The prognostic equation for biogeochemical tracers has no unique solution.

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Abstract. In this paper, a tracer prognostic differential equation related to the marine chemistry HAMOCC model is studied. Recently the present author found that the Navier - Stokes equation has no unique general solution [*Geurdes*, 2017]. The following question can therefore be justified. Do numerical solutions, found from prognostic equations akin to the Navier Stokes equation, provide unique nutrient distribution information.

1. Introduction.

Recent insights into global biogeochemical cycles, such as the key limiting role of iron as a nutrient [*Martin and Fitzwater*, 1988], justify a computational model search for the distribution of iron in the Oceans. Other elements like phosphorus also play a key role. Both iron as well as phosphorus influence the growth of phytoplankton. In addition, factors such as light are, obviously, important [*Popova, Ryabchenko and Fasham*, 2000].

In the present paper, we concentrate on chemical tracers that can be described in a prognostic equation. The prognostic is considered the kernel of numerical computations. This equation is also to be found in studies on marine aggregates [*Liu, Kindler and Khalili*, 2012], their equation (10).

One thing that catches the eye when looking at prognostic equations for tracers is that their mathematical structure resembles somewhat the Navier Stokes equation (NSE). There are numerical models such as HAMOCC [*Ilynia, Six, Segschneider, Maier-Reimer, Li and Nunez-Riboni*, 2013] based on those kernel equations. We note that in the field of biogeochemical cycle study, numerical studies based on the prognostic equations are supposed to fill in the gaps where data is sparse [*Manizza, Follows, Dutkiewicz, McClelland, Menemenlis, Hill, Townsend-Small and Peterson*, 2009].

Recently, the present author [*Geurdes*, 2017] showed that the NSE in three dimensions does not have a general solution. Such a conclusion leaves room for so called weak solutions. However, the weak solutions and their numerical equivalents are most likely not unique in the absence of an exact solution. Therefore it is methodologically interesting to turn the attention to tracer prognostics equations as a "NSE next of kin" equation. We are, therefore, allowed to ask if models based on that type of equation can have unique numerical solutions. If there is no unique solution associated to such an equation, the validation of the global cycle models perhaps do not allow the conclusions attached to them. Most likely more empirical validation via nutrient sampling is then necessary. A similar case can be made for CO_2 concentration models based on the prognostic.

2. Mathematical model prognostic.

The prognostic equation used by *Dutkiewicz, Follows and Parekh* [2005] can be written as,

$$\frac{\partial A(x)}{\partial t} + \nabla \cdot (\mathbf{u}(x)A(x)) - \nabla \cdot (\mathbf{K}(x)\nabla A(x)) = S_A(x) \quad (1)$$

This equation is the kernel of the HAMOCC model equation. Accordingly, we refer to equation (1) in *Ilynia, Six, Segschneider, Maier-Reimer, Li and Nunez-Riboni* [2013]. In our equation (1), the abbreviation, $x = (\mathbf{x}, t) \in \mathbb{R}^3 \times \mathbb{R}^+$, represents the space and time coordinates. Let us take t in a finite interval $[0, T]$ with $0 < T < \infty$. Furthermore, $\mathbf{u}(x)$, is the "transformed mean Eulerian" circulation and $\mathbf{K}(x)$ the mixing tensor. $S_A(x)$ describes the sources and sinks of the tracer, which concentration in (1) is denoted by $A(x)$.

2.1. Approximate interval

In this sub section we look at an approximation which appears to be valid given the context of oceanic bio-cycle studies. Let us concentrate on a certain fully connected finite space and time set $\Omega \subset \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^+$. Suppose, $\mathbf{x} = (x_1, x_2, x_3)$ and $t \in [0, T]$, then Ω is the cartesian product of intervals $\Omega = [x_{1,0}, x_{1,1}] \times [x_{2,0}, x_{2,1}] \times [x_{3,0}, x_{3,1}] \times [t_0, t_1]$. Here, $x_{m,0} < x_{m,1}$, for $m = 1, 2, 3$, and $t_0 < t_1$. Now for, $x \in \Omega$ we may approximate the tracer equation in (1) with $\mathbf{u}(x) \approx \mathbf{u}^0$ a constant vector in Ω approximating the Eulerian circulation in Ω. Subsequently, $\mathbf{K}(x) \approx \mathbf{K}^0$ the approximate constant mixing tensor and approximate constant source-sink function $S_A(x) \approx S_A^0$. Those approximate constant $\mathbf{u}, \mathbf{K}$ and S_A are reasonable given the physical problem at hand. We also suppose symmetrical mixing $K^0_{m,j} = K^0_{j,m}$. Subsequently, $A^0 = A - tS^0_A$, allows the prognostic equation to be written like

$$\frac{\partial A^0(x)}{\partial t} + \left(\mathbf{u}^0 \cdot \nabla\right) A^0(x) - \nabla \cdot \left(\mathbf{K}^0 \nabla A^0(x)\right) = 0 \tag{2}$$

Here the right hand approximation is replaced with an equal sign for ease of notation. Let us, further, introduce a spectral parameter breakdown of the solution such as in [*Quispel*, 1983]. E.g. we can have a (real) measure ν and real parameter $k \in \mathbb{R}$ such that

$$A^0(x) = \int_{\mathbb{R}} d\nu(k) A^0_k(x) \tag{3}$$

Subsequently let us look at functions carrying a spectral parameter, k. One would certainly agree that if we find two different functions, $A^0_{1,k}(x)$ and $A^0_{2,k}(x)$ that solve (2), then, via the integration over the measure ν in (3), two different functions $A^0_1(x)$ and $A^0_2(x)$ are obtained.

Before showing such two function let us start by introducing the plane wave [*Quispel*, 1983], factor for $x \in \Omega$,

$$\rho_{k,\mathbf{b}}(x) = \exp\left[-||\mathbf{b}||^2 k^2 t - ik\left(\mathbf{b} \cdot \mathbf{L}\mathbf{x}\right)\right] \tag{4}$$

In the definition of the plane wave the $||\mathbf{b}||^2$ represents the Euclidean norm of the constants vector $\mathbf{b}$. Furthermore, $\mathbf{L}$ is a real 3×3 constant-in-x tensor (matrix) such that, $p = 1, 2, 3,\ n = 1, 2, 3$

$$\mathbf{b} \cdot \mathbf{L}\mathbf{x} = \sum_p b_p \sum_n L_{p,n} x_n \tag{5}$$

Hence we find that, for $m = 1, 2, 3$,

$$\frac{\partial}{\partial x_m} (\mathbf{b} \cdot \mathbf{L}\mathbf{x}) = \sum_p b_p L_{p,m} \tag{6}$$

2.2. Integral equation

Starting from $x \in \Omega$, it is possible to write down an integral equation that resembles the ones that are employed for linearizion of nonlinear partial differential equations [*Nijhoff, Quispel, van der Linden and Capel*, 1983]

$$A^0_{1,k}(x) + iA^0_{2,k}(x) = \sum_q r_{q,k}\rho_{iq,\mathbf{a}}(x) + i\sum_q s_{q,k}\rho_{iq,\mathbf{b}}(x) \tag{7}$$

In this equation, $q \in \mathbb{Z}^+ - \{0\}$, together with, $r_{q,k} \in \mathbb{C}$ and, $s_{q,k} \in \mathbb{R}$. Note that **a** and **b** are different vectors. Moreover, the $A^0_{2,k}(x)$ in equation (7) is defined by

$$A^0_{2,k}(x) = \iint_{\mathbb{R}^2} d\lambda(\ell)d\lambda(\ell')\frac{\rho_{\ell-\ell',\mathbf{b}}(x)}{\ell+\ell'}A^0_{1,k+\ell+\ell'}(x) \tag{8}$$

Hence, equation(7) is an integral equation comparable to what is employed by *Quispel* [1983]. In the equation (8) the λ is a real measure and ℓ and ℓ' are real numbers. If, for arbitrary k, we have $A^0_{1,k}(x) \in \mathbb{R}$ then it follows, because of symmetry in exchange of ℓ and ℓ', while noting $\{\rho_{\ell-\ell',\mathbf{b}}(x)\}^* = \rho_{-\ell+\ell',\mathbf{b}}(x)$ and $*$ is complex conjugation, that $A^0_{2,k}(x)$ is a real number too. Moreover, it is noted that if a real (domain and co-domain) operator $\mathcal{O}$ exists such that the right-hand side of the integral equation (5) vanishes, then we find that, $\mathcal{O}A^0_{1,k}(x) + i\mathcal{O}A^0_{2,k}(x) = 0$. If $\mathcal{O}A^0_{j,k}(x) \in \mathbb{R}$, for $j = 1, 2$, then if $\mathcal{O}A^0_{1,k}(x) \in \mathbb{R}$ and $\mathcal{O}A^0_{2,k}(x) \in \mathbb{R}$ it follows that, $\mathcal{O}A^0_{1,k}(x) = \mathcal{O}A^0_{2,k}(x) = 0$.

2.3. Operations and parameters for linearization.

2.3.1. The $\mathbf{u^0} \cdot \nabla$ operator

Let us start the investigation of the approximate prognostic by noting that the operation $\mathbf{u}^0 \cdot \nabla$ can be written in a sum format $\sum_m u_m^0 \frac{\partial}{\partial x_m}$, with $m = 1, 2, 3$. Hence applying this operation to the left and right hand side of the integral equation in (7) we find

$$\sum_m u_m^0 \frac{\partial A_{1,k}^0(x)}{\partial x_m} + i \sum_m u_m^0 \frac{\partial A_{2,k}^0(x)}{\partial x_m} =$$
$$\sum_m u_m^0 \frac{\partial}{\partial x_m} \left[\sum_q r_{q,k} \rho_{iq,\mathbf{a}}(x) + i \sum_q s_{q,k} \rho_{iq,\mathbf{b}}(x) \right] \quad (9)$$

If we for the moment write $\mathbf{y}$ for $\mathbf{a}$ or $\mathbf{b}$ it can be acknowledged that

$$\frac{\partial}{\partial x_m} \rho_{iq,\mathbf{y}}(x) = q \rho_{iq,\mathbf{y}}(x) \frac{\partial}{\partial x_m} (\mathbf{y} \cdot \mathbf{L}\mathbf{x}) \quad (10)$$

With the use of the expression in (6) it follows

$$\frac{\partial}{\partial x_m}\rho_{iq,\mathbf{y}}(x) = q\rho_{iq,\mathbf{y}}(x)\sum_p y_p L_{p,m} \tag{11}$$

with, for completeness, $q \in \mathbb{Z}^+ - \{0\}$ and $p, m = 1, 2, 3$. Remembering that $\mathbf{y}$ can represent $\mathbf{a}$ or $\mathbf{b}$ the following observation can be made. We are in $\mathbb{R}^3$. Hence it is possible to have three mutual orthogonal vectors. Hence, looking at $\mathbf{u}^0$, it is possible to select $\mathbf{a}$ and $\mathbf{b}$ such that in combination with $\mathbf{L}$

$$\sum_m u_m^0 \sum_p a_p L_{p,m} = \sum_m u_m^0 \sum_p b_p L_{p,m} = 0 \tag{12}$$

Returning to equation (8) and bearing in mind the above exercise with the operator $\mathbf{u}^0 \cdot \nabla$, it can be concluded that $\mathbf{u}^0 \cdot \nabla$ is a $\mathcal{O}$-type operator aluded to in section-2.2. Hence, $\mathbf{u}^0 \cdot \nabla A_{1,k}^0(x) = \mathbf{u}^0 \cdot \nabla A_{2,k}^0(x) = 0$.

2.3.2. The $\frac{\partial}{\partial t} - \nabla \cdot (\mathbf{K}^0 \nabla)$ operator

With the result of section-2.3.1 in mind let us turn the attention to the rest of the operator employed in the approximate prognostic equation. The operator $\frac{\partial}{\partial t} - \nabla \cdot (\mathbf{K}^0 \nabla)$ is complicated. Let us start by looking at $\frac{\partial}{\partial t}$. This operation is applied to the right hand side of the integral equation (7). The tactics of how to approach the operator $\frac{\partial}{\partial t}$ on the right hand side of the integral equation resembles that of the section-2.3.1. So we may look at

$$\frac{\partial}{\partial t}\rho_{iq,\mathbf{y}}(x) = ||\mathbf{y}||^2 q^2 \rho_{iq,\mathbf{y}}(x) \tag{13}$$

Applying the operator $\nabla \cdot (\mathbf{K}^0 \nabla)$ to the right hand of the integral equation (7) implies

$$\sum_{m,j} \frac{\partial}{\partial x_m} K_{m,j}^0 \frac{\partial}{\partial x_j}\rho_{iq,\mathbf{y}}(x) = q \sum_{m,j} \frac{\partial}{\partial x_m}\rho_{iq,\mathbf{y}}(x) K_{m,j}^0 \sum_p y_p L_{p,j} =$$

$$q^2 \rho_{iq,\mathbf{y}}(x) \sum_{m,j} K_{m,j}^0 \sum_{p'} y_{p'} L_{p',m} \sum_p y_p L_{p,j} \tag{14}$$

Subsequent reshuffling of the sums gives

$$\sum_{m,j} \frac{\partial}{\partial x_m} K^0_{m,j} \frac{\partial}{\partial x_j} \rho_{iq,\mathbf{y}}(x) = q^2 \rho_{iq,\mathbf{y}}(x) \sum_{p'} \sum_{p} y_{p'} y_p \sum_{m,j} K^0_{m,j} L_{p',m} L_{p,j} =$$

$$q^2 \rho_{iq,\mathbf{y}}(x) \sum_{p'} \sum_{p} y_{p'} y_p \delta_{p,p'} = q^2 \rho_{iq,\mathbf{y}}(x) ||\mathbf{y}||^2 \qquad (15)$$

Here, the $\mathbf{L}$ matrix is such that

$$\sum_{m,j} K^0_{m,j} L_{p',m} L_{p,j} = \delta_{p,p'} \qquad (16)$$

This is the condition for (15), $m, j, p, p' = 1, 2, 3$ and $q \in \mathbb{Z}^+ - \{0\}$. Subsequently, looking at (13) and the result of (15) then

$$\left(\frac{\partial}{\partial t} - \nabla \cdot \left(\mathbf{K}^0 \nabla \right) \right) \rho_{iq,\mathbf{y}}(x) = 0 \qquad (17)$$

Hence, the operator $\frac{\partial}{\partial t} - \nabla \cdot (\mathbf{K}^0 \nabla)$ is also of the type of $\mathcal{O}$ alluded to in section-2.2. Looking at the integral equation (7) this implies that

$$\left(\frac{\partial}{\partial t} - \nabla \cdot \left(\mathbf{K}^0 \nabla \right) \right) A^0_{n,k}(x) = 0 \qquad (18)$$

and here, $n = 1, 2$. Hence, combining this result with $\mathbf{u}^0 \cdot \nabla A^0_{1,k}(x) = \mathbf{u}^0 \cdot \nabla A^0_{2,k}(x) = 0$, it must be possible for the approximate prognostic equation to have two solutions at least. One solution is derived from $A^0_{1,k}(x)$ the other from $A^0_{2,k}(x)$, which in fact is an integral transformation, given in (8), of $A^0_{1,k}(x)$.

2.4. Consistency for $A^0_{2,k}$ under $\frac{\partial}{\partial t} - \nabla \cdot (\mathbf{K}^0 \nabla)$

In order to study the consistency of the claim, we need to look at the transformation in (8). The two x dependent terms are of the greatest importance in the evaluation. Therefore let us introduce the abbreviation $\langle \cdot \rangle$ to denote the weighted integration in the transformation (8). So,

$$A^0_{2,k}(x) = \left\langle \rho_{\ell-\ell',\mathbf{b}}(x) A^0_{1,g}(x) \right\rangle \qquad (19)$$

with the abbreviation, $g = k + \ell + \ell'$. Looking at the operator form we find

$$\frac{\partial}{\partial t} A^0_{2,k}(x) = \langle -(\ell - \ell')^2 ||\mathbf{b}||^2 \rho_{\ell-\ell',\mathbf{b}}(x) A^0_{1,g}(x)\rangle + \left\langle \rho_{\ell-\ell',\mathbf{b}}(x) \frac{\partial}{\partial t} A^0_{1,g}(x) \right\rangle \tag{20}$$

The second part of the operator

$$\nabla \cdot \left(\mathbf{K}^0 \nabla\right) A^0_{2,k}(x) = \left\langle \sum_{m,j} \frac{\partial}{\partial x_m} K^0_{m,j} \frac{\partial}{\partial x_j} \rho_{\ell-\ell',\mathbf{b}}(x) A^0_{1,g}(x) \right\rangle =$$

$$\left\langle \sum_{m,j} \frac{\partial}{\partial x_m} K^0_{m,j} \rho_{\ell-\ell',\mathbf{b}}(x) \left\{ (-i)(\ell - \ell') \sum_p b_p L_{p,j} A^0_{1,g}(x) + \frac{\partial}{\partial x_j} A^0_{1,g}(x) \right\} \right\rangle \tag{21}$$

Subsequently, the $\frac{\partial}{\partial x_m}$ gives, suppressing for the moment, the x dependence in the notation

$$\nabla \cdot \left(\mathbf{K}^0 \nabla\right) A^0_{2,k} =$$

$$\left\langle \rho_{\ell-\ell',\mathbf{b}} \sum_{m,j} K^0_{m,j} (-i)(\ell - \ell') \sum_{p'} b_{p'} L_{p',m} \left\{ (-i)(\ell - \ell') \sum_p b_p L_{p,j} A^0_{1,g} + \frac{\partial A^0_{1,g}}{\partial x_j} \right\} \right\rangle +$$

$$\left\langle \rho_{\ell-\ell',\mathbf{b}} \sum_{m,j} K^0_{m,j} \left\{ (-i)(\ell - \ell') \sum_p b_p L_{p,j} \frac{\partial A^0_{1,g}}{\partial x_m} + \frac{\partial^2 A^0_{1,g}}{\partial x_j \partial x_m} \right\} \right\rangle \tag{22}$$

In the previous equation the form of equation (16) can be recognized. This gives $\delta_{p,p'}$ in the first term. Hence ,

$$W_1 = \left\langle (-1)(\ell - \ell')^2 ||\mathbf{b}||^2 \rho_{\ell-\ell',\mathbf{b}}(x) A^0_{1,g}(x) \right\rangle \tag{23}$$

Because $K^0_{m,j}$ is symmetric $m, j = 1, 2, 3$, the second term in (22) is

$$W_2 = \left\langle \rho_{\ell-\ell',\mathbf{b}}(x) \sum_{m,j} K^0_{m,j} \left\{ (-2i)(\ell - \ell') \sum_p b_p L_{p,j} \frac{\partial A^0_{1,g}(x)}{\partial x_m} \right\} \right\rangle \tag{24}$$

The third term then is equal to

$$W_3 = \left\langle \rho_{\ell-\ell',\mathbf{b}}(x) \nabla \cdot \left(\mathbf{K}^0 \nabla\right) A^0_{1g}(x) \right\rangle \tag{25}$$

The previous three forms allow the following conclusion

$$\left(\frac{\partial}{\partial t} - \nabla \cdot \left(\mathbf{K}^0 \nabla\right) \right) A^0_{2,k}(x) = 2i \left\langle \rho_{\ell-\ell',\mathbf{b}}(x) \sum_{m,j} K^0_{m,j} \left\{ (\ell - \ell') \sum_p b_p L_{p,j} \frac{\partial A^0_{1,g}(x)}{\partial x_m} \right\} \right\rangle \tag{26}$$

The second term of (20) cancels against W_3 of (25), because $\left(\frac{\partial}{\partial t} - \nabla\cdot(\mathbf{K}^0\nabla)\right) A^0_{1,k}(x) = 0$, for all, $k \in$.

This implies that for consistency we need to have, $W_2 = 0$ from (24)

$$\left\langle (\ell-\ell')\rho_{\ell-\ell',\mathbf{b}}(x) \sum_{m,j,p} b_p K^0_{m,j} L_{p,j} \frac{\partial A^0_{1,g}(x)}{\partial x_m} \right\rangle = 0 \tag{27}$$

2.4.1. The operator $\sum_{m,j,p} b_p K^0_{m,j} L_{p,j} \frac{\partial}{\partial x_m}$.

The result in (27) and therefore the consistency, $\left(\frac{\partial}{\partial t} - \nabla\cdot(\mathbf{K}^0\nabla)\right) A^0_{2,k}(x) = 0$ can be studied by applying the operator, $\sum_{m,j,p} b_p K^0_{m,j} L_{p,j} \frac{\partial}{\partial x_m}$ to the integral equation in (7). Hence,

$$\sum_{m,j,p} b_p K^0_{m,j} L_{p,j} \frac{\partial}{\partial x_m} A^0_{1,k}(x) + i \sum_{m,j,p} b_p K^0_{m,j} L_{p,j} \frac{\partial}{\partial x_m} A^0_{2,k}(x) =$$
$$\sum_{m,j,p} b_p K^0_{m,j} L_{p,j} \frac{\partial}{\partial x_m} \left(\sum_q r_{q,k}\rho_{iq,\mathbf{a}}(x) + i \sum_q s_{q,k}\rho_{iq,\mathbf{b}}(x) \right) \tag{28}$$

Similar to e.g. section-2.3.1, the use of $\rho_{iq,\mathbf{y}}(x)$ is justified looking at (28). So, it follows that, ignoring the q sums and coefficients for the moment,

$$\sum_{m,j,p} b_p K^0_{m,j} L_{p,j} \frac{\partial}{\partial x_m} \rho_{iq,\mathbf{y}}(x) = q\rho_{iq,\mathbf{y}}(x) \sum_{m,j,p} b_p K^0_{m,j} L_{p,j} \sum_{p'} y_{p'} L_{p',m} \tag{29}$$

In this equation we may recognize, $\sum_{m,j} K^0_{m,j} L_{p',m} L_{p,j} = \delta_{p,p'}$ from (16). This implies that

$$\sum_{m,j,p} b_p K^0_{m,j} L_{p,j} \frac{\partial}{\partial x_m} \rho_{iq,\mathbf{y}}(x) = q\rho_{iq,\mathbf{y}}(x) \sum_p b_p y_p \tag{30}$$

We note again that $\mathbf{y} = \mathbf{a}$ resembles the right hand side of (28). Hence if $\mathbf{a}$ and $\mathbf{b}$ are orthogonal, the first term in (28) on the right hand side vanishes. Then, because we work with real functions and coefficients, it follows that

$$\sum_{m,j,p} b_p K^0_{m,j} L_{p,j} \frac{\partial}{\partial x_m} A^0_{1,k}(x) = 0 \tag{31}$$

If this is combined with the result from (27) the consistency, $\left(\frac{\partial}{\partial t} - \nabla \cdot (\mathbf{K}^0 \nabla)\right) A^0_{2,k}(x) = 0$ applies. Comparing the imaginary parts on left and right hand of (28) it is also found that, using $\mathbf{y} = \mathbf{b}$ in (30), the consistency requires that

$$\sum_{m,j,p} b_p K^0_{m,j} L_{p,j} \frac{\partial}{\partial x_m} A^0_{2,k}(x) = \sum_q q ||\mathbf{b}||^2 s_{q,k} \rho_{iq,\mathbf{b}}(x) \tag{32}$$

Let us return to the use of the form in (8) and observe the truth of (31). It then clearly follows that the requirement in (32), in terms of $A^0_{1,k}$, is

$$\iint_{\mathbb{R}^2} d\lambda(\ell) d\lambda(\ell')(-i)(\ell - \ell') \frac{\rho_{\ell-\ell',\mathbf{b}}(x)}{\ell + \ell'} A^0_{1,k+\ell+\ell'}(x) = \sum_q q s_{q,k} \rho_{iq,\mathbf{b}}(x) \tag{33}$$

Here equation (16) is employed to the result of the $\frac{\partial}{\partial x_m}$ differentiation on the left hand side, such that the $||\mathbf{b}||^2$ drops off. It can be easily verified, taking the complex conjugate and interchanging the ℓ and the ℓ', that the left hand of (33) is real. So, this requirement can be met in principle for $x \in \Omega$. We also note that from $\left(\frac{\partial}{\partial t} - \nabla \cdot (\mathbf{K}^0 \nabla)\right) \rho_{iq,\mathbf{y}}(x) = 0$ in (17) and from equation (31) together with $\left(\frac{\partial}{\partial t} - \nabla \cdot (\mathbf{K}^0 \nabla)\right) A^0_{1,k}(x) = 0$, the equation in (33) is consistent.

3. Conclusion

It was demonstrated that the solution of an approximation of the tracer prognostic equation of *Dutkiewicz, Follows and Parekh* [2005] has no unique solution. This agrees with another finding that the Navier Stokes equation (NSE) in $d = 3$ spatial dimensions has no general exact solution [*Geurdes*, 2017]. The form of the prognostic of e.g. *Dutkiewicz, Follows and Parekh* [2005] resembles the NSE.

The question can therefore be raised what the numerical solution of [*Dutkiewicz, Follows and Parekh*, 2005] is telling us.

Of course it has to be acknowledged that the infinite sequence that can be derived from the prognostic equation is approximative. The marine biogeochemistry, however, seems to allow the approximation. Moreover, it also has to be acknowledged that the present infinite sequence of solutions is not tested with numerical analysis. Our argument uncovers to a principle. Numerical studies will not take the principle objection of a multitude of different solutions, away.

Subsequently, we have to acknowledge that the infinite series of solutions has a particular form. The operators $\mathbf{u}^0 \cdot \nabla$ and $\frac{\partial}{\partial t} - \nabla \cdot (\mathbf{K}^0 \nabla)$ working on the start-point solution, $A^0 = A_1^0$, must identically vanish. We note that if $A^0 \approx A - tS_A^0$, this is an approximate solution to the prognostic for the domain $x \in \Omega$. Of course, if it is argued that only the A's with $(\mathbf{u}^0 \cdot \nabla) A = \left(\frac{\partial}{\partial t} - \nabla \cdot (\mathbf{K}^0 \nabla)\right) A = 0$, show the infinite transformation characteristic, then one already agrees that the prognostic has not got a unique solution. Then the question can rightfully be raised why the numerical outcomes of the model computations are not checked for uniqueness.

Another point to raise is that the obtained infinite series of solutions all are bounded by a certain finite time interval. Hence, a possible test of correctness of HAMOCC numerical computations could be to check the numerical solution at "infinite time". It can be true that stable behavior at infinite time also implies uniqueness. However, the fact that the NSE does not have a general exact solution [*Geurdes*, 2017] raises doubt about such an approach.

The final question then is how the result of this paper affects the conclusions of HAMOCC-like model computations. It appears to make sense to claim that when there is no unique solution to the kernel equation of the model, then the verification of validity of the model computations cannot come from numeric model computations such as claimed by [*Manizza, Follows, Dutkiewicz, McClelland, Menemenlis, Hill, Townsend-Small and Peterson*, 2009].

The subsequent question might then be: How bad is the result of this mathematical exercise for model computations? Here we will answer this question conceptually and pose the real question. We have demonstrated that

$$A^0_{1,k}(x) + iA^0_{2,k}(x) = \sum_q r_{q,k}\rho_{iq,\mathbf{a}}(x) + i\sum_q s_{q,k}\rho_{iq,\mathbf{b}}(x)$$

contains two different *forms* of solution to the *same* prognostic differential equation. This equation is key to the HAMOCC model. To be more specific

$$A^0_{2,k}(x) = \iint_{\mathbb{R}^2} d\lambda(\ell)d\lambda(\ell')\frac{\rho_{\ell-\ell',\mathbf{b}}(x)}{\ell+\ell'}A^0_{1,k+\ell+\ell'}(x)$$

is a spectral integral transformation of $A^0_{1,k}(x)$. It implies that if one has one (numerical) solution, another one can be obtained. From that one, yet another one can be obtained etc, etc. The single question we ask is. How can the reseachers that use model computations of nutrients distribution in the ocean, be certain that a consistent "filling the gaps of measurement" [*Manizza, Follows, Dutkiewicz, McClelland, Menemenlis, Hill, Townsend-Small and Peterson*, 2009] description is obtained? here

References

Dutkiewicz, S., M.J. Follows and P. Parekh, Interactions of the iron and phosphorus cycles: A three-dimensional model study, *Global Biogeochemistry Cycles, 19*, doi:10.1029/2004GB002342, 2005.

Geurdes, H, A type D breakdown of the Navier Stokes equation in d=3 spatial dimensions, *Cogent Mathematics 4*, 1284293, doi:10.1080/23311835.2017.1284293, 2017.

Ilyina,T., K.D. Six, J. Segschneider, E. Maier-Reimer, H. Li and I. Nunez-Riboni, Global ocean biogeochemistry model HAMOCC, *J. Adv. Mod. Earth Syst, 5*, 286-315, 2013.

Liu, B, K. Kindlre, A. Khalili, Dynamic solute release from marine aggregates, *Limnol. Oceanogr. Fluids & Environment 2*, 109-120, 2012, doi 10.1215/21573689-2016772.

Nijhoff, F.W., G.R.W. Quispel, J. van der Linden and H.W. Capel, On some linear integral equations generating solutions of non-linear partial differential equations, *Physica 119A*, 101-142, 1983.

Manizza, M., M.J. Follows, S. Dutkiewicz, J.W. McClelland, D. Menemenlis, C.N. Hill, A. Townsend-Small, B.J. Peterson, Modeling transport and fate of riverine dissolved organic carbon in the Artic Ocean, *Global Biogeochemistry Cycles, 23*, doi:10.1029/2008GB003396, 2009.

Martine, J.H. and S. Fitzwater, Iron deficiency limits phytoplankton growth in the north-east Pacific subarctic, *Nature, 331*, 341-343, 1988.

Popova, E.E., V.A. Ryabchenko and M.J.R. Fasham, Biological pump and vertical mixing in the Southern Ocean: Their impact on atmospheric CO_2, *Global Biogeochemistry Cycles, 14*, 477-498, 2000.

Quispel, G.R.W. Linear integral equations and soliton systems, PhD thesis Leiden, 3-5, 1983.