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Decompression, self-leveling, and wobble decay of bubble rings released by scuba divers

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Abstract

Scuba divers blow bubble rings, hollow vortex rings with a gas core that expands as the ring rises because the water pressure drops with every meter of ascent. Classical theory keeps the gas volume fixed and treats only level rings, and the one earlier estimate of the expansion, for a laboratory tank, found it small. Here the ring is modeled as a thin hollow vortex ring with a core that obeys Boyle's law and a motion set by the Kelvin impulse balance of a massless bubble, and this planar model is tested against a 3D vortex filament model. The ring radius has an exact solution in exponential integrals that holds for any tilt, and the extra growth relative to a ring of constant gas volume depends on a single release pressure number. This gain is near one in a tank but reaches 1.38 for release at 10 m and 1.98 at 40 m. A reference ring holding 10 mL of gas and released at 10 m grows from 5 to 18.5 cm in radius, compared with 14.8 cm at constant volume. Because its horizontal impulse is conserved, a tilted ring levels faster than predicted for a core that carries axial flow; a ring released 30 degrees from level ends nearly level after drifting 1.44 m sideways. Long bending waves decay as the ring grows. Along these trajectories, a viscous capillary stability criterion limits intact ascent to release depths of about 10 m. The two models agree to the expected order in the core slenderness.

Keywords: bubble ring; Kelvin impulse; scuba diving; vortex filament; vortex ring

1 Introduction

A bubble ring is a vortex ring with a gas-filled core. Such rings form when a liquid jet pierces a rising bubble from below [21, 13]. Recreational and professional scuba (self-contained underwater breathing apparatus) divers also make them on purpose by releasing a short puff of breathing gas from the mouth. The diver watches the ring rise, widen, and finally break into a string of small bubbles. The ring is a buoyant vortex ring [19], and the classical theory of the toroidal bubble predicts that a ring with a fixed gas volume, rising level through still liquid, widens as the square root of time while it slows down [15]. Later studies computed how the ring forms and deforms with boundary integral methods [13], described its Hamiltonian structure [16], and simulated its viscous evolution and breakup at laboratory Reynolds numbers [5].

Two features of a ring blown by a scuba diver are missing from this work. The first is decompression. Divers release rings at depths of a few to several tens of meters, where the ambient pressure is two to five times its surface value, so the gas in the core expands severalfold on the way up. The only estimate of this effect was made for a laboratory bubble released a few meters below the surface of a tank. It found a difference of about 7% in radius, judged the effect nearly negligible, and obtained the result by integrating the governing equation numerically for that one case [15]. Variational models have included a compressible core [16], but again without a closed form or an analysis of how the effect changes with depth. The second feature is tilt. A ring released from the mouth is rarely level. Thin filament theory for a buoyant vortex ring with a heavy or light core predicts that the tilt θ decays so that $R \sin \theta$ stays constant, where R is the ring radius [4]. That result, however, sends the component of buoyancy along the core into an axial flow inside the core, and a massless gas core cannot carry such a flow.

This study addresses four problems. The first is to find how much decompression changes the growth, rise, and shape of a bubble ring at diving depths, in closed form where possible. The second is to find the law by which a tilted ring with a massless gas core levels itself. The third is to determine whether long bending waves

on the ring grow or decay as it widens. The fourth is to estimate how deep a ring can be released and still reach the surface intact. These problems are addressed with two models that share the same description of the gas core. The first is a planar thin ring model built on the Kelvin impulse balance, which admits exact solutions. The second is a 3D vortex filament model that carries waves and tilt without assuming a flat circular ring, and it serves as an independent check on the first. The study is idealized and uses no observational data or output from other models. The ring parameters are round values that are typical of rings blown by divers, and every number reported here follows from the model equations.

2 Methods

2.1 Configuration, assumptions, and the gas core

Figure 1 shows the configuration. Sea water of uniform density $\rho = 1025 \text{ kg m}^{-3}$ fills the half-space below a flat free surface held at atmospheric pressure $p_{\text{atm}} = 101325 \text{ Pa}$. The vertical coordinate z points upward, the depth is $d \equiv -z$, and the gravitational acceleration is $g = 9.81 \text{ m s}^{-2}$. The hydrostatic pressure at depth d is

$$p(d) = p_{\text{atm}} + \rho g d. \quad (1)$$

A ring is released at depth d_0 with radius R_0 , gas volume V_0 , and circulation Γ about its core, and is followed until its center reaches the end depth $d_e = 0.5 \text{ m}$. The circulation is the line integral of the liquid velocity around any closed loop that threads the ring once. The reference ring has $d_0 = 10 \text{ m}$, $R_0 = 5 \text{ cm}$, $V_0 = 10 \text{ mL}$, and $\Gamma = 0.06 \text{ m}^2 \text{ s}^{-1}$, and the surface tension between air and sea water is $\gamma = 0.073 \text{ N m}^{-1}$. Table 1 lists these values together with the quantities derived from them. The symbol $\equiv$ marks a definition, and units of the International System of Units (SI) are used unless stated otherwise.

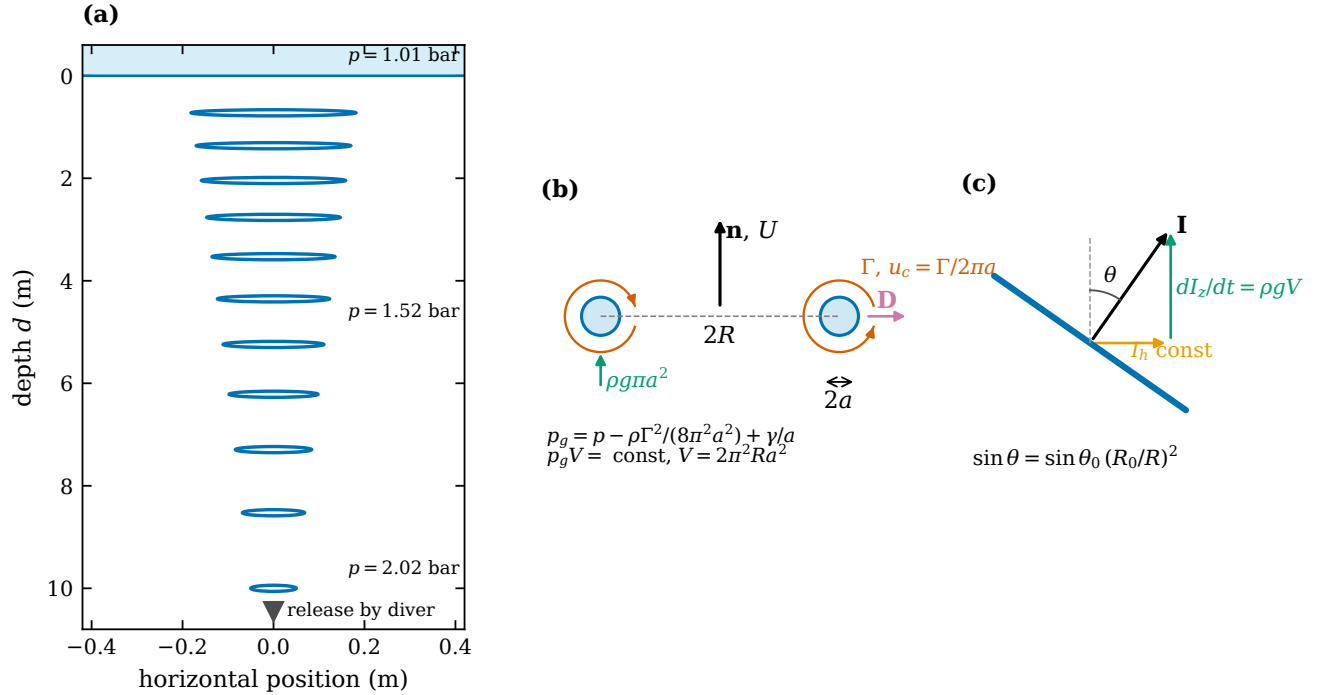


Figure 1: Configuration and notation. (a) Computed outlines of the reference ring every 4 s during its ascent from 10 m. The horizontal axis is at true scale and the vertical axis is compressed; the hydrostatic pressure is marked at three depths. (b) Cross-section of the ring through its axis, showing the ring radius R , the core radius a , the unit normal $\hat{n}$ along which the ring moves at speed U , the circulation Γ with swirl speed $u_c = \Gamma/(2\pi a)$ at the core surface, the buoyancy $\rho g \pi a^2$ per unit length of core, and the drift $\mathbf{D}$ of the core relative to the liquid that balances it. The core balance and gas law of Eqs. (2) and (3) are repeated. (c) A ring tilted by θ from level. Its Kelvin impulse $\mathbf{I}$ gains only a vertical component, so the horizontal component I_h is conserved.

Table 1: Reference parameters and derived quantities. The ring parameters are illustrative; no systematic measurement of rings blown by divers is known to the authors.

Symbol	Meaning	Value	Unit
ρ	density of sea water	1025	kg m^{-3}
g	gravitational acceleration	9.81	m s^{-2}
γ	surface tension, air and sea water	0.073	N m^{-1}
ν	kinematic viscosity of sea water	1.0×10^{-6}	$\text{m}^2 \text{s}^{-1}$
p_{atm}	atmospheric pressure	101 325	Pa
d_0	release depth	10	m
d_e	end depth	0.5	m
R_0	ring radius at release	0.05	m
V_0	gas volume at release	1.0×10^{-5}	m^3
Γ	circulation about the core	0.06	$\text{m}^2 \text{s}^{-1}$
a_0	core radius at release, Eq. (4)	3.18	mm
$\varepsilon_0 = a_0/R_0$	core slenderness	0.064	—
$u_c = \Gamma/(2\pi a_0)$	swirl speed at the core surface	3.00	m s^{-1}
We_0	swirl Weber number, Eq. (6)	402	—
$\text{Fr}_{c,0}$	core Froude number, Eq. (6)	288	—
$h_{s,0}$	swirl velocity head, Eq. (6)	0.46	m
Γ/ν	circulation Reynolds number	6×10^4	—
$\Pi_0 = \rho g d_0/p_{\text{atm}}$	release pressure number	0.99	—
U_0	rise speed at release	0.41	m s^{-1}

The liquid is taken as incompressible and inviscid, and its flow outside the core as irrotational, so by Kelvin’s circulation theorem [17] the circulation Γ stays constant. Viscosity enters only once, through the stability criterion of Eq. (21), which accounts for the viscous decay of the swirl at the core surface. The gas is treated as massless, ideal, and isothermal. This last choice is justified by comparing time scales: the thermal diffusion time of the core, $a_0^2/\kappa_g \approx 0.5$ s for a thermal diffusivity of air $\kappa_g = 2 \times 10^{-5} \text{ m}^2 \text{s}^{-1}$, is about one hundredth of the time $p(d_0)/(\rho g U_0) \approx 49$ s over which the ambient pressure changes, so heat exchange keeps the gas at the water temperature. Gas does not dissolve during the ascent.

The core is thin, with slenderness $\varepsilon \equiv a/R \ll 1$, and its cross-section is circular. In the planar model, corrections of relative order $\varepsilon^2 \ln \varepsilon$ are neglected and the ring is flat and circular; the filament model drops the second restriction. In both models the core radius is the same everywhere along the ring, and the gas pressure is set by the hydrostatic pressure at the mean height of the core. This holds as long as the vertical extent of the core stays below the swirl velocity head h_s defined in Eq. (6). The liquid is unbounded: the free surface is not modeled, and the integration stops at d_e . Finally, waves on the core that are shorter than its radius, with axial wavenumber k such that $ka > 1$, lie outside thin filament theory and are not represented.

Outside the core the liquid swirls irrotationally with azimuthal velocity $\Gamma/(2\pi r_c)$ at distance r_c from the core center. At the core surface the liquid pressure is therefore lower than the ambient value by the swirl dynamic pressure $\rho\Gamma^2/(8\pi^2 a^2)$, and the gas pressure p_g balances this pressure together with the capillary pressure γ/a of the curved surface,

$$p_g = p(d) - \frac{\rho\Gamma^2}{8\pi^2 a^2} + \frac{\gamma}{a}. \quad (2)$$

The same balance, for an adiabatic gas, sets the core volume in a variational model of the toroidal bubble [16]. The gas obeys Boyle’s law,

$$p_g V = C, \quad (3)$$

where the constant C is fixed by the state at release, and the volume of a thin torus is

$$V = 2\pi^2 R a^2. \quad (4)$$

Setting $V = V_0$ and $R = R_0$ at release gives $a_0 = [V_0/(2\pi^2 R_0)]^{1/2}$, and C then follows from Eq. (2) evaluated at d_0 and a_0 . Multiplying Eq. (2) by a^2 and using Eqs. (3) and (4) gives a quadratic equation in a with the positive root

$$a = \frac{2B}{\gamma + (\gamma^2 + 4p(d)B)^{1/2}}, \quad B \equiv \frac{\rho\Gamma^2}{8\pi^2} + \frac{C}{2\pi^2 R}. \quad (5)$$

This rationalized form avoids cancellation when $\gamma^2 \ll pB$. For a filament of length ℓ , the factor $2\pi^2 R$ is replaced by $\pi\ell$. Three dimensionless measures and one length describe the core,

$$\text{We} \equiv \frac{\rho\Gamma^2}{4\pi^2 \gamma a}, \quad \text{Fr}_c \equiv \frac{\Gamma^2}{4\pi^2 g a^3}, \quad h_s \equiv \frac{u_c^2}{2g}, \quad u_c \equiv \frac{\Gamma}{2\pi a}. \quad (6)$$

The swirl Weber number We is twice the ratio of the swirl dynamic pressure to the capillary pressure, a normalization chosen so that $\text{We} = 1$ marks the stability threshold of Eq. (20). The core Froude number Fr_c compares the centripetal acceleration u_c^2/a at the core surface with g . The swirl velocity head h_s is the height over which

the swirl suction can hold a uniform core against the hydrostatic pressure difference, and the ratio $h_s/(2R)$ compares this height with the ring diameter. When both corrections in Eq. (2) are dropped, the gas law reduces to $V = V_0 p(d_0)/p(d)$. This case is called the ideal closure, and Eqs. (2) to (5) the full closure. In the full closure the swirl suction lowers the gas pressure by about 2% of the ambient pressure at release, which enlarges the core by a corresponding amount.

2.2 Planar ring and vortex filament models

A massless bubble in an unbounded liquid transfers all of its buoyancy to the Kelvin impulse $\mathbf{I}$ of the liquid, whether or not its volume changes [1, 2, 3],

$$\frac{d\mathbf{I}}{dt} = \rho g V(t) \hat{\mathbf{z}}, \quad (7)$$

where $\hat{\mathbf{z}}$ is the upward unit vector. For a thin planar ring with unit normal $\hat{\mathbf{n}}$, the impulse is $\mathbf{I} = \rho\pi\Gamma R^2 \hat{\mathbf{n}} [1 + O(\varepsilon^2)]$ [17]. The vector $\mathbf{J} \equiv \mathbf{I}/(\rho\pi\Gamma) = R^2 \hat{\mathbf{n}}$ and the ring center $\mathbf{c}$ therefore obey

$$\frac{d\mathbf{J}}{dt} = \frac{gV}{\pi\Gamma} \hat{\mathbf{z}}, \quad \frac{d\mathbf{c}}{dt} = U \hat{\mathbf{n}}, \quad R = |\mathbf{J}|^{1/2}, \quad \hat{\mathbf{n}} = \frac{\mathbf{J}}{|\mathbf{J}|}. \quad (8)$$

The tilt θ is the angle between $\hat{\mathbf{n}}$ and $\hat{\mathbf{z}}$. For a level ring of constant volume, Eq. (8) gives $R^2 = R_0^2 + gVt/(\pi\Gamma)$, the classical square-root law [19, 15]; the circulation used in [15] equals $\Gamma/(2\pi)$ in the present notation.

The speed U at which a thin hollow ring moves through the liquid follows from the energy H of the system through $U = \partial H/\partial I$ at fixed Γ , as in the Hamiltonian theory of bubbles [2]. The energy consists of the kinetic energy $\frac{1}{2}\rho\Gamma^2 R [\ln(8R/a) - 2]$ of a hollow ring [17], the surface energy $4\pi^2\gamma Ra$, the work $2\pi^2 p R a^2$ done against the ambient pressure, and the free energy of the gas. H is stationary with respect to a exactly when Eq. (2) holds. Differentiating with respect to R at this stationary point and dividing by $\partial I/\partial R = 2\pi\Gamma R$ gives

$$U = \frac{\Gamma}{4\pi R} \left[\ln \frac{8R}{a} - c \right], \quad c \equiv \frac{1}{2} - \frac{1}{\text{We}}. \quad (9)$$

Without surface tension the core constant $c = 1/2$ is that of the hollow vortex ring [17]. Surface tension adds the small correction $1/\text{We} \approx 2.5 \times 10^{-3}$ for the reference ring, and the ideal closure takes $c = 1/2$. The planar model consists of Eqs. (1), (5), (8), and (9), a system of six ordinary differential equations (ODEs) for $\mathbf{J}$ and $\mathbf{c}$.

Two exact results follow from Eq. (8) before any closure is chosen. Because $d\mathbf{J}/dt$ is vertical, the horizontal component $J_h = R^2 \sin \theta$ is conserved,

$$R^2 \sin \theta = R_0^2 \sin \theta_0, \quad (10)$$

where θ_0 is the tilt at release, so a tilted ring levels as it grows. The vertical component obeys $dJ_z/dt = gV/(\pi\Gamma)$, while $dd/dt = -U \cos \theta$. Writing $J_z = (R^4 - J_h^2)^{1/2}$ and eliminating t gives

$$L dR = -\frac{2gV}{\Gamma^2} dd, \quad L \equiv \ln \frac{8R}{a} - c, \quad (11)$$

from which the tilt has dropped out. The radius as a function of depth is therefore the same for every initial tilt.

In the ideal closure, $V = V_0 p_0/p$ with $p_0 \equiv p(d_0)$. The logarithmic pressure coordinate $\lambda \equiv \ln(p_0/p)$ is zero at release and increases during the ascent, with $d\lambda = -\rho g dd/p$ by Eq. (1). Equation (11) then reads $L dR = \ell_B L_0 d\lambda$, where L_0 is the value of L at release and

$$\ell_B \equiv \frac{2V_0 p_0}{\rho \Gamma^2 L_0} \quad (12)$$

is a length referred to here as the Boyle length ($\ell_B = 25.2$ cm for the reference ring). If the logarithm is frozen at $L = L_0$, the radius grows linearly in λ ,

$$R = R_0 + \ell_B \lambda, \quad \frac{dR}{dh} = \frac{\ell_B \rho g}{p}, \quad (13)$$

where $h = d_0 - d$ is the height risen. The expansion per meter of rise grows as $1/p$, so the ring flares toward the surface. A ring of constant volume V_0 with the same frozen logarithm gains $2gV_0(d_0 - d)/(\Gamma^2 L_0)$ instead. The ratio of the two gains between d_0 and d_e depends on a single number,

$$\mathcal{G}(\Pi) \equiv \frac{1 + \Pi}{\Pi} \ln(1 + \Pi), \quad \Pi \equiv \frac{\rho g(d_0 - d_e)}{p(d_e)}, \quad (14)$$

which tends to one as $\Pi \rightarrow 0$ and grows like $\ln \Pi$ for large Π .

Freezing the logarithm overestimates the growth, because L increases as the core thins. Without this simplification, Eq. (4) and the ideal closure give $a^2 = V_0 e^\lambda/(2\pi^2 R)$, so that $L = L_0 + \frac{3}{2} \ln r - \frac{1}{2} \lambda$ with $r \equiv R/R_0$. Equation (11) becomes linear in $\lambda(r)$, $b d\lambda/dr + \lambda/2 = L_0 + \frac{3}{2} \ln r$ with $b \equiv \ell_B L_0/R_0$. With $\alpha \equiv 1/(2b)$ and $\lambda(1) = 0$ it integrates exactly to

$$\lambda(r) = 2L_0 \left[1 - e^{-\alpha(r-1)} \right] + 3 \left[\ln r - e^{-\alpha r} (\text{Ei}(\alpha r) - \text{Ei}(\alpha)) \right], \quad (15)$$

where Ei is the exponential integral, defined for $x > 0$ by the principal value $\text{Ei}(x) \equiv -\int_{-x}^{\infty} e^{-u} u^{-1} du$. Equation (15) gives the radius at any depth with a single root finding and holds for any tilt. The ascent time T of a

level ring and the horizontal travel x of a tilted ring each follow from one quadrature,

$$T = \frac{4\pi R_0^2 p_0}{\Gamma \rho g \ell_B L_0} \int_1^{r_e} r e^{-\lambda(r)} dr, \quad x = \frac{p_0 R_0}{\rho g \ell_B L_0} \int_1^{r_e} \frac{\sin \theta L e^{-\lambda(r)}}{\cos \theta} dr, \quad (16)$$

where r_e is the value of r at d_e and $\sin \theta = \sin \theta_0 / r^2$ by Eq. (10). With the logarithm frozen, the first integral has the closed form $T = 4\pi p_0 [R_0(1 - e^{-\lambda_e}) + \ell_B(1 - (1 + \lambda_e)e^{-\lambda_e})]/(\Gamma L_0 \rho g)$, with $\lambda_e \equiv \lambda(d_e)$. The full closure is integrated numerically.

The filament model represents each ring as a closed curve $\mathbf{X}(\varphi, t)$ parameterized by $\varphi \in [0, 2\pi)$, with arc length s , unit tangent $\hat{\mathbf{t}}$, curvature κ , and unit binormal $\hat{\mathbf{b}}$. The core radius $a(t)$ is uniform along the curve and follows from Eq. (5), using the length ℓ of the curve and the pressure at its mean height $\bar{z}$, weighted by arc length. Each point moves with

$$\frac{\partial \mathbf{X}}{\partial t} = \mathbf{u}_\delta(\mathbf{X}) + \mathbf{D}, \quad (17)$$

where $\mathbf{u}_\delta$ is the regularized Biot–Savart velocity induced by all rings, in the Rosenhead–Moore form [12, 17],

$$\mathbf{u}_\delta(\mathbf{x}) = \frac{\Gamma}{4\pi} \oint \frac{\mathbf{X}'_\varphi \times (\mathbf{x} - \mathbf{X}')}{(|\mathbf{x} - \mathbf{X}'|^2 + \delta^2)^{3/2}} d\varphi', \quad \delta = a e^{c-1}. \quad (18)$$

Here primes denote evaluation at φ' and the subscript φ denotes a derivative. A circular filament with this kernel moves at $\Gamma[\ln(8R/\delta) - 1]/(4\pi R)$, so this choice of the core parameter δ reproduces Eq. (9).

Buoyancy enters through the force balance on the massless core. Per unit length, the core carries the buoyancy $\rho g \pi a^2 \hat{\mathbf{z}}$. The part of this force along $\hat{\mathbf{t}}$ cannot be balanced by lift, which acts normal to the core. In a filament with a massive core, this part drives an axial flow inside the core [4]. In a gas core it is balanced instead by a gradient of the axial tension $\mathcal{T}(s)$ that the pressure difference across the core wall exerts along the tube. Setting the tangential force to zero gives $\partial \mathcal{T} / \partial s = -\rho g \pi a^2 t_z$, with $t_z \equiv \partial z / \partial s$, and hence $\mathcal{T} = -\rho g \pi a^2 (z - \bar{z})$, which is the hydrostatic variation of the pressure difference across the wall. The mean of $\mathcal{T}$ is already included in the core constant c of Eq. (9). The remaining normal force per unit length, $\mathbf{f} \equiv \rho g \pi a^2 \hat{\mathbf{z}} + \partial(\mathcal{T} \hat{\mathbf{t}}) / \partial s$, has no component along $\hat{\mathbf{t}}$. It is balanced by the Kutta–Joukowski lift $-\rho \Gamma \mathbf{D} \times \hat{\mathbf{t}}$ that the liquid exerts on a core moving at velocity $\mathbf{D}$ relative to it, with Γ taken positive in the sense of $\hat{\mathbf{t}}$. The balance $\mathbf{f} = \rho \Gamma \mathbf{D} \times \hat{\mathbf{t}}$, with $\mathbf{D}$ normal to $\hat{\mathbf{t}}$, gives the drift of the core relative to the liquid,

$$\mathbf{D} = \frac{\hat{\mathbf{t}} \times \mathbf{f}}{\rho \Gamma} = \frac{g \pi a^2}{\Gamma} [\hat{\mathbf{t}} \times \hat{\mathbf{z}} - (z - \bar{z}) \kappa \hat{\mathbf{b}}]. \quad (19)$$

The impulse of a filament is $\mathbf{I} = \frac{1}{2} \rho \Gamma \oint \mathbf{X} \times \mathbf{X}_\varphi d\varphi$. Because the regularized self-induced motion conserves impulse [12], its rate of change is $\rho \Gamma \oint \mathbf{D} \times \hat{\mathbf{t}} ds$. Substituting Eq. (19) and integrating the tension term by parts gives $\rho g \pi a^2 \oint \hat{\mathbf{z}} ds = \rho g V \hat{\mathbf{z}}$, so the filament satisfies Eq. (7) exactly for any shape. For a level circle, Eq. (19) gives $dR/dt = g \pi a^2 / \Gamma$, which is Eq. (8). The filament model never uses Eq. (7) directly, and it carries waves, tilt, and nonuniform curvature, so it is an independent test of the planar model.

To isolate the role of the tension, the filament model is also run with the transverse buoyancy alone, $\mathbf{D} = (g \pi a^2 / \Gamma) \hat{\mathbf{t}} \times \hat{\mathbf{z}}$. This is the closure for a core without axial flow [4]. The impulse then changes at the rate $\rho g \pi a^2 \oint (\hat{\mathbf{z}} - t_z \hat{\mathbf{t}}) ds$, which for a tilted circle contains a horizontal source, and the tilt law becomes $R \sin \theta = R_0 \sin \theta_0$ [4].

2.3 Diagnostics and numerical methods

Linearization about the irrotational swirl outside a gas column of radius a with surface tension, for disturbances proportional to $\exp[i(kz' + m\phi - \omega t)]$, where z' is the axial coordinate, ϕ the azimuthal angle about the core, and m the azimuthal mode number, gives the dispersion relation $(\omega - m\Omega)^2 = [\Omega^2 + \gamma(k^2 a^2 + m^2 - 1)/(\rho a^3)]/G_m$, where $\Omega \equiv \Gamma/(2\pi a^2)$ is the angular velocity of the swirl at the core surface, $G_m \equiv -K_m(ka)/[ka K'_m(ka)] > 0$, and K_m is the modified Bessel function of the second kind. Only $m = 0$ can be unstable, and its growth rate σ satisfies

$$\left(\frac{\sigma}{\Omega}\right)^2 = \left[\frac{1 - (ka)^2}{\text{We}} - 1\right] \frac{ka K_1(ka)}{K_0(ka)}, \quad (20)$$

so the swirl stabilizes every wavelength once $\text{We} > 1$. These stability conditions for a rotating flow with a cylindrical free surface [14] also hold for the core of a toroidal bubble [15]. In a real liquid, vorticity diffuses from the free-slip core surface and reduces the swirl there to $\Gamma[1 - \exp(-3a^2/(8\nu t))]/(2\pi a)$, where ν is the kinematic viscosity and t is the time since release [15]. The stability parameter becomes

$$M(t) \equiv \text{We} \left[1 - \exp\left(-\frac{3a^2}{8\nu t}\right)\right]^2, \quad (21)$$

and the core becomes unstable when M drops below one [15]. The margin M is evaluated along the computed trajectories, with $a(t)$ from the full closure, and the time and depth at which $M = 1$ are recorded. This is the only place where viscosity enters, and the result is an estimate: it ignores the stretching of the core and the effect of the weakening swirl on the speed and growth of the ring.

For the wobble diagnostics, a ring is described by its center $\mathbf{c}$, its impulse radius $R \equiv (|\mathbf{I}|/\rho \pi \Gamma)^{1/2}$, and its

normal $\hat{\mathbf{n}} = \mathbf{I}/|\mathbf{I}|$. Each point of the core then has an azimuth χ about the ring axis, a radial deviation $r'(\chi)$ from R in the plane normal to $\hat{\mathbf{n}}$, and an axial deviation $w(\chi)$ along $\hat{\mathbf{n}}$. The complex amplitudes of mode n are $\hat{r}_n \equiv \pi^{-1} \oint r' e^{-in\chi} d\chi$ and $\hat{w}_n \equiv \pi^{-1} \oint w e^{-in\chi} d\chi$, the mode amplitude is $A_n \equiv (|\hat{r}_n|^2 + |\hat{w}_n|^2)^{1/2}$, and the mode energy is $E_n \equiv A_n^2/2$. Modes $n = 0$ and $n = 1$ describe the radius, position, and tilt and are excluded. The spread of energy over modes $n = 2$ to $n_{\max} = 24$ is measured by the normalized spectral entropy [18]

$$H_s \equiv -\frac{1}{\ln(n_{\max} - 1)} \sum_{n=2}^{n_{\max}} P_n \ln P_n, \quad P_n \equiv \frac{E_n}{\sum_m E_m}, \quad (22)$$

which is zero when one mode holds all the energy and one when the energy is spread evenly. A ring that expands slowly compared with the period of its bending waves should conserve the action of each wave, that is, the ratio of its energy to its frequency. For a bending wave of mode $n \gg 1$ with amplitude A_n , the energy scales as $\rho \Gamma^2 A_n^2 n^2 L/R$ and the frequency as $\Gamma n^2 L/R^2$, so the action scales as $\rho \Gamma A_n^2 R$ and

$$A_n \propto R^{-1/2}. \quad (23)$$

This is a scaling argument rather than a derivation for low modes, and it is tested numerically. For the reference ring the period of the $n = 2$ wave, of order $8\pi^2 R^2/(\Gamma n^2 L) \approx 0.2$ s at release, is much shorter than the tens of seconds over which the ring doubles its radius, so the time scales are well separated.

The planar model, Eq. (8), is integrated with the explicit Runge–Kutta method of order 8(5,3) of Dormand and Prince (DOP853), with relative and absolute tolerances of 10^{-11} and 10^{-14} and an event that stops the integration at d_e . Equation (15) is inverted with Brent’s method, and Eq. (16) is evaluated by adaptive quadrature to a relative tolerance of 10^{-12} .

The filament is discretized at N points $\mathbf{X}_j$, $j = 0, \dots, N-1$, equally spaced in φ with spacing $h_\varphi = 2\pi/N$. All derivatives along the curve are computed spectrally with the fast Fourier transform (FFT), and the integral in Eq. (18) is evaluated with the periodic trapezoidal rule. When the point spacing exceeds δ , the integrand is sharply peaked near $\varphi' = \varphi_i$ and the plain rule converges slowly. The local expansion of the integrand,

$$g_i(\tau) = \frac{\mathbf{X}_{\varphi,i} \times \mathbf{X}_{\varphi\varphi,i}}{2} \frac{\tau_p^2}{(q_i^2 \tau_p^2 + \delta^2)^{3/2}}, \quad \tau_p \equiv 2 \sin \frac{\tau}{2}, \quad q_i \equiv |\mathbf{X}_{\varphi,i}|, \quad (24)$$

with $\tau = \varphi' - \varphi_i$, is therefore subtracted, the trapezoidal rule is applied to the smooth remainder, and the exact integral $\oint g_i d\tau = \frac{1}{2} (\mathbf{X}_\varphi \times \mathbf{X}_{\varphi\varphi})_i q_i^{-3} F(\delta/q_i)$ is added back. Here $F(\beta) \equiv \int_{-\pi}^{\pi} \tau_p^2 (\tau_p^2 + \beta^2)^{-3/2} d\tau$ is tabulated once by adaptive quadrature and interpolated with a cubic spline in $\ln \beta$.

Only the normal component of the velocity changes the shape of the curve. The tangential component is replaced with the one that keeps the points equally spaced in arc length [8], and any remaining nonuniformity is relaxed on a time scale of 0.05 s. Because waves shorter than the core are excluded, the velocity of each ring is projected onto the azimuthal Fourier modes $|n| \leq n_c \equiv \lfloor \ell_0/(2\pi a_0) \rfloor$, fixed at release. These modes have $ka \leq 1$ at release and smaller values later, because a/R decreases. Without this projection, regularized filaments develop a spurious instability near $k\delta \sim 1$. This is the filament counterpart of the short-wave instability of vortex rings [22], which a filament cannot represent faithfully. Because positions and velocities share the same band of modes, the projection keeps Eq. (7) exact and does not reduce the temporal order. Time stepping uses the classical fourth-order Runge–Kutta method (RK4) with step $\Delta t = \Delta t_0 \min[(\ell/\ell_0)^2, 3]$, where ℓ is the current length of the shortest ring, ℓ_0 is its length at release, and $\Delta t_0 = 2$ ms for $N = 128$ or 4 ms for $N = 64$. The factor $(\ell/\ell_0)^2$ follows the frequency of the fastest retained wave, which is of order $\Gamma n_c^2/(4\pi R^2)$. Wavy rings are started with radial and axial perturbations $\sum_{n=2}^8 [A_{r,n} \cos(n\varphi + \psi_n), A_{w,n} \sin(n\varphi + \psi_n)]$, with Gaussian amplitudes of standard deviation $\sqrt{2}$ mm in each component and uniformly random phases ψ_n , and the curve is then resampled at equal arc length.

The solvers are verified against closed forms, conserved quantities, and measured convergence rates, as in earlier idealized solvers [7, 10, 11]. The code is written in Python with NumPy [6], SciPy [20], and Matplotlib [9], and is organized as the library `bubble_rings`.

3 Results

The planar model reproduces the closed forms to the tolerance of the integrator (Table 2). The radius from Eq. (15) differs from the integrated radius in the ideal closure by at most 7×10^{-12} in relative terms, and Eqs. (10) and (16) agree with the integration to 2×10^{-12} . The radius of a ring released 30° from level differs from that of a level ring at the same depth by 8×10^{-13} , which confirms that the tilt drops out of Eq. (11). As an outside check, the one earlier hydrostatic calculation was repeated for a bubble of 21 cm^3 with ring radius 5 cm and circulation $2\pi \times 50 \text{ cm}^2 \text{ s}^{-1}$, released in fresh water 3.43 m below the surface [15]. The core radius at release from Eq. (4), 4.61 mm, matches the reported 0.46 cm. After 40 s the radius with Boyle expansion exceeds the constant-volume radius by 7.7%, compared with the reported value of just under 7% [15]. The small difference may come from the use of the speed of a viscous core in that calculation in place of Eq. (9).

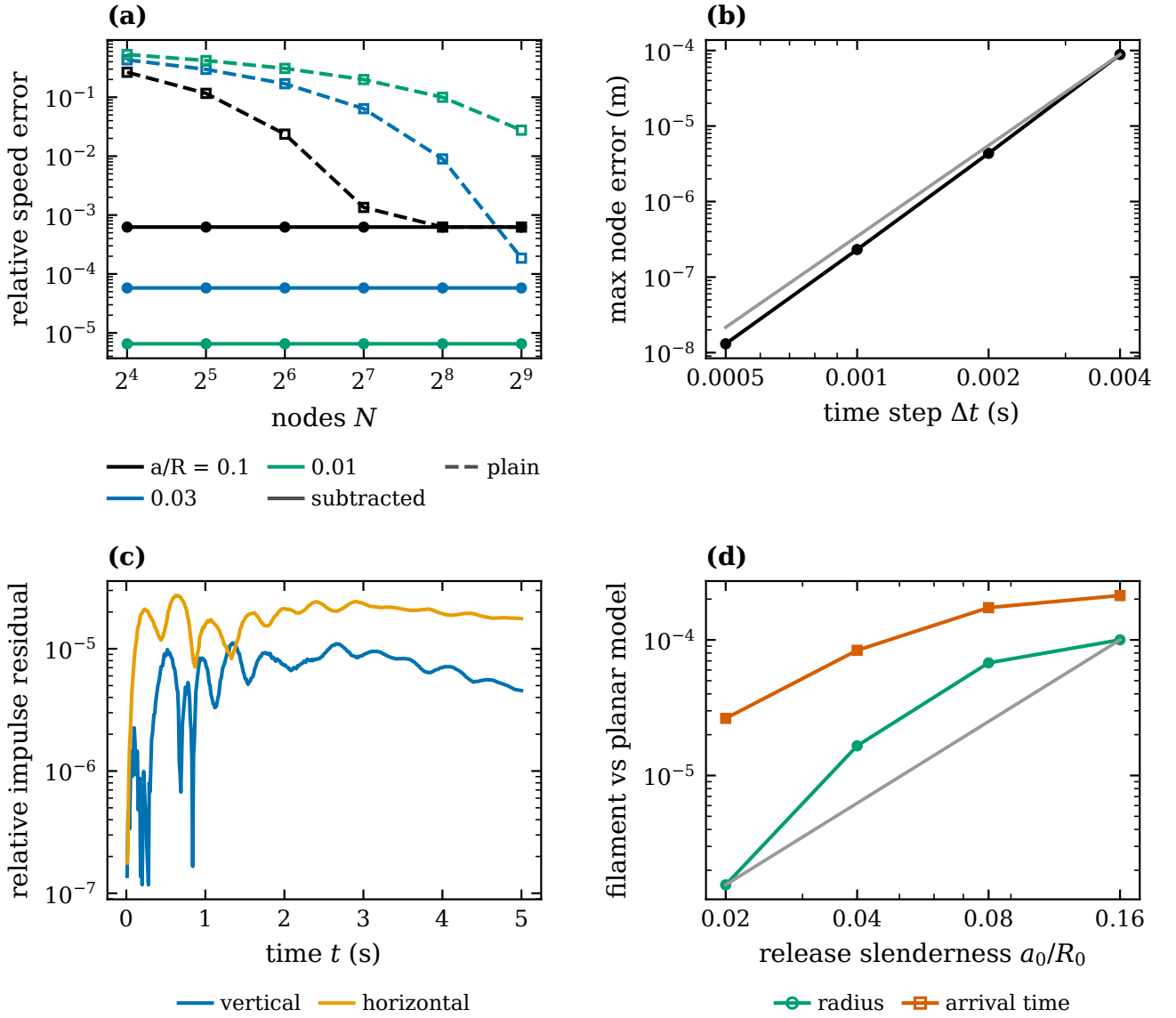


Figure 2: Verification of the filament model. (a) Speed of a circular filament relative to Eq. (9) against the number of points N , for slenderness $a/R = 0.1$, 0.03 , and 0.01 , with the subtraction of Eq. (24) (solid) and with the plain trapezoidal rule (dashed). (b) Maximum point error of a wavy ring ($N = 64$) after 0.4 s against the RK4 step, relative to a run with $\Delta t = 62.5 \mu\text{s}$; the gray line has slope 4. (c) Residual of the impulse balance, $|\mathbf{I}(t) - \mathbf{I}(0) - \rho g \hat{\mathbf{z}} \int_0^t V dt'|/|\mathbf{I}(t)|$, for a wavy ring tilted by 30° , in the vertical and horizontal components, with V integrated by the composite Simpson rule on 0.01 s samples. (d) Difference between the filament and planar models after rising from 10 to 7 m, in radius (relative) and in arrival time (relative to the ascent time), against the release slenderness a_0/R_0 ; the gray line has slope 2.

The filament model passes four tests (Fig. 2). With the subtraction of Eq. (24), the speed of a circular filament no longer depends on N from $N = 16$ upward, and it differs from Eq. (9) by 6.3×10^{-4} , 5.8×10^{-5} , and 6.5×10^{-6} for $a/R = 0.1$, 0.03 , and 0.01 . These plateaus are the difference, of order $\varepsilon^2 \ln \varepsilon$, between the regularized ring and its thin-core limit. Without the subtraction, the error at $a/R = 0.01$ is still 2.8×10^{-2} at $N = 512$. The RK4 integration converges at observed orders of 4.35, 4.23, and 4.15. The impulse balance of a tilted wavy filament holds to 1.1×10^{-5} in the vertical and 2.7×10^{-5} in the horizontal over 5 s, which is the accuracy of the quadrature of $\int V dt$. The filament and planar models differ by 1.6×10^{-6} to 1.0×10^{-4} in radius for $a_0/R_0 = 0.02$ to 0.16 , and the difference falls roughly as ε^2 at small slenderness, the order that thin ring theory neglects. Finally, a wavy ring computed with $N = 64$, 128 , and 256 has the same radius to 2×10^{-5} and the same wave amplitude to 0.3% after 3 s.

Table 2: Measured residuals of the verification tests.

Test	Comparison	Residual
Exact radius	Eq. (15) vs planar ODE, ideal closure	7.1×10^{-12}
Frozen-logarithm radius, ascent time	Eq. (13) and T vs ODE	1.4×10^{-12} , 4.6×10^{-12}
Horizontal travel	Eq. (16) vs ODE, $\theta_0 = 30^\circ$	2.0×10^{-12}
Tilt invariance of $R(d)$	tilted vs level ring at d_e	7.8×10^{-13}
Core radius	Eq. (5) vs bracketing root	2.2×10^{-16}
Earlier laboratory bubble [15]	Boyle vs constant volume at 40 s	7.7% (reported: below 7%)
Ring speed, $a/R = 0.01$	filament vs Eq. (9), any N	6.5×10^{-6}
RK4 order	$\Delta t = 4$ to 0.5 ms	4.35, 4.23, 4.15
Impulse balance	filament vs Eq. (7), 5 s	1.1×10^{-5} , 2.7×10^{-5}
Thin-ring limit	filament vs planar, $a_0/R_0 = 0.02$	1.6×10^{-6}
Point number	$N = 64, 128, 256$ at 3 s	2×10^{-5} in R

The reference ring rises from 10 m to 0.5 m in 41.4 s and grows from 5 to 18.5 cm in radius (Fig. 3a). A ring of constant gas volume reaches only 14.8 cm, so decompression makes the final radius 25% larger. The exact solution, Eq. (15), gives 18.1 cm. The 2% difference from the full closure comes from the swirl suction in Eq. (2), which enlarges the core and therefore the buoyancy. The frozen-logarithm law, Eq. (13), gives 21.2 cm because it ignores the increase of L as the core thins, and it overestimates the ascent time as 56 s instead of 41 s.

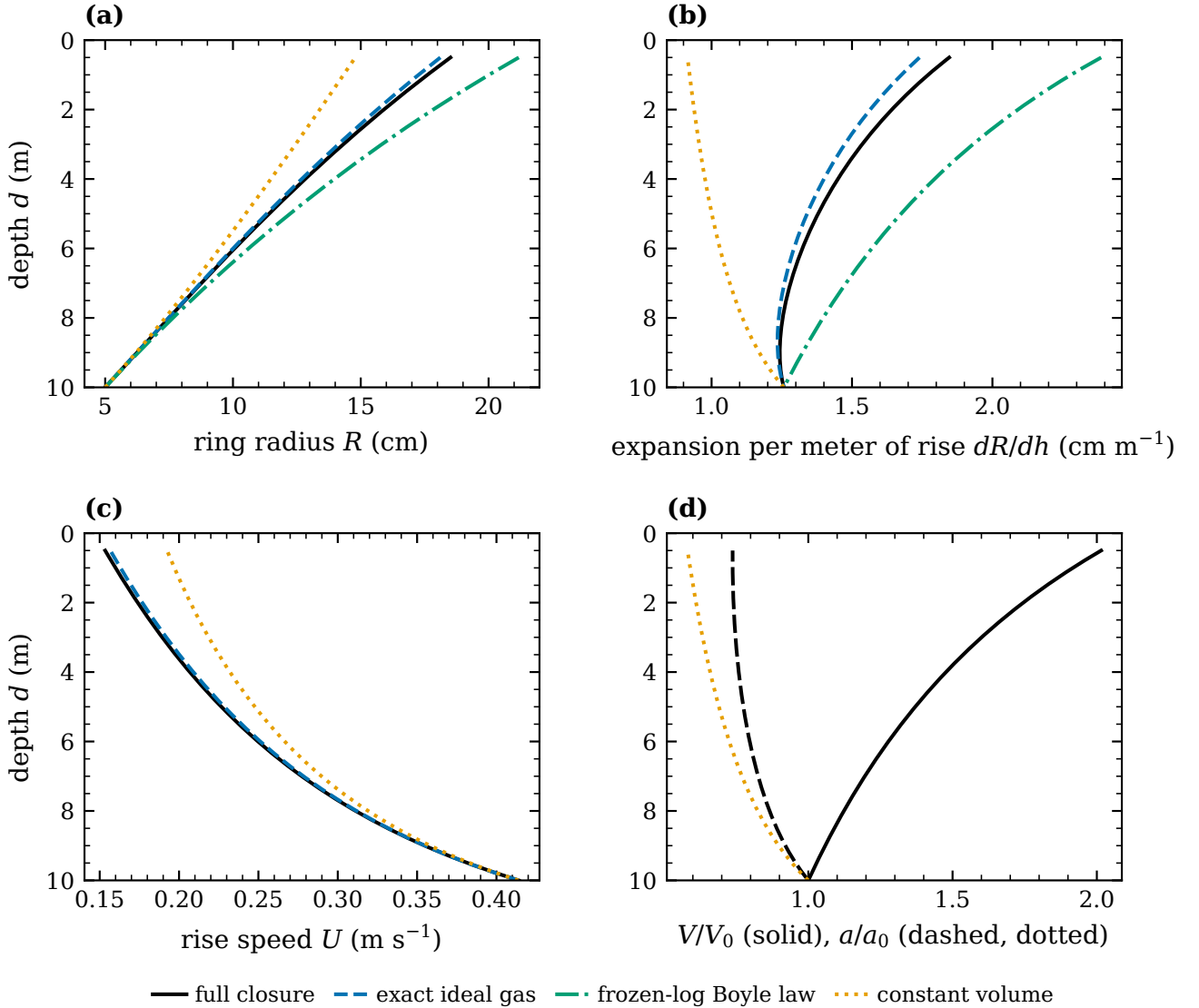


Figure 3: Ascent of the reference ring from 10 m. (a) Ring radius against depth for the full closure, the exact ideal-gas solution of Eq. (15), the frozen-logarithm law of Eq. (13), and a ring of constant gas volume. (b) Expansion per meter of rise, dR/dh . (c) Rise speed U . (d) Gas volume V/V_0 (solid) and core radius a/a_0 (dashed for the full closure, dotted for constant volume).

The expansion per meter of rise increases from 1.26 cm m^{-1} at release to 1.85 cm m^{-1} at 0.5 m (Fig. 3b), so the ring flares toward the surface. A ring of constant volume does the opposite. Its expansion per meter decreases, because the logarithm L in Eq. (11) grows as its core thins. The rise speed drops from 0.41 to 0.15 m s^{-1} , more steeply than at constant volume, because a larger ring induces a weaker velocity on itself (Fig. 3c). The gas volume doubles, but the core radius shrinks to 0.74 of its initial value because the gas is spread along a longer ring (Fig. 3d). The slenderness therefore decreases during the ascent, and the thin-core approximation improves.

The effect of decompression grows with release depth (Fig. 4). The ratio of final to initial radius increases from 1.13 for release at 1 m to 4.3 at 12 m and 14.4 at 40 m , compared with 1.12 , 3.3 , and 7.8 at constant volume (Fig. 4a). The gain over a ring of constant volume (Fig. 4b) closely follows Eq. (14) for all three closures. It is 1.03 for $\Pi = 0.05$, which corresponds to release at 1 m , 1.38 for release at 10 m , and 1.98 for release at 40 m , while the frozen-logarithm prediction is 1.35 and 1.97 for the last two. Equation (14) therefore captures the effect of decompression even though the frozen-logarithm radius itself is off by 14% , because the errors in the two radii largely cancel in the ratio. The ascent time grows from 1.3 s for release at 1 m to 383 s at 40 m (Fig. 4c). The flaring, measured as the ratio of dR/dh at 0.5 m to its value at release, rises from 1.02 to 3.1 over the same range, whereas Eq. (13) predicts the larger ratio $p_0/p(d_e)$ (Fig. 4d). For releases deeper than about 29 m the final radius exceeds d_e , so the last part of these ascents takes place where the free surface, which the model leaves out, would affect the ring.

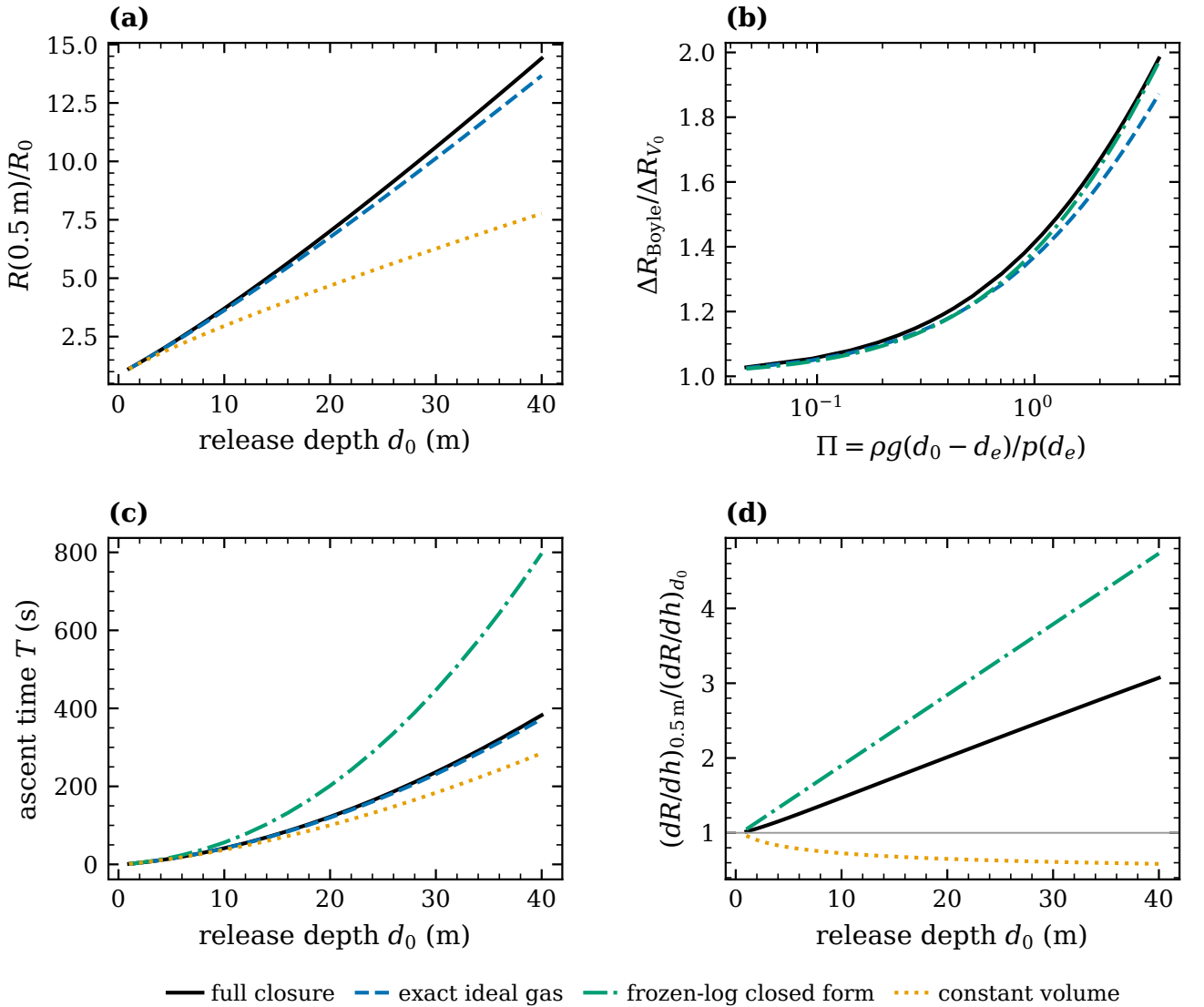


Figure 4: Dependence on release depth d_0 , from 1 to 40 m , for the reference gas volume, radius, and circulation. (a) Radius at 0.5 m divided by R_0 . (b) Radius gain with decompression divided by the gain at constant volume, against Π , with the universal function of Eq. (14). (c) Ascent time; the frozen-logarithm value uses the closed form given below Eq. (16). (d) Ratio of the expansion per meter of rise at 0.5 m to its value at release.

Rings released at 10 m with initial tilts of 15° to 60° level quickly (Fig. 5a). A ring released at 30° is tilted by 13° after rising 2 m and by 2.1° at 0.5 m . The filament model, which assumes neither a flat circular ring nor Eq. (7), agrees with the planar model at every sampled depth. For all four initial tilts, the filament values of

$\sin \theta / \sin \theta_0$ fall on $(R_0/R)^2$, as Eq. (10) predicts (Fig. 5b). When the axial tension is removed from Eq. (19), the same filament code follows R_0/R instead, the law for a core that carries axial flow [4]. The two laws differ by the factor R_0/R , which reaches 0.27 for the reference ring at 0.5 m. The closure of the core therefore controls how fast a ring levels, and measuring the tilt at two depths would distinguish the two laws.

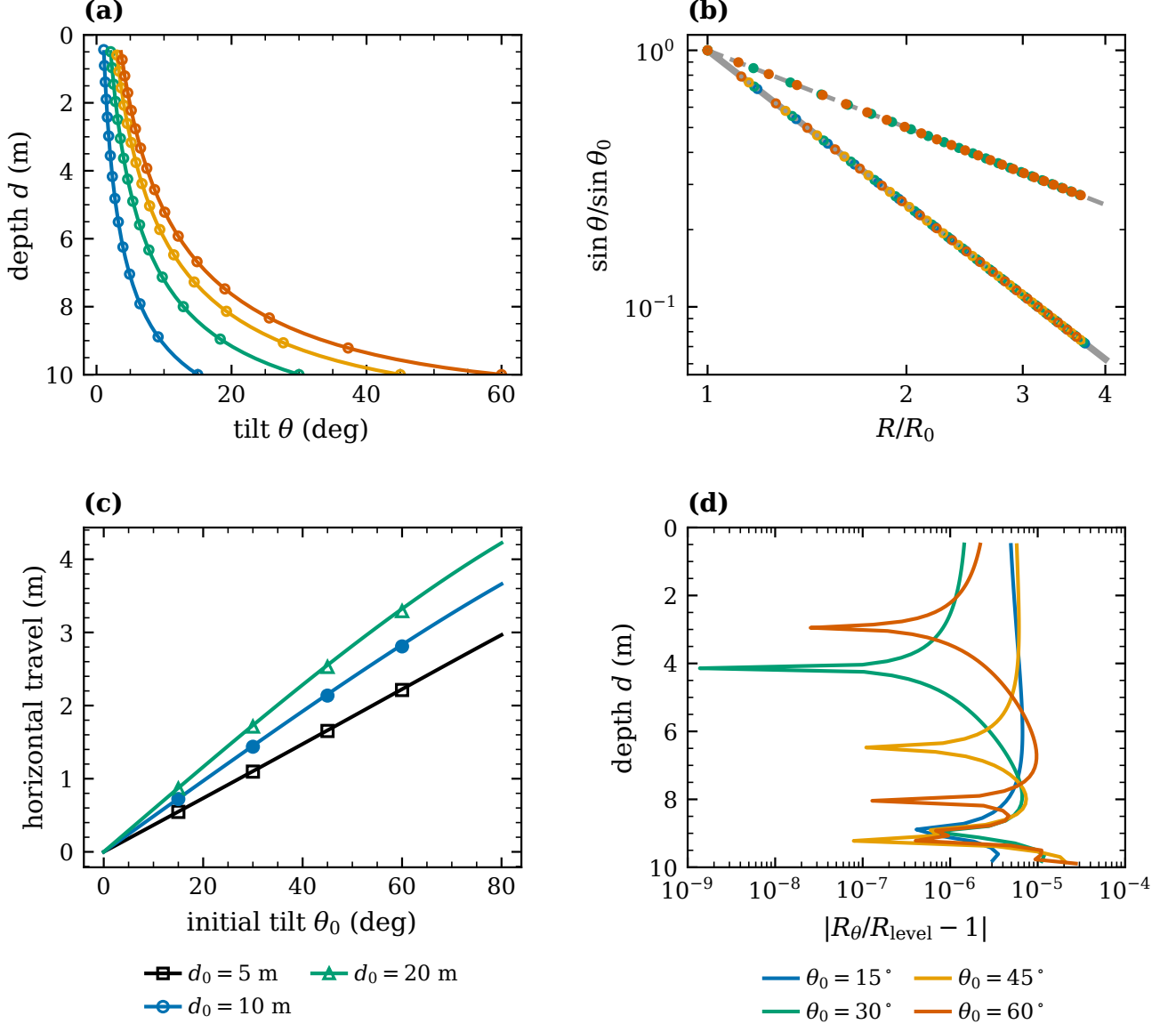


Figure 5: Tilted rings released at 10 m. (a) Tilt against depth for initial tilts of 15°, 30°, 45°, and 60°, from the filament model (64 points, symbols) and the planar model (lines). (b) $\sin \theta / \sin \theta_0$ against R/R_0 for the filament rings (open symbols) with the law of Eq. (10) (gray line), and for filament rings at 30° and 60° with the transverse buoyancy alone (filled symbols) with the law R_0/R (gray dashed line). (c) Horizontal travel by 0.5 m against initial tilt for release at 5, 10, and 20 m: Eq. (16) (lines), planar model with the full closure (open symbols), and filament model at 10 m (filled symbols). (d) Radius of each tilted filament ring relative to that of the level ring at the same depth.

Because the ring moves along its normal, tilt turns part of the rise into horizontal travel. The travel is finite because the tilt decays as R^{-2} while the speed decays as R^{-1} . A ring released at 30° from 10 m travels 1.44 m sideways in the full closure and 1.45 m according to Eq. (16). The travel increases almost linearly with the initial tilt and slowly with release depth, from 1.10 m for release at 5 m to 1.73 m for release at 20 m at the same tilt (Fig. 5c). The radius of each tilted filament ring matches that of the level ring at the same depth to better than 3×10^{-5} (Fig. 5d), as Eq. (11) requires.

In an inviscid liquid the gas core of the reference ring is far from capillary instability. The swirl Weber number is 402 at release and rises to 545 at 0.5 m as the core thins, and the core Froude number rises from 288 to 718. The dispersion relation, Eq. (20), shows the band of unstable axial wavenumbers that opens below $We = 1$ (Fig. 6a). A circulation of only $3.0 \times 10^{-3} \text{ m}^2 \text{ s}^{-1}$, one twentieth of the reference value, would keep $We > 1$ at every release depth, and $3.5 \times 10^{-3} \text{ m}^2 \text{ s}^{-1}$ keeps $Fr_c > 1$ (Fig. 6c).

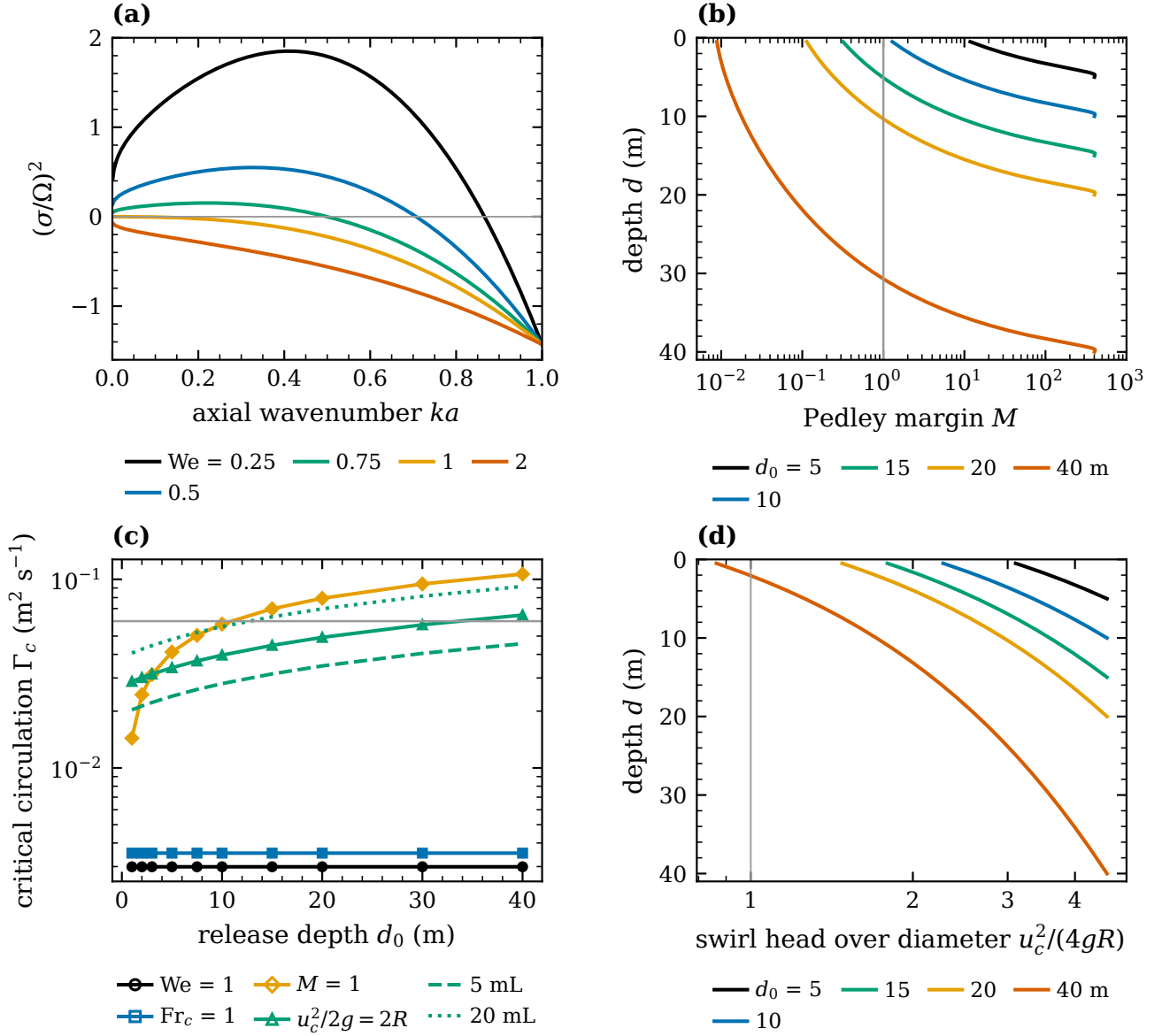


Figure 6: Stability of the gas core. (a) Axisymmetric growth rate of capillary waves on a hollow vortex, Eq. (20), against ka for We from 0.25 to 2; positive values are unstable. (b) Viscous capillary margin M of Eq. (21) along the ascent from release depths of 5, 10, 15, 20, and 40 m; the core becomes unstable where M falls below one (gray line). (c) Smallest circulation that keeps each margin above one over the whole ascent: We , Fr_c , M , and the swirl head h_s at least equal to the ring diameter, for 10 mL of gas (solid), with the swirl-head bound also shown for 5 and 20 mL; the gray line is the reference circulation. (d) Swirl head divided by ring diameter, $h_s/(2R)$, along the ascent for the release depths of (b).

Viscous diffusion changes this result. The margin M of Eq. (21) decreases with time from its initial value We (Fig. 6b) and crosses one between 37 and 47 s after release, almost independently of the release depth. Rings released at 10 m or shallower reach 0.5 m before that time, and the reference ring arrives with a margin of 1.3. A ring released at 15 m becomes unstable at a depth of 5.1 m, one released at 20 m at 10.3 m, and one released at 40 m at 30.7 m. For the reference ring, this estimate limits intact ascent to release depths of about 10 m. The circulation needed to keep $M > 1$ all the way to 0.5 m increases from $0.041 \text{ m}^2 \text{s}^{-1}$ for release at 5 m to $0.107 \text{ m}^2 \text{s}^{-1}$ at 40 m (Fig. 6c). A third bound tests the assumption of a uniform core. The swirl head exceeds the ring diameter throughout the ascent for releases down to about 30 m, so a uniform core can be held in hydrostatic balance at any tilt. For deeper releases the ring grows wide enough near the surface for $h_s/(2R)$ to drop below one (Fig. 6d), and a strongly tilted ring would then collect its gas at its highest point.

Rings released level at 10 m with random bending and breathing waves in modes 2 to 8 lose the wobble during growth (Fig. 7). The total wave amplitude $A \equiv (\sum_n A_n^2)^{1/2}$ falls to about half its initial value while the radius grows 3.6-fold (Fig. 7a). Least-squares fits of $\ln A$ against $\ln R$ over the whole ascent give exponents of -0.53 (range -0.55 to -0.52 over four random seeds) with decompression and -0.54 (-0.56 to -0.52) at constant volume, close to the $-1/2$ of Eq. (23). The modes that carry most of the energy, $n = 2$ to 4, have exponents between -0.53 and -0.49 (Fig. 7c). The weaker modes 5 to 8 scatter more widely because of energy exchange

with the stronger modes. The decay is the same with and without decompression, as expected from Eq. (23), in which only the growth of the ring appears. The spectral entropy of Eq. (22) stays within about 0.1 of its initial value for each seed, between 0.35 and 0.66 overall, and shows no trend (Fig. 7d). Energy therefore does not cascade to shorter waves within the band that the model resolves.

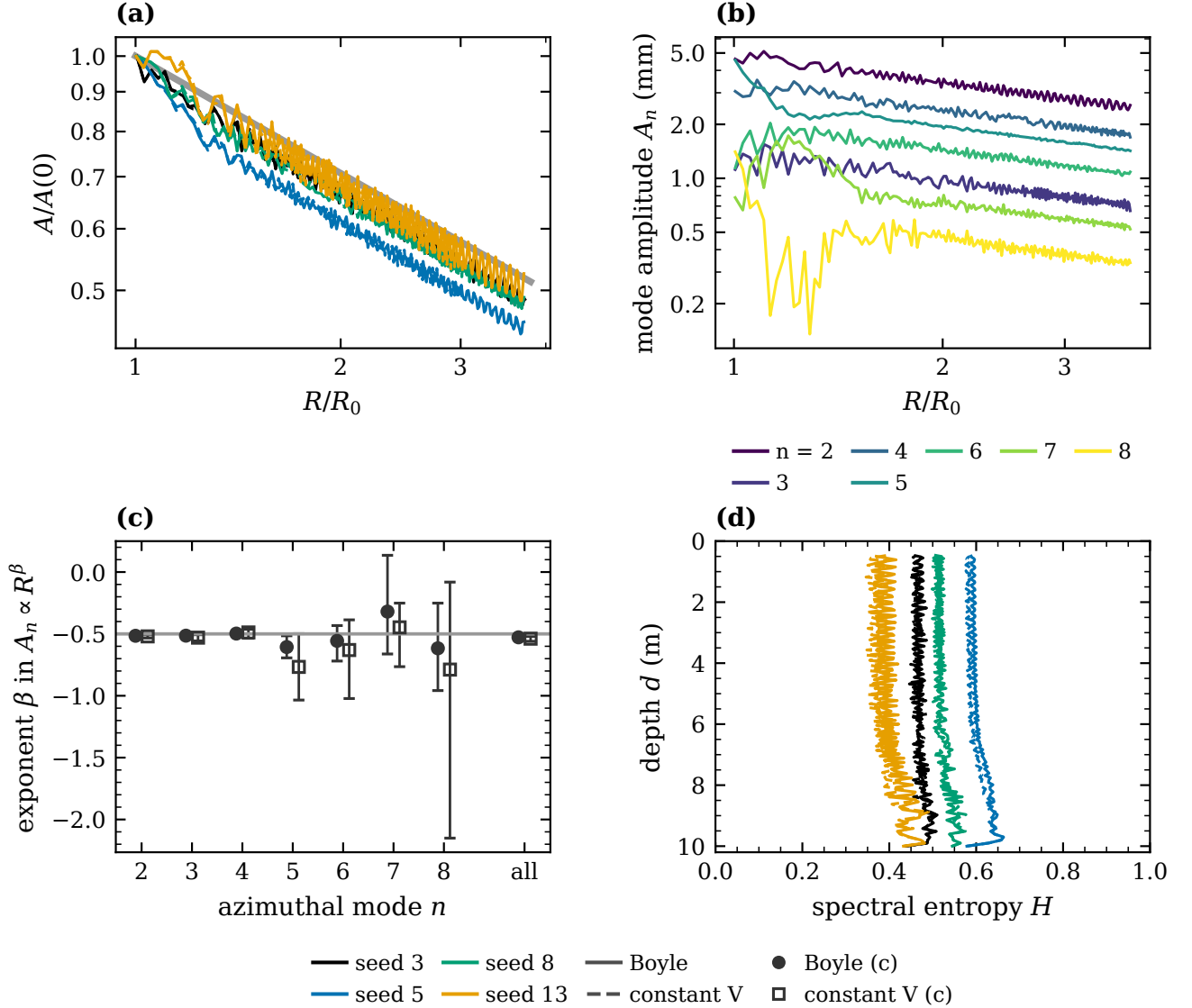


Figure 7: Wobble of rings released level at 10 m with random waves in modes 2 to 8, followed with the filament model (128 points) for four random seeds. Solid lines: gas volume from the full closure; dashed lines: gas volume held at V_0 . (a) Total wave amplitude divided by its initial value against R/R_0 , with the slope $-1/2$ of Eq. (23) (gray). (b) Amplitude A_n of each mode for the first seed with decompression. (c) Fitted exponent β of $A_n \propto R^\beta$ for each mode and for the total amplitude; symbols are seed means and bars show the range over seeds. (d) Spectral entropy of Eq. (22) against depth.

4 Discussion

The model recovers the classical results it shares with earlier work: the square-root growth of a ring of constant volume [19, 15], the speed of a hollow ring [17], the core balance of a compressible gas core [16], and the capillary criterion [14, 15]. The first new result is that decompression, found nearly negligible for a laboratory ring released 3.4 m below the surface [15], is a leading-order effect at diving depths. The single number Π of Eq. (14) explains why. The laboratory bubble had $\Pi \approx 0.3$, for which $\mathcal{G}$ is close to one, whereas release at 10 to 40 m gives Π between about 1 and 4 and gains of 1.4 to 2. The exact solution, Eq. (15), replaces the numerical integration of earlier work with a closed form. The frozen-logarithm law, Eq. (13), gives a simple estimate of the flaring that divers see near the surface, although it overestimates the radius.

The second result concerns tilt. The law $R^2 \sin \theta = \text{const}$, Eq. (10), follows from the exact impulse balance of a massless bubble and does not depend on how the core is modeled. The law $R \sin \theta = \text{const}$, in contrast, comes from a filament model in which the core stores the momentum supplied by the tangential buoyancy as axial flow

[4]. A gas core has almost no mass, and the tension term in Eq. (19) returns that momentum to the liquid. The faster leveling of Eq. (10) should therefore apply to bubble rings, and the slower law may apply to rings with a liquid or heavy core. The two laws can be separated experimentally, since a ring filmed from the side at two depths gives R and θ at each, and the two laws predict different ratios of $\sin \theta$.

The third result is that long bending waves decay as $R^{-1/2}$. This agrees with conservation of wave action and means that the growth of the ring smooths out its large-scale wobble. It also means that breakup, which rings blown by divers are observed to undergo, does not proceed through the long waves resolved here. Viscous simulations and tank experiments at Reynolds numbers of 30 to 100 show breakup through an azimuthal instability of short waves on the scale of the core [5], in the band $ka \gtrsim 1$ that the present models exclude. Equation (21) adds a time scale to this description. Viscous diffusion weakens the swirl at the core surface until surface tension dominates, about 40 s after release, which limits the useful release depth of the reference ring to about 10 m. Because the margin crosses one at nearly the same time for every release depth, this limit is set mainly by the ascent time. A larger circulation both strengthens the swirl and shortens the ascent, and so allows deeper release.

These results share the limitations of an idealized model. The liquid is inviscid except in Eq. (21), so the circulation does not decay, and the breakup estimate ignores stretching of the core and the feedback of a weakening swirl on the motion of the ring. The gas does not dissolve. The core is uniform along the ring and circular in cross-section, whereas computed and observed cores deform [13, 5]. Waves shorter than the core and the breakup itself are not represented. The free surface is not modeled, and for releases deeper than about 29 m the final radius exceeds the end depth. The gas volume, radius, and circulation of rings blown by divers do not appear to have been measured systematically, so the absolute radii, times, and travel distances are illustrative. The results stated as ratios or exponents, namely the gain of Eq. (14), the flaring, the tilt law, and the decay of the wobble, do not depend on these values, and the sensitivity of the absolute results is tabulated in the accompanying reports. Measurements of ring radius, tilt, and rise speed at two or more depths, which a diver with a camera and a depth gauge can make, would fix the three parameters and test the tilt law directly.

5 Conclusions

A bubble ring released by a scuba diver was modeled as a thin hollow vortex ring with a gas core that obeys Boyle’s law. A planar model built on the exact Kelvin impulse balance of a massless bubble agrees with an independent vortex filament model in which buoyancy acts through the lift on the core and a hydrostatic tension along it. Written in terms of the logarithm of the pressure, the ring radius has an exact solution that is the same for every tilt, and the extra growth caused by decompression depends on a single release pressure number. Decompression is small in a laboratory tank but becomes a leading effect at diving depths, where it makes the ring much wider and makes it flare as it nears the surface. A tilted ring conserves its horizontal impulse, so it levels faster than a ring with axial flow in the core and drifts only a finite distance sideways. Long bending waves on the ring weaken as the ring grows, with or without decompression, so the visible wobble fades during the ascent. Viscous weakening of the swirl at the core surface sets a time limit on the life of the ring and therefore a limit on how deep a ring can be released and still reach the surface intact. Measurements of rings at several depths would test these predictions, and adding viscous decay of the circulation, a free surface, and the short waves that break the ring would extend the model to the end of the life of the ring.

Declaration of competing interest

The authors declare no known competing financial interests or personal relationships that could have appeared to influence the work reported in this article.

Declaration of generative AI use

During the preparation of this work, the authors used Claude Sonnet 5 solely for English grammar and vocabulary refinement. The conceptualization, model development, mathematical derivations, computational execution, analysis, and interpretation presented in this study were carried out by the authors, who take full responsibility for the content of this article.

Code and data availability

This study uses no external data. The library `bubble_rings`, the scripts that produce every figure, animation, and report, CSV (comma-separated values) tables of the data behind each figure panel, the figures in vector and raster form, plain-text computation reports, and three animations in GIF (graphics interchange format) are openly available at https://github.com/sandyherho/bubble_rings under the MIT License.

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