 = 2G_{ij}^2, \quad (11)$$

This is a substantial simplification: the Gaussian-increment assumption reduces the fourth-order statistics required for SF2 uncertainty to second-order increment covariances G_{ij} , which can be related directly to SF2 itself.²

To express G_{ij} in terms of second-order structure function D_2 , consider two pairs with increments

$$\delta X_i = X(\mathbf{x}_2) - X(\mathbf{x}_1), \quad \delta X_j = X(\mathbf{x}_4) - X(\mathbf{x}_3). \quad (12)$$

For a homogeneous, isotropic field,

$$G_{ij} = \frac{1}{2} [D_2(r_{23}) + D_2(r_{14}) - D_2(r_{24}) - D_2(r_{13})], \quad (13)$$

where $r_{ab} = |\mathbf{x}_a - \mathbf{x}_b|$. This expression highlights an important feature of G_{ij} and the SF uncertainty itself: it depends on the full four-point geometry of the two contributing increments, not only on their nominal separation.

Under this closure, (5) becomes

$$V_G(r) = \frac{2}{N_r^2} \sum_{i=1}^{N_r} \sum_{j=1}^{N_r} G_{ij}^2. \quad (14)$$

Equivalent Gaussian expressions for the sampling variance of empirical variograms have been derived in the geostatistical literature (Bogaert and Russo 1999; Pardo-Igúzquiza and Dowd 2001; Ortiz and Deutsch 2002).

It is useful to separate the magnitude of the increments from their covariance structure by defining the normalized increment covariance

$$g_{ij}(r) = \frac{G_{ij}}{D_2(r)} = \frac{1}{2D_2(r)} [D_2(r_{23}) + D_2(r_{14}) - D_2(r_{24}) - D_2(r_{13})]. \quad (15)$$

²Similar expressions can be obtained for SFs of other integer orders. Here, we focus on SF2 for simplicity.

Equation (14) can then be written

$$V_G(r) = D_2^2(r) F_G(r), \quad (16)$$

where

$$F_G(r) = \frac{2}{N_r^2} \sum_{i=1}^{N_r} \sum_{j=1}^{N_r} g_{ij}^2. \quad (17)$$

If D_2 is known at all separations required by the observation geometry, the Gaussian contribution to the sampling variance, V_G , is fully specified. The factor $D_2^2(r)$ sets the variance scale, while the dimensionless factor F_G accounts for the number, dependence, and geometry of the SFCs. In terms of the effective sample size introduced in (7), the Gaussian result gives $N_{\text{eff,G}} = 2/F_G$, since $C_0 = 2D_2^2(r)$. Equivalently, the fractional standard uncertainty is

$$\frac{\sqrt{V_G(r)}}{D_2(r)} = \sqrt{F_G(r)}. \quad (18)$$

In applying (16) to observations, the unknown $D_2(r)$ must be replaced by its empirical estimate $\widehat{D}_2(r)$. Direct substitution of $\widehat{D}_2^2$ introduces a positive bias, since under the Gaussian model

$$\mathbb{E}[\widehat{D}_2^2(r)] = D_2^2(r) + V_G(r) = D_2^2(r) [1 + F_G(r)]. \quad (19)$$

Treating the model-derived $F_G(r)$ as fixed, the positive bias from squaring $\widehat{D}_2(r)$ can be removed by estimating the Gaussian sampling variance as

$$\widehat{V}_G(r) = \frac{F_G(r)}{1 + F_G(r)} \widehat{D}_2^2(r). \quad (20)$$

For $F_G \ll 1$, the correction is negligible and $\widehat{V}_G \simeq F_G \widehat{D}_2^2$.

a. Regular one-dimensional sampling

For simplicity, we take the grid spacing as unity, so that r and s denote the increment length and starting-point separation in grid intervals. The Gaussian closure gives $C_r(s) = 2G_r^2(s)$, where $G_r(s) = D_2(r)g_r(s)$ is the covariance between increments of length r whose starting points are

separated by s . For the regularly sampled one-dimensional case, (17) can be simplified further. For increments of length r whose starting points are separated by s , the four endpoint separations entering (15) are

$$r_{23} = |r - s|, \quad r_{14} = r + s, \quad r_{13} = r_{24} = s. \quad (21)$$

Therefore, the normalized increment covariance depends only on the lag $s = |i - j|$,

$$g_r(s) \equiv g_{i,i+s}(r) = \frac{D_2(|r - s|) + D_2(r + s) - 2D_2(s)}{2D_2(r)}. \quad (22)$$

Here, it is interesting to note that for $s > r$, the numerator in (22) is the centered second finite difference of $D_2(s)$ with step r . Thus, covariance between well-separated increments is controlled by the curvature of the structure function with separation. Equation (17) then becomes

$$F_G(r) = \frac{2}{N_r} \left[1 + 2 \sum_{s=1}^{N_r-1} \left(1 - \frac{s}{N_r} \right) g_r^2(s) \right]. \quad (23)$$

In the well-resolved, long-record limit, the Gaussian variance factor can be expressed in terms of the integral SFC correlation scale as

$$F_G(r) \simeq \frac{2\lambda_r}{N_r}, \quad (24)$$

where the Gaussian closure gives

$$\lambda_r = 2 \int_0^\infty g_r^2(s) ds, \quad (25)$$

provided that the integral converges.

In principle, the empirical SF2 estimate $\widehat{D}_2(r)$ can be used directly in (14). At large separations, however, it is typically poorly constrained. A parametric model fitted to the observed field covariance C_X (or, equivalently, $\widehat{D}_2$) can therefore provide a more stable representation, at the cost of introducing additional model assumptions.

b. Covariance models

We consider two classes of second-order models: finite-scale and scale-free. Finite-scale covariance models, such as the exponential and Gaussian models, introduce a characteristic scale

ℓ beyond which spatial dependence of X decays. The corresponding SF2 curves consequently approach a constant for $r \gg \ell$. The two models differ at small separations: exponential covariance describes a rough field with $D_2(r) \propto r$, whereas Gaussian covariance describes a smooth field with $D_2(r) \propto r^2$. This distinction affects increment dependence and hence SF2 sampling uncertainty. A characteristic scale, however, is not always resolved in observations. Velocity and tracer spectra and SF2 in ocean turbulence often exhibit approximately self-similar power-law behavior across portions of the finescale, mesoscale, and submesoscale ranges, although the observed slopes vary considerably among regions and dynamical regimes (e.g., Callies and Ferrari 2013; Rocha et al. 2016; Qiu et al. 2017). Such behavior motivates a scale-free model for the SF2,

$$D_2(r) \propto r^\zeta, \quad 0 < \zeta < 2, \quad (26)$$

over the range of scales considered. The limiting exponent $\zeta = 2$ describes the small-separation behavior of a mean-square differentiable field. If this quadratic form holds at all separations, however, the increments are fully coherent and the process is a random linear function. For a power-law spectrum $E(k) \propto k^{-q}$ with $1 < q < 3$, the corresponding structure-function exponent is $\zeta = q - 1$ (e.g., Nichols-Pagel et al. 2008).

These alternatives should not be viewed as mutually exclusive physical descriptions. A field may have an outer scale while its structure function is approximately scale free over a restricted range of smaller scales. The relevant choice therefore depends on the scales resolved by the observations and on what can reasonably be inferred about the continuation of D_2 beyond the separation of interest.

Figure 1 compares four synthetic Gaussian processes: two exponential-covariance models with characteristic scales $\ell = N/10,000$ and $N/100$ (small and large relative to the number of observations N), a Gaussian-covariance model with $\ell = N/100$, and a scale-free model with $\zeta = 2/3$ ($q = 5/3$). Representative realizations are shown in Fig. 1a, and the corresponding structure functions in Fig. 1b. The finite-scale structure functions saturate at large separations, but their small-separation slopes differ. The scale-free structure function continues to grow as $r^{2/3}$ throughout the range of scales considered.

Figure 1c shows distinct shapes of the normalized SFC covariance at a fixed separation $r = N/1000$. For the short-scale exponential model ($\ell \ll r$), the central peak decays over ℓ , followed

by a separate peak near $s = r$, where the increments share an endpoint. Both peaks have widths of order ℓ . For the longer-scale exponential model ($r \ll \ell$), much of the field variance resides at scales larger than r and largely cancels when the endpoints are differenced. The central lobe therefore extends to $s \simeq r$, with only weak covariance beyond it. The Gaussian-covariance model has a broader, flatter central lobe, followed by a secondary peak at lags of order ℓ . Small increments in this field satisfy $\delta X \simeq r dX/dx$ (i.e., the field is locally-smooth). Reducing r therefore rescales the increments without changing their normalized spatial dependence. The normalized SFC covariance consequently remains appreciable at $s \sim \ell$, even when $r \ll \ell$.

For the scale-free model, r is the only intrinsic length scale: the normalized covariance is a function of s/r , with its shape determined by ζ . The illustrated $\zeta = 2/3$ case has a central lobe concentrated at $s \lesssim r$, a weak secondary peak at $s = r$, and a rapidly decaying tail.

These differences determine the Gaussian variance factor $F_G(r)$ in Fig. 1d. Below ℓ , the exponential models follow approximately r/N_r scaling and level off above ℓ . In contrast, the Gaussian-covariance model gives a much weaker dependence on r , with F_G remaining of order ℓ/N_r throughout. The scale-free model shown here approximately follows r/N_r scaling over most of the plotted range. Monte Carlo estimates from 1,000 Gaussian realizations closely follow the exact finite-record predictions for all four models. The following subsections derive asymptotic expressions explaining these behaviors.

1) EXPONENTIAL COVARIANCE MODEL

We first consider a stationary Gaussian process with exponential covariance

$$C_X(s) = \sigma_X^2 \exp\left(-\frac{|s|}{\ell}\right), \quad (27)$$

where ℓ is the characteristic correlation scale. The corresponding SF2 is

$$D_2(r) = 2\sigma_X^2 \left[1 - \exp\left(-\frac{r}{\ell}\right)\right]. \quad (28)$$

Thus, $D_2(r) \simeq 2\sigma_X^2 r/\ell$ for $r \ll \ell$ and approaches $2\sigma_X^2$ for $r \gg \ell$, as illustrated in Fig. 1b.

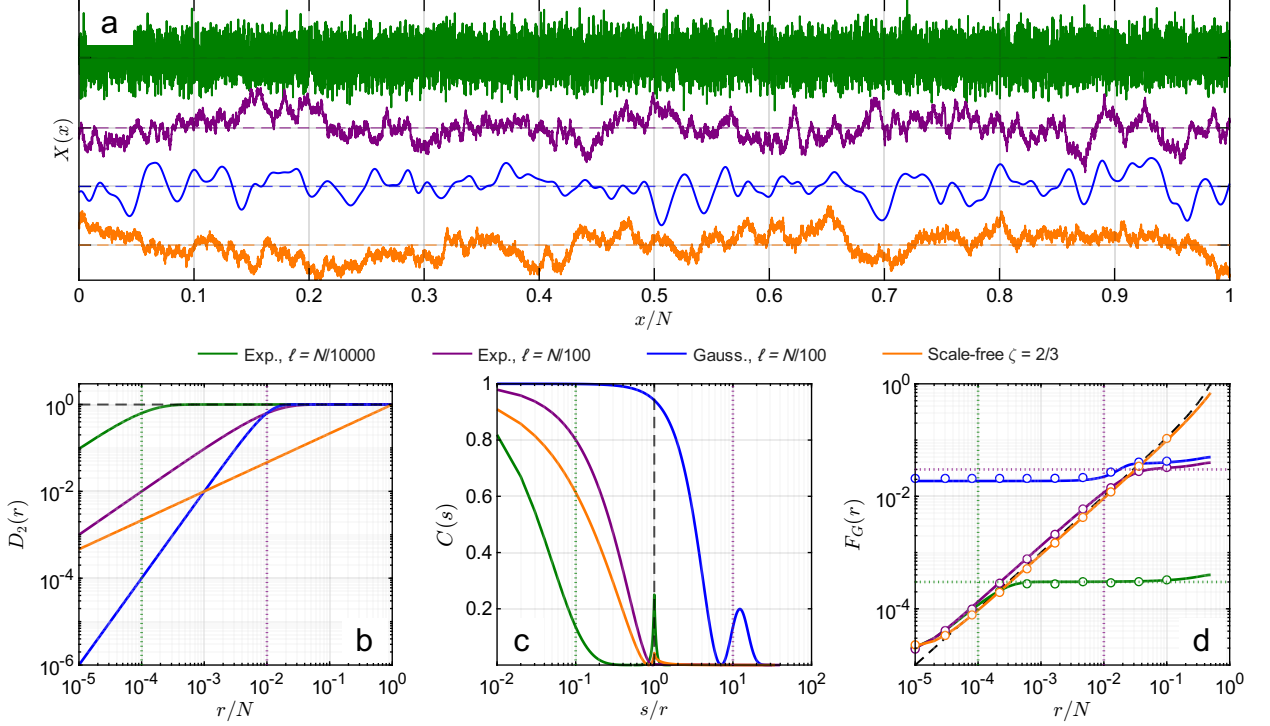


FIG. 1. Dependence of structure-function sampling uncertainty on the field covariance model. (a) Representative Gaussian realizations for exponential covariance with $\ell/N = 1/10000$ and $1/100$, Gaussian covariance with $\ell/N = 1/100$, and a scale-free structure function with $\zeta = 2/3$ ($q = 5/3$). Realizations are vertically offset for clarity. (b) Corresponding second-order structure functions. (c) Normalized SFC covariance, $C_r(s)/C_r(0) = g_r^2(s)$, at $r = N/1000$. The black dashed line marks $s = r$. (d) Gaussian sampling-variance factor $F_G(r) = V_G(r)/D_2^2(r)$; symbols show Monte Carlo estimates from 1,000 Gaussian realizations. The black dashed line shows r/N_r , and colored horizontal dotted lines show $3\ell/N$ for the exponential models. Vertical dotted lines in (b)–(d) mark their characteristic scales ℓ ; the larger scale also applies to the Gaussian-covariance model.

For regularly spaced observations in one dimension, this covariance model gives

$$g_r(s) = \frac{2e^{-s/\ell} - e^{-|r-s|/\ell} - e^{-(r+s)/\ell}}{2(1 - e^{-r/\ell})}. \quad (29)$$

Substituting (29) into (25) gives the asymptotic SFC correlation scale (Appendix A),

$$\lambda_r \simeq \begin{cases} \frac{2}{3}r, & r \ll \ell, \\ \frac{3}{2}\ell, & r \gg \ell. \end{cases} \quad (30)$$

Thus, the integral SFC correlation scale grows linearly with separation for $r \ll \ell$ and approaches a plateau for $r \gg \ell$. Through (24), these limits explain the corresponding r/N_r scaling and plateau in fractional sampling variance (Fig. 1d), provided that the covariance is well resolved and the record is sufficiently long.

2) GAUSSIAN COVARIANCE MODEL

To examine the effect of small-scale smoothness, we consider

$$C_X(s) = \sigma_X^2 \exp \left[- \left(\frac{s}{\ell} \right)^2 \right], \quad (31)$$

corresponding to SF2

$$D_2(r) = 2\sigma_X^2 \left\{ 1 - \exp \left[- \left(\frac{r}{\ell} \right)^2 \right] \right\}. \quad (32)$$

Thus, $D_2(r) \simeq 2\sigma_X^2(r/\ell)^2$ for $r \ll \ell$, and $D_2(r) \rightarrow 2\sigma_X^2$ for $r \gg \ell$. The normalized increment covariance is

$$g_r(s) = \frac{2e^{-(s/\ell)^2} - e^{-[(s-r)/\ell]^2} - e^{-[(s+r)/\ell]^2}}{2[1 - e^{-(r/\ell)^2}]}. \quad (33)$$

For this covariance model, (33) gives the asymptotic SFC correlation scale (Appendix A),

$$\lambda_r \simeq \begin{cases} \frac{3}{4}\sqrt{\frac{\pi}{2}}\ell, & r \ll \ell, \\ \frac{3}{2}\sqrt{\frac{\pi}{2}}\ell, & r \gg \ell. \end{cases} \quad (34)$$

Thus, the integral SFC correlation scale remains proportional to ℓ at both small and large separations, with the large-separation value twice the small-separation value. Importantly, the integral SFC correlation scale remains of order ℓ even as $r/\ell \rightarrow 0$. Through (24), these limits explain the two plateaus in fractional sampling variance (Fig. 1d).

294 3) SCALE-FREE COVARIANCE MODEL

295 For the scale-free model (26), the normalized increment covariance in one dimension, (22),
 296 becomes

$$g_r(s) = g_\zeta(s^*) \equiv \frac{1}{2} [|1 - s^*|^\zeta + (1 + s^*)^\zeta - 2(s^*)^\zeta], \quad s^* = \frac{s}{r}. \quad (35)$$

297 Thus, the normalized increment covariance is self-similar: $g_\zeta(s^*)$ depends only on the scaled
 298 lag $s^* = s/r$ and the exponent ζ . This self-similarity implies that, when finite, the integral SFC
 299 correlation scale is proportional to r :

$$\lambda_r = 2 \int_0^\infty g_r^2(s) ds = 2r \int_0^\infty g_\zeta^2(s^*) ds^* \equiv r \mathcal{G}_\zeta. \quad (36)$$

300 Here, $\mathcal{G}_\zeta$ is a dimensionless coefficient determined only by ζ , provided that the integral converges.

301 For the special case $\zeta = 1$ ($q = 2$), (35) reduces to the triangular form $g_\zeta(s^*) = \max(1 - s^*, 0)$. The
 302 SFC covariance therefore vanishes for nonoverlapping increments ($s \geq r$), so dependence arises
 303 solely from overlap. Integrating the squared triangular covariance gives

$$\lambda_r = \frac{2}{3} r. \quad (37)$$

304 The same correlation scale arises asymptotically for the exponential covariance model at $r \ll \ell$,
 305 where the normalized increment covariance approaches the same triangular form (A1).

306 The existence of a finite λ_r depends on the covariance decay at large lags. For $s \gg r$ and $\zeta \neq 1$,
 307 (35) gives

$$g_\zeta(s^*) \sim \frac{\zeta(\zeta - 1)}{2} (s^*)^{\zeta-2}, \quad (38)$$

308 so the normalized SFC covariance decays as

$$g_\zeta^2(s^*) \propto (s^*)^{2\zeta-4}. \quad (39)$$

309 The integral defining λ_r therefore converges only for $\zeta < 3/2$, or equivalently $q < 5/2$.

310 For $\zeta \geq 3/2$, the integral approximation (24) no longer applies, and F_G must be evaluated from the
 311 finite covariance sum (23). At $\zeta = 3/2$, the cumulative covariance grows logarithmically with the

maximum scaled lag, giving $F_G \propto (r/N_r) \log(N_r/r)$ in the long-record limit. For steeper spectra, long-range covariance causes F_G to decrease even more slowly with increasing record length at fixed separation.

Figure 2 illustrates the distinction between finite and divergent integral SFC correlation scales. For $\zeta < 3/2$, the compensated variance approaches the constant $2\mathcal{G}_\zeta = 2\lambda_r/r$ in the well-resolved, long-record limit. The plateaus evident for the shallower spectral slopes illustrate this behavior. As ζ approaches $3/2$ from below, however, the increasingly slow decay of SFC covariance delays convergence, and substantial finite-record effects can persist even at small r/N_r . For $\zeta \geq 3/2$, the compensated variance continues to grow as r/N_r decreases, reflecting the increasing contribution of long-range dependence and the absence of a finite λ_r .

In the opposite limit, $\zeta \rightarrow 0$ at fixed sampling geometry, correlation between distinct increments persists only when they share an endpoint. For $r \ll N_r$, (23) therefore gives $F_G \simeq 3/N_r$.

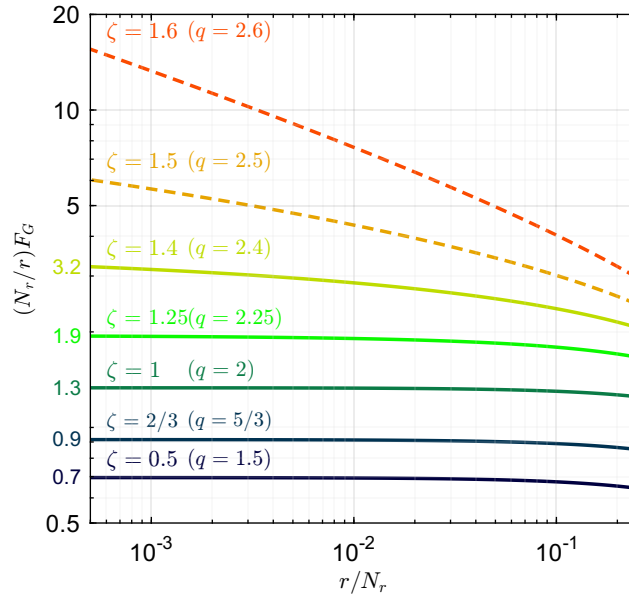
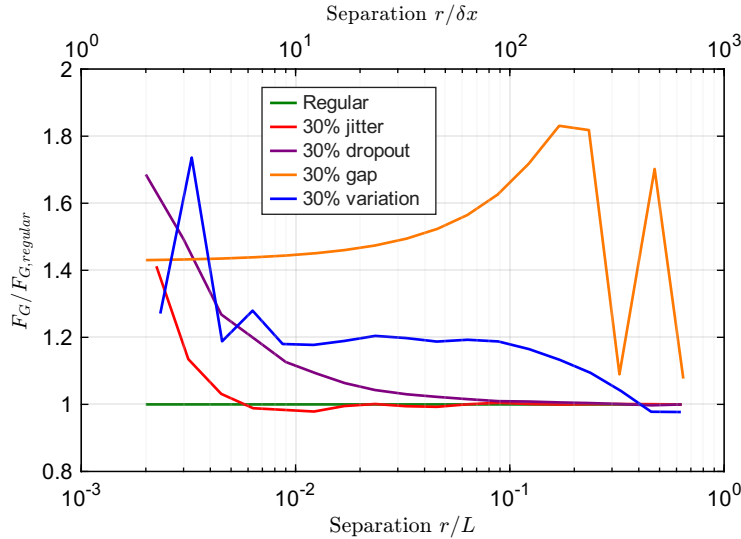


FIG. 2. Compensated Gaussian reference variance $(N_r/r)F_G$ dependence on normalized separation r/N_r for regularly sampled one-dimensional records with $D_2(r) \propto r^\zeta$. Labels give ζ and the corresponding spectral exponent $q = \zeta + 1$.

327 *c. Weakly-irregular one-dimensional sampling*

328 When the integral SFC correlation scale λ_r is finite, approximation (24) expresses the leading-
 329 order fractional variance in terms of the ratio of this model-dependent scale to the record length.
 330 Once the SFC covariance is adequately resolved and the record is sufficiently long, this ratio
 331 is insensitive to sampling resolution. This suggests that strict sampling regularity may not be
 332 necessary for this approximation to remain valid, potentially extending its applicability to quasi-
 333 regular observations such as those collected during underway oceanographic surveys.

339 Figure 3 examines this possibility for the scale-free model with $\zeta = 1$. The Gaussian uncertainty
 340 factor F_G is calculated for weakly-irregular sampling scenarios and normalized by the correspond-
 341 ing factor for regular sampling over the same domain. Random positional jitter with 30% amplitude
 342 and removal of 30% of observations produce appreciable departures only at separations compara-
 343 ble to the nominal sampling interval. At larger separations, their uncertainty factors approach the



334 FIG. 3. Effect of irregular one-dimensional sampling on the Gaussian uncertainty factor for the scale-free
 335 model with $\zeta = 1$ ($q = 2$). The uncertainty factor is normalized by its value for regular sampling over the same
 336 domain. Sampling configurations include random positional jitter of amplitude $0.3\Delta x$, random removal of 30%
 337 of observations, a central gap occupying 30% of the domain, and smoothly varying sampling intervals with 30%
 338 amplitude. The left edge of the plotted range corresponds to the nominal sampling interval $\delta x/L$.

regular-sampling reference. In contrast, a contiguous observational gap and systematic variations in sampling density produce larger departures that persist across a broader range of separations.

These results suggest that the regular-sampling asymptotic expressions remain relevant for sufficiently resolved, quasi-regular observations, provided the contributing increments retain approximately uniform spatial coverage. The actual limitation is therefore not the sampling irregularity itself, but spatial concentration of the contributing increments, which reduces their effective coverage of the sampled domain and alters SFC covariance structure. For strongly nonuniform sampling, F_G should therefore be calculated directly from the observed geometry using the general Gaussian formulation (17).

d. Irregular and multidimensional sampling

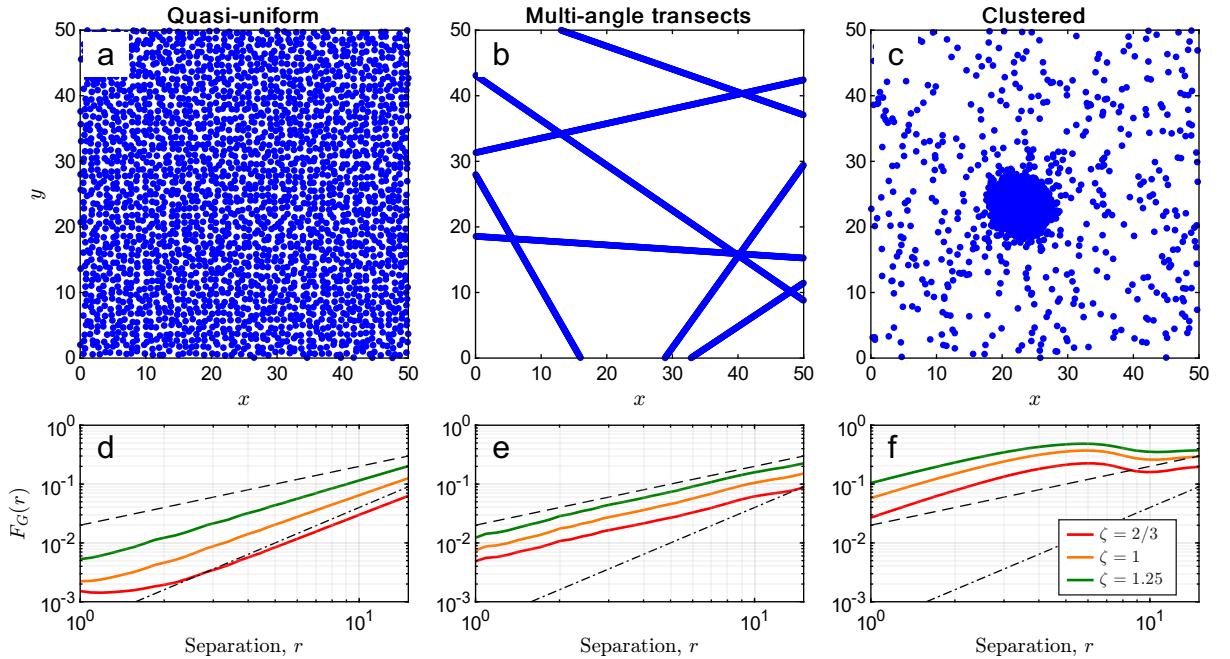
For irregular or multidimensional sampling, the double sum in (17) generally cannot be reduced to a sum over a scalar lag. Instead, F_G depends on the spatial arrangement of the observations and contributing pairs, so sampling patterns with the same domain size and number of observations can have different sampling uncertainties, as also recognized in geostatistical studies of variogram estimation (Bogaert and Russo 1999; Marchant and Lark 2004). For any specified covariance or structure-function model, F_G can nevertheless be evaluated directly using the actual sampling geometry. Appendix B presents a graph-based formulation that avoids constructing the full $N_r \times N_r$ increment-covariance matrix.

Figure 4 illustrates this geometric dependence for several scale-free structure-function models. The three sampling patterns represent quasi-uniform sampling, a set of intersecting transects resembling a typical oceanographic survey, and strongly clustered sampling. All three use the same domain size L and number of observations $n = 2,500$, so differences among the columns reflect the effect of sampling geometry, while differences among the F_G curves within each column reflect the dependence on the scaling exponent ζ .

For quasi-uniform two-dimensional sampling, F_G varies smoothly with separation and its magnitude and slope depend systematically on the structure-function exponent ζ . The transect geometry behaves differently: the observations occupy predominantly one-dimensional sampling supports, and the plotted uncertainty factors lie closer to the r/L reference scaling. This resemblance is descriptive; covariance between increments on different transects remains part of the full geometric

377 calculation. Clustered sampling departs more strongly from any simple power-law scaling. The
 378 cluster size introduces an additional geometric scale, producing a change in slope and eventual
 379 flattening of $F_G(r)$.

380 These examples show that, for irregular sampling, SF uncertainty can vary substantially with
 381 the detailed geometry of the contributing pairs, even under the simplified assumptions of Gaussian
 382 increments and scale-free power-law statistics. No universal scaling applies across sampling
 383 geometries; F_G must therefore be evaluated using the actual observation coordinates.



362 FIG. 4. Dependence of the Gaussian uncertainty factor F_G on sampling geometry and structure-function
 363 scaling exponent. (a-c) Example quasi-uniform, multi-angle transect, and clustered sampling geometries. (d-f)
 364 Corresponding $F_G(r)$ for $\zeta = 2/3$, 1, and 1.25. The dashed and dot-dashed lines indicate the reference scaling
 365 (r/L) and $(r/L)^2$.

384 4. Non-Gaussian contribution

385 Under the jointly Gaussian assumption, the SFC covariance is determined by D_2 alone. More
386 generally, for zero-mean increments, the exact decomposition is

$$C_{ij} = 2G_{ij}^2 + K_{ij}, \quad (40)$$

387 where $K_{ij} \equiv \text{cum}(\delta X_i, \delta X_i, \delta X_j, \delta X_j)$ is the fourth-order increment cumulant. Accordingly,

$$V(r) = V_G(r) + V_4(r), \quad (41)$$

388 where

$$V_4(r) = \frac{1}{N_r^2} \sum_{i,j} K_{ij}. \quad (42)$$

389 The contribution V_4 need not be positive: departures from Gaussianity can either increase or
390 decrease the sampling variance relative to V_G . The Gaussian reference is therefore not a general
391 lower bound. The Gaussian term V_G is determined entirely by second-order statistics and can
392 therefore be obtained from D_2 through (13). V_4 contains additional fourth-order information not
393 recoverable from D_2 alone.

394 Higher-order statistics and intermittency have been examined in a number of ocean-turbulence
395 studies, typically through higher-order structure functions, flatness, or multifractal scaling (e.g.,
396 Lozovatsky et al. 2010; Balwada et al. 2016; Isern-Fontanet and Turiel 2021). Much less is
397 known observationally about the joint fourth-order statistics of increments needed to estimate K_{ij} .
398 Estimating V_4 therefore requires direct assessment of these higher-order statistics from the data,
399 and is likely to be considerably less robust than estimating V_G .

400 5. From sampling variance to confidence intervals

401 The sampling variance $V(r)$ characterizes the variability of the structure-function estimator $\widehat{D}(r)$
402 among repeated realizations, but does not by itself determine a confidence interval. Constructing
403 confidence limits requires an additional assumption about the sampling distribution of $\widehat{D}(r)$.

404 If this distribution is approximated as Gaussian, a two-sided confidence interval with confidence
 405 level $1 - \alpha$ is

$$\text{CI}_{\widehat{D},\alpha} = \widehat{D} \pm z_{1-\alpha/2} \sqrt{\widehat{V}}, \quad (43)$$

406 where $z_{1-\alpha/2}$ is the corresponding standard-normal quantile (1.96 for a 95% interval). This
 407 approximation is most appropriate when many effectively independent SFCs contribute to the
 408 estimate.

409 For the second-order structure function under the Gaussian-increment reference model, F_G is
 410 the fractional sampling variance. Using the leading-order estimate $\widehat{V}_G \simeq F_G \widehat{D}_2^2$, valid for $F_G \ll 1$,
 411 gives

$$\text{CI}_{\widehat{D}_2,\alpha} \simeq \widehat{D}_2(r) \left[1 \pm z_{1-\alpha/2} \sqrt{F_G(r)} \right]. \quad (44)$$

412 For finite records, however, the sampling distribution of $\widehat{D}_2(r)$ may be appreciably non-Gaussian,
 413 particularly when only a small number of effectively independent SFCs enter the estimate. A scaled
 414 chi-squared distribution provides a natural moment-matched approximation (Satterthwaite 1946):

$$\widehat{D}_2(r) \approx D_2(r) \frac{\chi_\nu^2}{\nu}, \quad (45)$$

415 where ν is an effective number of degrees of freedom. This form is exact for the mean of indepen-
 416 dent, zero-mean Gaussian increments squared with equal variances. For dependent increments, it
 417 is generally an approximation. Its fractional variance is $2/\nu$, so matching the Gaussian reference
 418 variance gives

$$\nu_{\text{eff}}(r) = \frac{2}{F_G(r)} = N_{\text{eff,G}}(r). \quad (46)$$

419 Thus, the moment-matched degrees of freedom coincide with the Gaussian effective sample size
 420 defined in (7), and are determined by the second-order covariance model and sampling geometry.

421 Inverting the scaled chi-squared approximation, with ν_{eff} treated as fixed, gives the approximate
 422 lower and upper confidence limits for confidence level $1 - \alpha$

$$D_{\text{lo}}(r) = \frac{\nu_{\text{eff}}(r)}{\chi_{\nu_{\text{eff}}, 1-\alpha/2}^2} \widehat{D}_2(r), \quad D_{\text{hi}}(r) = \frac{\nu_{\text{eff}}(r)}{\chi_{\nu_{\text{eff}}, \alpha/2}^2} \widehat{D}_2(r), \quad (47)$$

where $\chi_{\nu,p}^2$ denotes the p quantile of a chi-squared distribution with ν degrees of freedom. Unlike Eq. (43), this interval is asymmetric in linear space and remains positive. As ν_{eff} becomes large, the scaled chi-squared distribution approaches a Gaussian and Eq. (47) approaches the symmetric Gaussian approximation.

6. Uncertainty of derived quantities

Many quantities of scientific interest (turbulence dissipation rates, scaling exponents, etc.) are inferred from structure functions evaluated at one or more separations. The uncertainty of such quantities generally depends on the joint sampling variability of the structure-function estimates across separations. Estimates at different r are typically correlated because they are constructed from the same observations and because increments at different scales are themselves correlated.

Sampling errors correlated across separations can produce coherent shifts, tilts, or curvature in the estimated SF (Fig. 5). An apparently well-defined scaling regime may therefore have an erroneous amplitude or exponent, even when deviations from a fitted power law are small. Pointwise error bars describe uncertainty at individual separations but do not convey these coherent variations of the estimated curve. Quantifying uncertainty in parameters derived from the SF therefore requires the cross-separation covariance.

Let $\hat{\mathbf{D}}$ denote the vector of structure-function estimates across a range of separations

$$\hat{\mathbf{D}} = \left[\hat{D}_2(r_1), \dots, \hat{D}_2(r_m) \right]^T, \quad \mathbf{\Sigma} = \text{Cov}(\hat{\mathbf{D}}). \quad (48)$$

For a derived quantity $\hat{\theta} = f(\hat{\mathbf{D}})$, first-order uncertainty propagation gives

$$\text{Var}(\hat{\theta}) \simeq \nabla f^T \mathbf{\Sigma} \nabla f, \quad (49)$$

provided f is approximately linear over the sampling fluctuations. The gradient is evaluated at the population SF2 vector and, in practice, approximated at $\hat{\mathbf{D}}$. Neglecting cross-scale covariance can misrepresent the uncertainty of quantities inferred from multiple separations.

The Gaussian-increment formulation developed above extends directly to these cross-separation covariances. Consider two separation bins centered at r_a and r_b , with pair index sets $\mathcal{I}_a$ and $\mathcal{I}_b$

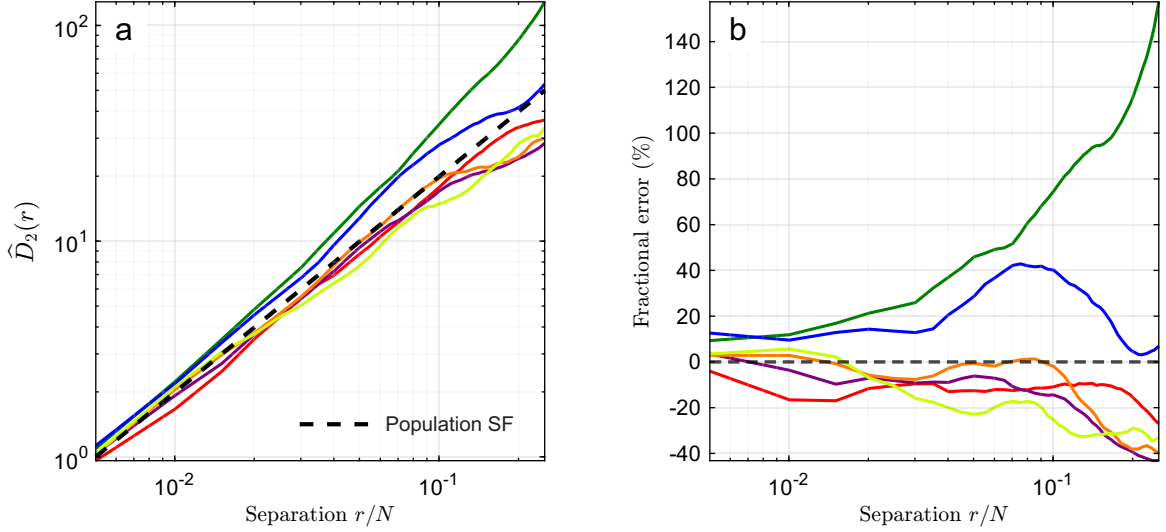


FIG. 5. Correlated sampling errors in structure-function estimates. (a) Second-order SF estimates from six independent realizations of a Gaussian process with a population SF $D_2(r) \propto r$, corresponding to the scale-free model with $q = 2$ (colors). The dashed black line shows the true population SF. (b) Corresponding fractional errors, $\hat{D}_2(r)/D_2(r) - 1$, with matching colors. Separation is normalized by the record length. Despite their visual smoothness, individual estimates exhibit substantial, coherent departures from the population SF.

containing N_a and N_b pairs, respectively. The covariance between their SF2 estimates is

$$\Sigma_{ab} = \frac{1}{N_a N_b} \sum_{i \in \mathcal{I}_a} \sum_{j \in \mathcal{I}_b} C_{ij}^{ab}, \quad (50)$$

where C_{ij}^{ab} is the covariance between SFCs from the two bins. Under the zero-mean, jointly Gaussian increment assumption,

$$\Sigma_{G,ab} = \frac{2}{N_a N_b} \sum_{i \in \mathcal{I}_a} \sum_{j \in \mathcal{I}_b} \left(G_{ij}^{ab} \right)^2, \quad (51)$$

where G_{ij}^{ab} follows from the same four-point expression (13), now applied to increments from different bins.

Defining

$$g_{ij}^{ab} = \frac{G_{ij}^{ab}}{\sqrt{D_2(r_a) D_2(r_b)}}, \quad F_{G,ab} = \frac{2}{N_a N_b} \sum_{i \in \mathcal{I}_a} \sum_{j \in \mathcal{I}_b} \left(g_{ij}^{ab} \right)^2, \quad (52)$$

457 gives

$$\Sigma_{G,ab} = F_{G,ab} D_2(r_a) D_2(r_b). \quad (53)$$

458 The dimensionless cross-separation covariance matrix F_G is determined by the same second-order
459 model and observation geometry as before. Its diagonal elements satisfy $F_{G,aa} = F_G(r_a)$.

460 *a. Structure function slope*

461 As a common application, consider estimating an SF2 scaling exponent ζ by an ordinary least-
462 squares (OLS) fit in log-log space over a set of separations (e.g., Nichols-Pagel et al. 2008;
463 McCaffrey et al. 2015).³ Writing $x_a = \ln r_a$ and $y_a = \ln \widehat{D}_2(r_a)$, the fitted slope is

$$\widehat{\zeta} = \sum_a w_a y_a, \quad w_a = \frac{x_a - \bar{x}}{\sum_b (x_b - \bar{x})^2}. \quad (54)$$

464 The gradient with respect to the SF2 estimates, evaluated at the population SF2 vector, is
465 $w_a/D_2(r_a)$. For sufficiently small fractional SF2 fluctuations, first-order uncertainty propaga-
466 tion therefore gives

$$\text{Var}_G(\widehat{\zeta}) \simeq \mathbf{w}^T \mathbf{F}_G \mathbf{w}. \quad (55)$$

467 Thus, within this approximation, the uncertainty of the fitted scaling exponent is independent
468 of an overall SF2 amplitude and is determined by the chosen covariance model, the separations
469 included in the fit, and the sampling geometry.

470 *b. Turbulence dissipation*

471 Another common application is estimating the turbulent kinetic energy dissipation rate ε by
472 fitting the inertial-range relationship to velocity SF2 estimates (e.g., Wiles et al. 2006; Thomson
473 2012):

$$D_2(r_a) = A t_a, \quad t_a = (r_a)^{2/3}, \quad A = C_2 \varepsilon^{2/3}, \quad (56)$$

³The corresponding covariance matrix in log space, approximately F_G for small fractional SF fluctuations, could also be used for a generalized least-squares fit. Here, we focus on OLS for simplicity.

where C_2 is the second-order structure-function Kolmogorov constant appropriate to the measured velocity component. An unweighted OLS fit through the origin ⁴ gives

$$\hat{A} = \frac{\sum_a t_a \hat{D}_2(r_a)}{\sum_a t_a^2}, \quad \hat{\varepsilon} = \left(\frac{\hat{A}}{C_2} \right)^{3/2}. \quad (57)$$

Because $\hat{A}$ is linear in the SF2 estimates, its sampling variance is exactly

$$\text{Var}(\hat{A}) = \frac{\mathbf{t}^T \boldsymbol{\Sigma} \mathbf{t}}{(\mathbf{t}^T \mathbf{t})^2}. \quad (58)$$

Under the Gaussian reference, substituting (53) and $D_2(r_a) = A t_a$ gives

$$\frac{\text{Var}_G(\hat{A})}{A^2} = \mathbf{b}^T \mathbf{F}_G \mathbf{b}, \quad b_a = \frac{t_a^2}{\sum_c t_c^2} = \frac{r_a^{4/3}}{\sum_c r_c^{4/3}}. \quad (59)$$

For sufficiently small fractional fluctuations of the fitted amplitude, first-order propagation to dissipation therefore yields

$$\frac{\text{Var}_G(\hat{\varepsilon})}{\varepsilon^2} \simeq \frac{9}{4} \mathbf{b}^T \mathbf{F}_G \mathbf{b}. \quad (60)$$

The uncertainty thus depends on cross-scale covariance throughout the fitted range.

7. Discussion and conclusions

Quantifying the sampling uncertainty of a structure-function estimate is not straightforward. The uncertainty is controlled by the covariance among SFCs, arising from both shared endpoints and spatial dependence of the underlying field. For second-order SFs, approximating the increments as jointly Gaussian gives the reference variance $V_G(r) = D_2^2(r) F_G(r)$, where F_G is the fractional sampling variance determined by the normalized increment covariance and the geometry of the contributing pairs. For a specified model of D_2 , this factor can be evaluated directly for arbitrary sampling geometry.

For well-resolved, regularly sampled one-dimensional records, the Gaussian reference provides a simple physical interpretation: the effective number of independent SF contributions is approxi-

⁴A similar procedure could be designed for fits that include a measurement-noise offset (e.g., Thomson 2012), but it is more complicated.

491 mately the record length divided by the integral SFC correlation scale λ_r . Specifically,

$$\begin{aligned} \nu_{\text{eff}}(r) &\simeq \frac{N_r}{\lambda_r}, \\ F_G(r) &= \frac{V_G(r)}{D_2^2(r)} \simeq \frac{2\lambda_r}{N_r}, \end{aligned} \quad (61)$$

$$\text{SE}_G[\widehat{D}_2(r)] = \sqrt{V_G(r)} \simeq D_2(r) \sqrt{\frac{2\lambda_r}{N_r}}.$$

492 Here, lengths are expressed in sampling intervals. These approximations require a finite λ_r ,
 493 sampling sufficiently fine to resolve the SFC covariance, and a record long enough to capture
 494 the lags contributing appreciably to its integral. Once these conditions are satisfied, increasing
 495 sampling density over a fixed domain provides little further reduction in sampling uncertainty.

496 The SFC correlation scale λ_r generally differs from the correlation scales of the underlying field
 497 and its increments, and its dependence on separation determines how SF uncertainty varies with
 498 r . For exponential field covariance, $\lambda_r \simeq 2r/3$ at $r \ll \ell$ and $\lambda_r \simeq 3\ell/2$ at $r \gg \ell$. For Gaussian field
 499 covariance, λ_r remains proportional to ℓ in both limits. For scale-free covariance with SF2 slope
 500 $0 < \zeta < 3/2$ (spectral slope $1 < q < 5/2$), self-similarity gives $\lambda_r = r\mathcal{G}_\zeta$, where $\mathcal{G}_\zeta$ depends only
 501 on ζ . The fractional variance therefore scales as r/N_r . In particular, $\zeta = 1$ ($q = 2$), gives $\mathcal{G}_1 = 2/3$,
 502 $\nu_{\text{eff}} \simeq (3/2)N_r/r$, and a standard error $\text{SE}_G[\widehat{D}_2(r)] \simeq D_2(r)\sqrt{(4/3)r/N_r}$. This scaling provides
 503 a useful reference for upper-ocean submesoscale velocity and tracer spectra near k^{-2} (Callies and
 504 Ferrari 2013).

505 The assumptions underlying these predictions form a hierarchy of increasing specialization.
 506 The general covariance-summation expression for SF sampling variance is exact, provided the
 507 required moments exist. The Gaussian-increment closure then expresses SFC covariance in terms
 508 of SF2, while a prescribed covariance model supplies the information needed for quantitative
 509 evaluation. Finally, the simple length-ratio asymptotics require additional assumptions about
 510 sampling regularity, covariance resolution, and record length.

511 The Gaussian-increment closure is a deliberate simplification that does not fully represent the
 512 known characteristics of turbulent fields. Ocean turbulence, in particular, exhibits fronts, filaments,
 513 and other intermittent structures whose statistics can depart substantially from Gaussianity. Nev-
 514 ertheless, the closure makes the problem tractable by expressing SFC covariance entirely in terms

515 of second-order statistics, allowing the effects of the underlying field covariance and sampling
516 geometry to be examined systematically. The resulting predictions provide a useful reference and
517 help develop physical intuition about the factors controlling SF sampling uncertainty. Although
518 the quantitative predictions may change when higher-order statistics are included, the underlying
519 dependence on SFC covariance and sampling geometry remains general.

520 For non-Gaussian increments, SFC covariance contains an additional fourth-order joint cumulant
521 that cannot be inferred from D_2 alone. Its contribution to sampling variance need not be small and
522 cannot generally be assigned a predictable sign or magnitude. Higher-order statistics can modify
523 both the magnitude and spatial dependence of SFC covariance, potentially changing its effective
524 correlation scale and the resulting uncertainty scaling. The Gaussian result should therefore be
525 interpreted as a reference prediction, not as a controlled approximation with necessarily small
526 corrections.

527 Sensitivity of the Gaussian variance prediction to the prescribed second-order covariance model
528 may also be significant, but can be assessed directly by evaluating F_G for alternative models
529 consistent with the observed SF. Extrapolation beyond reliably observed separations may require
530 additional physical assumptions about the underlying process, since the covariance structure at
531 unobserved scales cannot be determined from a single finite record.

532 The asymptotic scaling relations provide physical insight into the spatial scales controlling SF
533 sampling uncertainty and offer useful order-of-magnitude estimates. Tests with weakly irregular
534 one-dimensional sampling suggest that these scalings remain applicable when contributing incre-
535 ments provide approximately uniform spatial coverage. For quantitative predictions, however, direct
536 numerical evaluation of F_G is generally preferable, as it automatically accounts for finite-record
537 effects and the actual contributing-pair geometry without relying on asymptotic approximations.
538 The calculation is straightforward to incorporate into existing SF estimation procedures, with
539 computational optimizations discussed in Appendix B.

540 Structure function estimates at different separations tend to be strongly correlated, making
541 empirical SF curves appear smoother than their pointwise uncertainties would suggest. The
542 Gaussian reference framework described here can be used to obtain the cross-scale covariance of
543 SF estimation errors. Accounting for this covariance is essential for reliable uncertainty estimates

544 of quantities derived from the empirical SF curves, such as scaling exponents and turbulence
545 dissipation rates.

546 A complementary approach is to estimate sampling variance directly from the observed SFCs
547 using dependence-preserving methods such as overlapping batch means or block bootstrap (Meke-
548 ton and Schmeiser 1984; Flegal and Jones 2010; Künsch 1989). Unlike the Gaussian closure
549 discussed here, these methods can retain higher-order dependence represented in the observations,
550 but require adequate sampling of the relevant dependence and suitable choices of block size.
551 For multidimensional or irregular sampling, converting variability among block estimates into
552 full-record uncertainty also requires accounting for differences in pair geometry and dependence
553 within and between blocks. These issues are examined in a follow-up paper.

554 It is important to keep in mind that any single-record uncertainty estimate, including those
555 developed here, relies on assumptions about statistics that the observations do not adequately
556 constrain. Long-range dependence beyond the sampled domain and contributions from rare events
557 not represented in the observations cannot be determined from that realization alone and therefore
558 remain fundamentally non-identifiable.

559 Within these limits, the Gaussian reference provides both a practical baseline for predicting
560 SF uncertainty for specified covariance models and sampling geometries, and a framework for
561 understanding how dependence limits the effective number of observations. In all cases, uncertainty
562 estimates should be reported together with the methodology, assumptions, and the range of spatial
563 scales over which the relevant dependence has been resolved.

564 *Acknowledgments.* This work was supported by NASA SASSIE project (award
565 80NSSC21K0832), by NASA S-MODE project (an EVS-3 investigation awarded under NASA
566 Research Announcement NNH17ZDA001N-EVS3), and by the NSF Anisotropy project (award
567 2344156). I acknowledge helpful discussions with OpenAI’s ChatGPT, which assisted in re-
568 fining the argument and presentation. I remain personally responsible for all scientific claims,
569 conclusions, and remaining errors.

570 **Data availability**

571 All numerical examples use synthetic data generated from the statistical models de-
572 scribed in the text. MATLAB demonstration code illustrating the calculation of second-

order structure functions and their Gaussian-reference sampling uncertainty is available at
<https://github.com/shcher2018/SF-Uncertainty-Gaussian>.

APPENDIX A

Asymptotic SFC correlation scales for finite-scale covariance models

We derive the limiting forms of the integral SFC correlation scale λ_r for the exponential and Gaussian covariance models using (25).

For exponential covariance, expanding (29) for $r \ll \ell$ gives

$$g_r(s) \simeq 1 - \frac{s}{r}, \quad 0 \leq s \leq r. \quad (\text{A1})$$

Beyond $s = r$, the normalized increment covariance has a weaker tail of amplitude $O(r/\ell)$ and decay scale ℓ . Neglecting this tail and higher-order corrections gives

$$\lambda_r \simeq 2r \int_0^1 (1-u)^2 du = \frac{2}{3}r, \quad (\text{A2})$$

where $u = s/r$.

For $r \gg \ell$, the normalized increment covariance has two well-separated contributions,

$$g_r(s) \simeq \begin{cases} e^{-s/\ell}, & s = O(\ell), \\ -\frac{1}{2}e^{-|s-r|/\ell}, & |s-r| = O(\ell). \end{cases} \quad (\text{A3})$$

Neglecting their overlap and extending the integration about $s = r$ to $(-\infty, \infty)$ gives

$$\lambda_r \simeq 2\ell \left[\int_0^\infty e^{-2u} du + \frac{1}{4} \int_{-\infty}^\infty e^{-2|u|} du \right] = \frac{3}{2}\ell, \quad (\text{A4})$$

where $u = s/\ell$ in the first integral and $u = (s-r)/\ell$ in the second.

For Gaussian covariance, expanding (33) for $r \ll \ell$ gives

$$g_r(s) \simeq [1 - 2(s/\ell)^2] e^{-(s/\ell)^2}. \quad (\text{A5})$$

587 This limiting covariance is independent of r and varies with lag on the scale ℓ . Substituting into
 588 (25) and neglecting higher-order corrections gives

$$\lambda_r \simeq 2\ell \int_0^\infty (1-2u^2)^2 e^{-2u^2} du = \frac{3}{4} \sqrt{\frac{\pi}{2}} \ell, \quad (\text{A6})$$

589 where $u = s/\ell$.

590 For $r \gg \ell$, the normalized increment covariance has two well-separated contributions,

$$g_r(s) \simeq \begin{cases} e^{-(s/\ell)^2}, & s = O(\ell), \\ -\frac{1}{2} e^{-[(s-r)/\ell]^2}, & |s-r| = O(\ell). \end{cases} \quad (\text{A7})$$

591 Neglecting their overlap and extending the integration about $s = r$ to $(-\infty, \infty)$ gives

$$\lambda_r \simeq 2\ell \left[\int_0^\infty e^{-2u^2} du + \frac{1}{4} \int_{-\infty}^\infty e^{-2u^2} du \right] = \frac{3}{2} \sqrt{\frac{\pi}{2}} \ell, \quad (\text{A8})$$

592 where $u = s/\ell$ in the first integral and $u = (s-r)/\ell$ in the second. Thus, the integral SFC
 593 correlation scale remains proportional to ℓ in both limits, with the large-separation value twice the
 594 small-separation value.

595 APPENDIX B

596 Efficient evaluation of F_G for arbitrary sampling geometries

597 Direct evaluation of (17) requires summing $O(N_r^2)$ squared normalized increment covariances g_{ij}^2
 598 and, if implemented naively, storage of an $N_r \times N_r$ matrix. This can become prohibitive because
 599 the number of contributing pairs N_r may greatly exceed the number of observations n . The
 600 calculation can be simplified by recognizing the contributing increments as pairwise differences on
 601 a graph whose vertices are the n observations and whose edges are the N_r contributing pairs. The
 602 corresponding edge–vertex incidence matrix provides a compact representation of these differences
 603 (e.g., Merris 1994; Reinisch et al. 2017).

Let $\mathbf{B}$ be the $N_r \times n$ pair-incidence matrix, with row i containing -1 and $+1$ at the two endpoints of pair i . For a specified SF2 model $D_2(s)$, define the symmetric normalized SF2 matrix

$$R_{kl} = \frac{D_2(|\mathbf{x}_k - \mathbf{x}_l|)}{D_2(r)}. \quad (\text{B1})$$

The matrix of normalized increment covariances g_{ij} can then be represented as

$$\mathbf{g} = -\frac{1}{2}\mathbf{B}\mathbf{R}\mathbf{B}^T. \quad (\text{B2})$$

Equation (B2) is an exact matrix rewriting of (15). The double sum in (17) can also be expressed in a matrix form

$$F_G(r) = \frac{2}{N_r^2} \text{tr}(\mathbf{g}^T \mathbf{g}) = \frac{1}{2N_r^2} \text{tr}(\mathbf{B}\mathbf{R}\mathbf{B}^T \mathbf{B}\mathbf{R}\mathbf{B}^T). \quad (\text{B3})$$

Defining the graph Laplacian $\mathbf{L} = \mathbf{B}^T \mathbf{B}$ and using the cyclic invariance of the trace gives

$$F_G(r) = \frac{1}{2N_r^2} \text{tr}(\mathbf{R}\mathbf{L}\mathbf{R}\mathbf{L}). \quad (\text{B4})$$

Thus, the dense $N_r \times N_r$ matrix $\mathbf{g}$ need not be formed. Instead, the calculation uses the $n \times n$ structure-function matrix $\mathbf{R}$ and the graph Laplacian $\mathbf{L}$, which can be stored sparsely when the contributing-pair graph is sparse. Exploiting this sparsity can substantially reduce computational cost and storage requirements.

The trace formulation also extends to cross-separation covariance (52). Let $\mathbf{B}_a$ denote the incidence matrix for the N_a pairs in separation bin a , define $\mathbf{L}_a = \mathbf{B}_a^T \mathbf{B}_a$, and let $\mathbf{R}_a$ be the normalized SF2 matrix in (B1) with $r = r_a$. Then

$$F_{G,ab} = \frac{1}{2N_a N_b} \text{tr}(\mathbf{R}_a \mathbf{L}_a \mathbf{R}_b \mathbf{L}_b). \quad (\text{B5})$$

For $a = b$, this reduces to (B4). Thus, the full cross-separation covariance matrix can be evaluated using observation-level matrices, without constructing the pair-level increment-covariance matrices.

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