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A discontinuous Galerkin method for sequences of earthquakes and aseismic slip on multiple faults using unstructured curvilinear grids

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SUMMARY
Physics-based simulations provide a path to overcome the lack of observational data which is hampering a holistic understanding of earthquake faulting and crustal deformation across the vastly varying space-time scales governing the seismic cycle. However, simulations of sequences of earthquakes and aseismic slip (SEAS) including more than one fault, complex geometries, and elastic heterogeneities are challenging. We present a symmetric interior penalty discontinuous Galerkin (SIPG) method to perform SEAS simulations accounting for the complex geometries and heterogeneity of the subsurface. Due to the discontinuous nature of the approximation, the spatial discretisation natively provides a mean to impose boundary and interface conditions associated with geometrically complex domains and embedded faults. The method accommodates two- and three-dimensional domains, is of arbitrary order, handles sub-element variations in material properties and supports isoparametric elements, i.e. high-order representations of the exterior and interior boundaries and interfaces including intersecting faults.

We provide an open-source reference implementation, \textit{Tandem}, that utilises highly efficient kernels for evaluating the SIPG linear and bilinear forms, is inherently parallel and well suited to perform high resolution simulations on large scale distributed memory architectures. Further flexibility and efficiency is provided by optionally defining the displacement evaluation via a discrete Green’s function, which is evaluated once in an embarrassingly parallel pre-computation step using algorithmically optimal and scalable sparse parallel solvers and preconditioners.

We illustrate the characteristics of the SIPG formulation via an extensive suite of verification problems (analytic, manufactured, and code comparison) for elasto-static and seismic cycle problems. Our verification suite demonstrates that high-order convergence of the discrete solution can be achieved in space and time and highlights the benefits of using a high-order representation of the displacement, material properties, and geometry.

Lastly, we apply \textit{Tandem} to realistic demonstration models consisting of a 2D SEAS multi-fault scenario on a shallowly dipping normal fault with four curved splay faults, and a 3D multi-fault scenario of elasto-static instantaneous displacement due to the 2019 Ridgecrest, CA, earthquake sequence. We exploit the curvilinear geometry representation in both application examples and elucidate the importance of accurate stress (or displacement gradient) representation on-fault. Our demonstrator models exploit advantages of both the boundary integral and volumetric methods and open new avenues to pursue extreme scale 3D SEAS simulations in the future.

\textbf{Key words:} Seismic cycle; Numerical approximations and analysis; Numerical modelling; Transient deformation; Earthquake dynamics; Earthquake interaction, forecasting, and prediction.

1 INTRODUCTION

Numerical modelling is an important tool to enhance our understanding of how faults slip, since the spatial and temporal scales involved in (sequences of) earthquakes render the direct observation of earthquake source processes difficult or infeasible (e.g., Lapusta et al. 2019). While historical and geological records have been used to construct catalogs of major earthquakes on known natural fault systems that extend over several thousands of years (Ben-Menahem 1991; Rockwell et al. 2015), the lack of space-time
A variety of numerical discretisation methods have been applied to SEAS type problems, including the boundary integral method (BIM) and boundary element method (BEM) (Barbot 2019; Segall & Bradley 2012; Bradley 2014; Li & Liu 2016, 2017; Liu & Rice 2005; Lapusta et al. 2000; Lapusta & Liu 2009; Luo et al. 2017), the finite difference method (FDM) (Erickson & Dunham 2014; Allison & Dunham 2018; Pranger 2021; Almquist & Dunham 2021), and the finite element method (FEM) (Aagaard et al. 2013; Kaneko et al. 2011; Kozdon 2019; Luo et al. 2020). In the BEM one only needs to discretise the domain boundaries and the fault (or only the fault for specific geometries such as the half-space), such that the dimensionality of the problem is effectively reduced by one. However, this method relies on the existence of the analytic fundamental solution (Chen & Zhou 2010), which is typically only available for homogeneous material parameters and single, planar faults. Therefore, BEM is not applicable for scenarios in general heterogeneous media (e.g., Erickson & Dunham 2014; Thakur et al. 2020). While heterogeneous material parameters can be easily included in the FDM and FEM, these methods inherently require more degrees of freedom than BEM since they discretise the volume.

In this paper, we evaluate the symmetric interior penalty Galerkin (SIPG) method, which is a discontinuous Galerkin finite element method (DG-FEM) with a particular choice of numerical flux (Arnold et al. 2002), on unstructured curvilinear grids for SEAS models. The discontinuous Galerkin (DG) method for SEAS problems is promising for the following reasons:

- The method is geometrically flexible, meaning there are few restrictions on the complexity and shape of the fault geometry. Such flexibility is realised as the method permits the domain to be decomposed into a set of triangles or quadrilaterals (2D), or a set of tetrahedra or hexahedra (3D), and allows for unstructured meshes to be used. Static spatial adaptivity can be applied to bridge spatial scales (e.g., Mazziere et al. 2013; Ulrich et al. 2022).
- Values defined on the faces of each cell are double-valued such that the DG function spaces naturally include the displacement discontinuity at the (mesh-aligned) fault. There is no need to split nodes at the fault or to introduce Lagrange multipliers as is the case in the continuous Galerkin FEM (Kaneko et al. 2011; Aagaard et al. 2013). These characteristics of the DG scheme have been successfully exploited in modelling faults in dynamic rupture applications (e.g., Tago et al. 2012; Pelties et al. 2014).
- The polynomial degree of the basis function can be chosen flexibly and the implementation is agnostic with respect to the polynomial degree. Thus, we obtain an arbitrary high-order method on unstructured grids. When high-order convergence is achieved for sufficiently smooth solutions, less degrees of freedom are required in comparison to low-order methods. Establishing whether high-order convergence can be achieved for SEAS problems using SIPG is central to this paper.
- DG methods are provably stable for a wide range of physical processes governed by elliptic, parabolic and hyperbolic partial differential equations (e.g., Hill 1973; Cockburn & Shu 1989; Grote et al. 2006; Hesthaven & Warburton 2008; Rivière 2008; Dumbser & Käser 2006; Reinarz et al. 2020; Duru et al. 2022).
- The physical characteristics of quasi-dynamic SEAS require an incredibly large number of time-steps to be performed (in the order of million time-steps). To enable both high resolution 3D simulations and rapid-to-solution, parallelism must be exploited. The DG discretisation is well suited to complex large scale simulations and scalable up to the largest-available supercomputers (Wilcox et al. 2010; Heinke et al. 2014; Kronbichler et al. 2017; Uphoff...
The basis of the slip-rate vector is given by the normal to the fault is defined to be zero (we ignore fault opening).

The evaluation of the SIPG method is realised through our open-source reference implementation Tandem, which exploits all of the aforementioned benefits. The novelty and functionality of the computational SEAS framework can be summarized as follows:

(i) Support for 2D and 3D spatial discretisations using unstructured meshes comprised of triangles (2D) and tetrahedra (3D);

(ii) Support for high-order (curvilinear) representation of exterior boundaries and interfaces (faults);

(iii) Sub-element (high-order accurate) representation of material properties and sub-element (high-order accurate) representation of slip, slip-rate in SEAS problems;

(iv) Efficient kernels for the assembly of discontinuous Galerkin operators and residual calculators;

(v) A fully parallel implementation, including mesh loading, solution stage, output and visualisation;

(vi) Access to state-of-the art solvers, preconditioners and time integrators by using the Portable, Extensible Toolkit for Scientific Computation (PETSc);

(vii) Support to optionally compute the discrete Green’s function (i.e., the affine function which maps slip to traction) via efficient scalable (algorithmic and parallel) solvers.

The remainder of this paper is organized as follows. We begin with reviewing the quasi-dynamic SEAS model in Section 2. The SIPG method supporting the slip boundary condition is introduced in Section 3, and analyzed in Section 4. Specific aspects of the DG implementation on curvilinear meshes are presented in Section 5 and coupling to rate-and-state friction is discussed in Section 6. In Section 7 we show how the discrete analogue to a Green’s function can be obtained. Results of our numerical experiments are reported in Sections 8 and 9, with the SIPG verification presented in Section 8 and the 2D SEAS splay fault and 3D elasto-static Ridge-crest demonstrators presented in Section 9. We discuss our results in Section 10 and conclude in Section 11.

2 QUASIDYNAMIC SEAS MODEL

We here present the equations of the quasi-dynamic SEAS model. The equations are formulated for $D = 2$ and $D = 3$, i.e. for two and three space dimensions. We note that summation over lower indices (but not upper indices) appearing twice is implied throughout the paper.

2.1 Rate and state friction

In the framework of rate and state friction the effective coefficient of friction $F$ depends on the slip rate vector $V$ and the scalar state variable $\psi$, such that fault strength $\tau_S$ is given by

$$\tau_S = \sigma_n F(|V|, \psi),$$

where $\sigma_n$ is the normal stress and $|\cdot|$ denotes the Euclidean norm of a vector. The evolution of the state variable is governed by the following ordinary differential equation (ODE):

$$\frac{d\psi}{dt} = G(|V|, \psi).$$

The slip-rate vector has $D - 1$ components, since the slip-rate normal to the fault is defined to be zero (we ignore fault opening). The basis of the slip-rate vector is given by the $D - 1$ unit tangents of the fault. By stacking the tangents in the $D \times (D - 1)$ matrix $B$ we relate the slip-rate vector to the velocity vector $v$ as $[v_i] = B_{ij} v_j$. The jump operator $[\cdot]$ is defined as $[v_i] := v_i^- - v_i^+$ and we define that the normal vector $n$ points from the “−” side to the “+” side of a fault discontinuity.

With the conventions adopted, the fault strength is related to shear traction $\tau$ and slip-rate by (Day et al. 2005)

$$|\tau| \leq \tau_S,$$

$$\tau_S V + \tau |V| = 0.$$

The first equation limits the traction by the fault strength and the second equation ensures that traction is antiparallel to slip-rate. Moreover, it follows from the second equation that either slip-rate is zero or the magnitude of shear traction is equal to fault strength. That is, if $|V| \neq 0$ then

$$-\tau = \sigma_n F(|V|, \psi) \frac{V}{|V|}.$$

In this paper, we use the regularized version of the rate and state friction law (Rice & Ben-Zion 1996; Lapusta et al. 2000)

$$F(V, \psi) = a \arcsinh \left( \frac{V}{2V_0} \exp \left( \frac{\psi}{a} \right) \right)$$

and the Dieterich-Ruina ageing law

$$G(V, \psi) = \frac{bV_0}{L} \left( \exp \left( \frac{f_0 - \psi}{b} \right) - \frac{V}{V_0} \right),$$

where $a, b, V_0, f_0, L$ are (possibly space-dependent) constants. We note that the form of the ageing law in Eq. (7) is equivalent to the more familiar form $d\theta/dt = 1 - V \theta / L$ with the substitution $\psi = f_0 + b \ln(V_0 \theta / L)$. However, we prefer (7) which results in the state variable being order one, i.e. $\psi \sim O(1)$.

2.2 Quasi-dynamic approximation

The theory of linear elasticity is used to compute the mechanic response to slip on a fault. Inertia is neglected in the elastodynamic equations in the quasi-dynamic approach such that one needs to solve the elliptic problem

$$-\frac{\partial \sigma_{ij}}{\partial x_j} = f_i,$$

where $\sigma_{ij}$ is the stress, $u$ the displacement and $f_i$ is the body-force vector. Given the isotropic stiffness tensor $c_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})$, $\lambda$ and $\mu$ are the Lamé parameters and $\delta$ is the Kronecker symbol, the linear stress-strain relation is

$$\sigma_{ij} = c_{ijkl} \varepsilon_{kl}(u) = \frac{1}{2} c_{ijkl} \left( \frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k} \right).$$

Slip is imposed as an interface condition. The slip vector $S$ on the fault is related to the displacement vector $u$ by $[u_i] = B_{ij} S_j$. Moreover, the free surface is modelled with a Neumann boundary condition and a Dirichlet boundary condition imposes tectonic loading.

Given the displacement $u$ that solves the boundary value problem, one computes the shear traction $T$ in the fault-aligned basis with $T_i = T^0_i + B_{ij} \sigma_{kl} (u) n_k$ and the compressive normal stress with $\sigma_n = \max (0, \sigma_n^0 - n_i \sigma_{ij}(u) n_j)$. Here, the loading stress field is included with $T^0_i$ and $\sigma_n^0$. One might consider requiring $\tau = T$, however, the resulting system of equations diverges (Rice 1993). Instead, the radiation damping term $-\eta V$ is added to mimic the outflow of energy due to seismic waves, with $\eta$ being half the
shear wave impedance. That is, one requires $T = \tau - \eta V$ such that
\[ -T = \sigma_n F(|V|, \psi) \frac{V}{|V|} + \eta V. \quad (10) \]

We note that the radiation damping term appears naturally when considering the exact Riemann problem at the fault, which can be used to implement provable stable fault boundary conditions for the fully dynamic equations (Duru & Dunham 2016; Uphoff 2020).

2.3 Summary of equations

We denote the physical domain by $\Omega$ and its boundary by $\partial \Omega$. We assume that $\Omega$ is defined in $\mathbb{R}^D$, where $D = 2$ or $D = 3$. The boundary of the domain $\partial \Omega$ is partitioned in three pairwise disjoint sets: a Dirichlet boundary $\Gamma^D$, Neumann boundary $\Gamma^N$, and internal fault boundary $\Gamma^F$.

On $\Gamma^F$ we solve the following rate and state friction relations
\[ -T_i = \sigma_n F(|V|, \psi) V_i \frac{V}{|V|} + \eta V_i, \]
\[ \frac{d\sigma_{ij}}{dt} = G(|V|, \psi), \]
\[ \frac{dS_i}{dt} = V_i, \quad \text{where} \]
\[ T_i = T_i^0 + B_{ij} \sigma_{jk} \langle u \rangle n_k, \]
\[ \sigma_n = \max \left(0, a_n^\alpha - n_i \sigma_{ij} \langle u \rangle n_j \right), \]
and the displacement is the solution of the boundary value problem
\[ \frac{\partial \sigma_{ij} \langle u \rangle}{\partial x_j} = f_i, \quad \text{in} \ \Omega, \]
\[ \sigma_{ij} \langle u \rangle = c_{ijkl} \varepsilon_{kl} \langle u \rangle \quad \text{in} \ \Omega, \]
\[ u_i = g_i^D \quad \text{on} \ \Gamma^D, \]
\[ \sigma_{ij} \langle u \rangle n_j = 0 \quad \text{on} \ \Gamma^N, \]
\[ [u_i] = g^F_i = B_{ij} S_j \quad \text{on} \ \Gamma^F. \quad (12) \]

3 SYMMETRIC INTERIOR PENALTY GALERKIN

In this section, we develop a discontinuous Galerkin (DG) method to solve the elasticity sub-problem (12) numerically. Discontinuous Galerkin schemes for elliptic PDEs are typically stated either in the flux formulation or in the primal formulation (Arnold et al. 2002). Central to the flux formulation is the numerical flux which approximates the stress tensor and the displacement on the boundary of an element. The numerical flux is typically single-valued but this is not a necessity. We introduce a double-valued numerical flux that is equivalent to the numerical flux of the symmetric interior penalty Galerkin (SIPG) method, except on the fault $\Gamma^F$. We then derive the primal formulation from the flux formulation. The primal formulation defines a variational problem common in finite elements, the first-order system
\[ \sigma_{ij} - c_{ijkl} \varepsilon_{kl} \frac{\partial u_i}{\partial x_j} = 0, \quad (13) \]
\[ -\frac{\partial \sigma_{ij}}{\partial x_j} = f_i. \]

The domain $\Omega$ is approximated using a conforming curvilinear mesh $\mathcal{T}_h = \{ T \}$. The mesh is defined via mapping $X_E : \tilde{E} \rightarrow E$ from the reference element $\tilde{E}$ to the physical element $E$. We require that $X_E$ is invertible. The reference element $\tilde{E}$ is given by
\[ \tilde{E} = \{ \xi : 0 \leq \xi, \sum_{i=1}^2 \xi_i \leq 1 \}, \quad (14) \]
thus we have a reference triangle for $D = 2$ and a reference tetrahedron for $D = 3$.

The mapping relates the quantities $(\sigma, u)$ on the physical element to the quantities $(\tilde{\sigma}, \tilde{u})$ on the reference element via the following transformation:
\[ \sigma = (\tilde{\sigma}(\nabla X_E)^{-1}) \circ X_E^{-1}, \quad u = \tilde{u} \circ X_E^{-1}. \quad (15) \]

We introduce the finite element spaces
\[ \Sigma_h = \{ \sigma \in [L^2(\Omega)]^{D \times D} : \tilde{\sigma} | E \in [P_N(\tilde{E})]^{D \times D} \forall E \in \mathcal{T}_h \}, \]
\[ V_h = \{ u \in [L^2(\Omega)]^D : \tilde{u} | E \in [P_N(\tilde{E})]^D \forall E \in \mathcal{T}_h \}, \quad (16) \]
where $P_N(\tilde{E})$ is the space of polynomials of degree at most $N$ on $\tilde{E}$. Note that we have the inclusion $\nabla \cdot V_h \subset \Sigma_h$, where $\nabla$ takes the gradient element-wise.

We multiply (13) with test functions $\tau \in \Sigma_h$ and $v \in V_h$, integrate over element $E$, integrate by parts, and introduce numerical fluxes $\tilde{\sigma}_{ij}$ and $\tilde{u}_i$:
\[ \int_E \tau_i \sigma_{ij} \frac{\partial u_j}{\partial x_j} \ dx + \int_E \frac{\partial (\tau_i c_{ijkl} \varepsilon_{kl} \langle u \rangle)}{\partial x_j} \ dx - \int_{\partial \Omega_E} \tau_i c_{ijkl} \varepsilon_{kl} \langle u \rangle n_j \ dx = 0; \]
\[ \int_E \frac{\partial v_i}{\partial x_j} \tilde{\sigma}_{ij} \ dx - \int_{\partial \Omega_E} v_i \tilde{\sigma}_{ij} n_j \ dx + \int_E v_i f_i \ dx. \quad (17) \]

We need a few definitions in order to define the numerical flux. First, let $\Gamma^i := \bigcup_{E \in \mathcal{T}_h} \partial E$ be the set of facets. The set of interior faces is given by $\Gamma^i := \Gamma^i \setminus \partial \Omega$ and the set of non-fault interior faces is given by $\Gamma^D = \Gamma^i \setminus \Gamma^F$. Second, we define averages and jumps. Let $E^1_i$ and $E^2_i$ be two elements that share an edge $e \in \Gamma^i$, that is, $e = E^1_i \cap E^2_i$. Then the average and jump of a quantity $v$ along $e$ is given by
\[ \|v\| = \frac{1}{2} (|v|_{E^1_i} + |v|_{E^2_i}), \quad [v] = |v|_{E^1_i} - |v|_{E^2_i}. \quad (19) \]

When the normal $n^e$ is included in the average or jump we define that it is oriented from $E^1_i$ to $E^2_i$. For convenience, we also define average and jump for $e \in \partial \Omega$:
\[ \|v\| = |v|_{E^1_i}, \quad [v] = |v|_{E^1_i}. \quad (20) \]

The numerical flux we adopt is equivalent to the usual SIPG numerical flux (Arnold et al. 2002; Rivière 2008), except on the fault. For $e \in \Gamma^F$ we define the fluxes according to
\[ \tilde{\sigma}_{ij} = \left\{ \left( c_{ijkl} \frac{\partial u_j}{\partial x_j} \right) - \delta_k \left( [u_i] - g_i^F \right) n_j, \right. \]
\[ \left. \left\| \tilde{u}_i \right\| = \|u_i\|, \quad \left[ \tilde{u}_i \right] = g_i^F. \right. \quad (21) \]
Table 1. Summary of the numerical flux. The numerical flux is double-valued on \(\Gamma^0 \cup \Gamma^F\) and single-valued on \(\Gamma^D \cup \Gamma^N\).

<table>
<thead>
<tr>
<th>Interior</th>
<th>(\tilde{\sigma}_{ij})</th>
<th>(\tilde{u}_i)</th>
<th>(\tilde{u}_j)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(e \in \Gamma^0)</td>
<td>(c_{ijrs} \frac{\partial u_r}{\partial x_s} - \delta_i(u_i\eta_j))</td>
<td>(u_i)</td>
<td>0</td>
</tr>
<tr>
<td>(e \in \Gamma^F)</td>
<td>(c_{ijrs} \frac{\partial u_r}{\partial x_s} - \delta_i((u_i - g^F)\eta_j))</td>
<td>(u_i)</td>
<td>(g^F_i)</td>
</tr>
<tr>
<td>Boundary</td>
<td>(\sigma_{ij})</td>
<td>(\tilde{u}_i)</td>
<td></td>
</tr>
<tr>
<td>(e \in \Gamma^D)</td>
<td>(c_{ijrs} \frac{\partial u_r}{\partial x_s} - \delta_i(u_i - g^D)\eta_j)</td>
<td>(g^D_i)</td>
<td></td>
</tr>
<tr>
<td>(e \in \Gamma^N)</td>
<td>0</td>
<td>(u_i)</td>
<td></td>
</tr>
</tbody>
</table>

By allowing the displacement flux to be double-valued we can prescribe the fault slip \(g^F\). Moreover, in the stress flux, we add a penalty term that becomes active when the displacement discontinuity differs from the prescribed slip. The penalty parameter \(\delta_e\) needs to be large enough to ensure stability. In Section 4 we give a lower bound on \(\delta_e\). All numerical fluxes used are summarised in Table 1.

### 3.2 Primal formulation

Arnold et al. (2002) explain in detail how to derive the primal formulation from the flux formulation. Inserting our numerical flux and following their procedure (using \(\nabla v V_h \subset \Sigma_h\)) we obtain

\[
\sum_{E \in T_h} \int_E c_{ijrs} \frac{\partial u_r}{\partial x_s} \frac{\partial v_i}{\partial x_j} \, dx + \sum_{e \in \Gamma^{i,j,t} E} \int_e \delta_i[v_i][u_i] \, ds \\
- \sum_{e \in \Gamma^{i,j,t} E} \int_e \left( [u_i] \left( c_{ijrs} \frac{\partial v_i}{\partial x_j} \right) + [v_i] \left( c_{ijrs} \frac{\partial u_i}{\partial x_j} \right) \right) \eta_j \, ds \\
= \int_\Omega f_i v_i \, dx + \sum_{e \in \Gamma^D E} \int_e \delta_i v_i g^D_i \, ds + \sum_{e \in \Gamma^F E} \int_e \delta_i v_i g^F_i \, ds \\
- \sum_{e \in \Gamma^{i,j,t} E} \int_e g^F_i \left( c_{ijrs} \frac{\partial v_j}{\partial x_s} \eta_i \right) \, ds - \sum_{e \in \Gamma^{D,j} E} \int_e g^D_i c_{ijrs} \frac{\partial v_j}{\partial x_s} \eta_i \, ds.
\]

(22)

From (22) we collect all terms which are function of both the test and trial function spaces (LHS of (22)) into a single bilinear form \(a(u, v)\). The linear form, denoted by \(L(v)\), collects all terms which are only a function of the test function space (RHS of (22)). The variational problem can be then stated as: Find \(u_h \in V_h\) which satisfies

\[
a(u_h, v) = L(v), \quad \forall v \in V_h.
\]

Finally, we note that for \(\Gamma^F = 0\) and \(g^D = 0\) the bilinear form \(a(u_h, v)\) and the linear form \(L(v)\) is equivalent to the SIPG scheme for elasticity given by Rivière (2008, Chapter 5).

### 4 COERCIVITY AND BOUNDEDNESS

The penalty \(\delta_e\) needs to be chosen large enough such that the bilinear form \(a(u, v)\) is coercive, because it follows from coercivity that (23) has a unique solution (Riviére 2008). However, the condition number of the matrix \(A\) resulting from (23) increases with the penalty parameter (Antonietti & Houston 2011). That is, \(\delta_e\) is ideally set to the lowest value that still ensures coercivity.

Explicit expressions for the penalty parameter are given by Shabhazi (2005) and Epshteyn & Rivière (2007) for a Poisson model problem. In both papers, numerical experiments demonstrate that the estimate is sharp, i.e. the lowest penalty parameter is close to its theoretical value. The estimates of the penalty parameter rely on the inverse trace inequality (Warburton & Hesthaven 2003)

\[
\forall u \in P_N(K) : ||u||_K \leq \sqrt{\frac{(N+1)(N+D)}{D}} \frac{|e|}{|K||u||_K},
\]

(24)

where \(K\) is a planar \(D\)-simplex, \(|K|\) is the volume of \(K\), \(|e|\) is a facet of \(K\), and \(|e|\) is the length of \((D = 2)\) or area \((D = 3)\) of \(e\).

In the following, we find bounds for the penalty parameter for linear elasticity. Similarly to Shabhazi (2005) and Epshteyn & Rivière (2007) we restrict the analysis to affine mapping functions, because there are some unresolved difficulties for curvilinear coordinates. At the end of this section, we remark on these difficulties.

For the analysis of coercivity, we introduce the semi-norm

\[
|u|_{e,h} = \left( \sum_{E \in T_h} \int_E \frac{\partial u_i}{\partial x_j} c_{ijrs} \frac{\partial v_j}{\partial x_s} \, dx \right)^{1/2}
\]

and the energy norm

\[
||u||_e = \left( |u|_{e,h}^2 + \sum_{e \in \Gamma^{i,j,t} E} ||u_i||_e \, ||u_i||_e \right)^{1/2}.
\]

Moreover, we assume that the stiffness tensor is bounded, i.e. let \(\kappa\) be a symmetric \(D \times D\) matrix then we have on element \(E\)

\[
\forall \kappa \neq 0 : \inf_{x \in E} 2\mu(x),
\]

(25)

The constants \(c_0^E\) and \(c_1^E\) are given by

\[
c_0^E = \inf_{x \in E} \frac{2\mu(x)}{\kappa},
\]

(26)

\[
c_1^E = \sup_{x \in E} (D\lambda(\kappa) + 2\mu(x)).
\]

for the isotropic stiffness tensor.

We obtain the following bounds on the penalty parameter:

**Theorem 4.1. (Coercivity)** Let \(X_E\) be an affine function (i.e. the Jacobian is constant). Moreover, let

\[
\delta_e^*(\zeta) = \left( \frac{\beta_{e,1} + \beta_{e,2}}{|e|} \right) \zeta,
\]

(27)

where \(0 < \zeta < 1\) and \(\beta_{e,1} = \frac{N(N - 1 + D)}{D} \frac{|e|}{|K|^2} \frac{c_0^E}{c_0^E} \zeta^2\).

Then choosing \(\delta_e > \delta_e^*\) ensures

\[
a(u, u) \geq K^*(\zeta)||u||_e^2
\]

with coercivity constant

\[
K^*(\zeta) = \min \left\{ 1 - \zeta, \frac{1}{\min_{e \in \Gamma^{i,j,t} E} \left( 1 - \frac{\delta_e^*(\zeta)}{\delta_e} \right)} \right\}
\]

**Proof.** See Appendix A. □

**Corollary 4.1. (Boundedness)**

\[
a(u, u) \leq K^*(\zeta)||u||_e^2
\]

with constant

\[
K^*(\zeta) = \max \left\{ 1 + \zeta, \frac{1}{\max_{e \in \Gamma^{i,j,t} E} \left( 1 + \frac{\delta_e^*(\zeta)}{\delta_e} \right)} \right\}
\]
tools to implement the DG scheme on unstructured curvilinear
tools on the reference element is required, and we detail the necessary
of the non-zero entities of the matrix
where
The solution to the variational problem (23) is obtained by solving
5 DISCONTINUOUS GALERKIN IMPLEMENTATION
with curvilinear coordinates and
and determine the constant in the inequality.
mine the constant
extended by establishing one of the following inequalities:
with little distortion although a bound of the penalty parameter for
penalty parameter might also be adequate for curvilinear meshes
The defining property of nodal basis functions is \( \phi_i(\xi) = \delta_{ij} \),
with Kronecker delta \( \delta_{ij} \) and nodes \( \xi_1, \ldots, \xi_B_N \in \mathbb{E} \), where
\( B_N = \binom{N + D}{D} \). We use the warp & blend nodes in this paper as they
have a small Lebesgue constant and are easy to construct on the triangle and the tetrahedron (Warburton 2006).
The warp & blend nodes are illustrated in Figures 1 and 2.
The points \( \mathbf{X}_E^{P} \in \mathbb{R}^D \) give the location of a node in the domain,
as \( \mathbf{X}_E(\xi) = \mathbf{X}_E^{P} \). In practice, these can be obtained in at least two ways:
Either, one generates a simplex mesh and applies a warp function to each node, such that \( \mathbf{X}_E \) is an interpolant of the warp function. Or, one can use high-order meshing software,
e.g. the open-source software Gmsh (Geuzaine & Remacle 2009; Johnen et al. 2013).
Volume integrals over the physical element \( E = \mathbf{X}_E(\mathbb{E}) \) can be computed using quadrature rules (Witherden & Vincent 2015; Jaskowiec & Sukumar 2020). For example, the integral of \( u \in V_h \) is computed with
\[
\int_E u_p(x) \, dx = \int_E u_p^E \phi_i(\xi) J^E(\xi) \, d\xi \\
\approx \sum_q W_q u_p \phi_i(\xi_q) J^E(\xi_q),
\]
where we use the nodal basis expansion \( u_p|_E = u_p^E \phi_i(\xi) \). \( W_q \) and \( \xi_q \) are quadrature weights and absicissa for \( \mathbb{E} \), and
\( J^E(\xi) = | \det(\nabla \mathbf{X}_E(\xi)) | \).
For the surface integral’s parameterisation, we introduce the maps
\( \mathbf{Y}_E^f : \bar{\mathbf{E}} \to \mathbb{E} \) from the reference facet \( \bar{\mathbf{E}} \) (i.e. reference edge in 2D and reference triangle in 3D) to the reference element \( \mathbb{E} \), given by
\[
\mathbf{Y}_E^f(\chi_j) = \left( 1 - \sum_{i=1}^{D-1} \chi_i \right) \xi_j^f + \sum_{i=1}^{D-1} \chi_i \xi_i^f,
\]
where \( \xi_j^f, \ldots, \xi_{D-1}^f \) are the vertices of the reference simplex that make up the \( f \)-th facet (\( f = 0, \ldots, D \)).
Surface integrals in DG involve information of the two simplices \( E_1 \) and \( E_2 \) adjoining the facet \( e = \partial E_1 \cap \partial E_2 \). Let \( f, g \) be the local face number of \( e \) in \( E_1, E_2 \), respectively, and \( \mathbf{Y}_E^f \) and \( \mathbf{Y}_g^e \) the respective facet maps. The parameterisation is greatly simplified by ensuring consistent local edge directions. For triangle or tetrahedral meshes, one can show that local edge directions are always consistent when the local vertex order is chosen according to

\[ \nabla = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

\[ \mathbf{X}_E(\xi) = \sum_j \phi_j(\xi) \mathbf{X}_E^j. \]
a global vertex order (Rognes et al. 2009). In this case, we have
\[ \forall \chi \in \bar{E} : X_{E_1} \circ \mathbf{Y} = X_{E_2} \circ \mathbf{Y}. \]
(34)

For example, for \( u, v \in V_{i1} \) we have
\[ \int_{E} u_{ij} v_{rs} \, ds = \int_{E} \frac{E_{1}}{2} \phi_{ij} \left( \mathbf{Y}'(\chi) \right) \mathbf{n}^e(\chi) \, d\mathbf{X} \approx \sum_{q} w_q \frac{E_1}{2} \phi_{ij} \left( \mathbf{Y}'(\chi_q) \right) \mathbf{n}^e(\chi_q), \quad \left(35\right) \]
where we have used the nodal basis expansions \( u_{ij}|_{E_1} \circ X_{E_1} = \sum_{j} \frac{E_1}{2} \phi_{ij} (\xi) \) and \( v_{rs}|_{E_2} \circ X_{E_2} = \sum_{r} \frac{E_2}{2} \phi_{rs} (\xi) \). Quadrature weights and abscessa for \( \bar{\epsilon} \) are given by \( w_q \) and \( \chi_q \) (Witherden & Vincent 2015) and
\[ \mathbf{n}^e(\chi) = \mathbf{J}^T(\mathbf{Y}'(\chi)) \left( \nabla X_{E_1}(\mathbf{Y}'(\chi)) \right)^{-T} \mathbf{n}', \quad \left(36\right) \]
where \( \mathbf{n}' \) are the outward pointing normals of the reference simplex (cf. Figures 1 and 2). We remark that the absolute value of the determinant of the face Jacobian is taken in (36), because the determinant might be negative as a consequence of the numbering convention. (E.g. in 2D the vertices might be ordered both clockwise and counter-clockwise.) The same reasoning also holds for the cell Jacobian in (32).

5.2 Residual evaluation and matrix-free implementation

We first eliminate the derivative in (17) through integration by parts:
\[ \int_{E} \tau_{ij} \sigma_{ij} \, d\mathbf{X} = \int_{E} \tau_{ij} c_{ijrs} \frac{\partial u_{i}}{\partial x_{s}} \, d\mathbf{X} \]
\[ + \int_{\partial E} \tau_{ij} c_{ijrs} (u_{i} - u_{r}) n_{s} \, ds. \quad \left(37\right) \]
We set \( \tau_{ij} = \partial v_{i} / \partial x_{j} \) in (37), which is valid due to \( \nabla n_{i} V_{h} \subset \mathbb{S}_{h} \). The result is plugged into (18):
\[ \int_{E} \frac{\partial u_{i}}{\partial x_{j}} c_{ijrs} \frac{\partial u_{r}}{\partial x_{s}} \, d\mathbf{X} + \int_{\partial E} v_{i} \cdot \mathbf{n} \, n_{j} \, ds = \int_{E} f_{i} v_{i} \, d\mathbf{X}. \quad \left(38\right) \]
The computation of the numerical flux on \( e = \partial E_{1} \cap \partial E_{2} \) is described next. Recall the basis expansion \( u = \sum_{E} \phi_{E}^{e} \, e_{n} W_{n}^{e} \), where \( \phi_{E}^{e} \) is only non-zero on \( E \). If \( e \in \mathbb{I}^{f} \) we have for \( \mathbf{x} \in e \)
\[ \sigma_{ij}^{e}(\mathbf{x}) = \left\{ c_{ijrs}^{e} \frac{\partial u_{i}}{\partial x_{s}} \right\} - \delta_{ij} u_{j} = \frac{1}{2} c_{ijrs}^{e} \frac{\partial \phi_{ij}^{e}}{\partial x_{s}} W_{i}^{e} + \frac{1}{2} \frac{E_{1}}{E_{2}} \frac{\partial E_{1}^{e}}{\partial x_{s}} \frac{E_{2}}{E_{1}} \frac{\partial E_{2}^{e}}{\partial x_{s}} - \delta_{ij} u_{j} \phi_{ij}^{e} - \delta_{ij} \phi_{ij}^{e} n_{j}. \quad \left(39\right) \]
For \( \mathbf{x} \in E \) the basis functions are given by \( \phi_{E}^{e} = \phi_{0} \circ X_{E}^{-1} \). Using the chain rule we obtain (Brezzi & Fortin 1991, III.1.3)
\[ \nabla \phi_{E}^{e}(\mathbf{x}) = (\nabla \phi_{0} \circ X_{E}^{-1}) \nabla X_{E}^{-1} = (\nabla \phi_{0}(\nabla X_{E}^{-1})) \circ X_{E}^{-1}. \quad \left(40\right) \]
Moreover, we introduce the shorthand notation
\[ G_{E}^{e}(\mathbf{x}) = (\nabla X_{E}(\mathbf{x}))_{ijrs}^{e}, \quad D_{k}^{e}(\mathbf{x}) = \frac{\partial \phi_{ij}^{e}}{\partial x_{k}}. \quad \left(41\right) \]

Moreover, the local facet numbers for \( e \) of elements \( E_{1}^{e} \) and \( E_{2}^{e} \) respectively. Hence, we get on the reference facet
\[ \sigma_{ij}^{e}(\chi) = \frac{1}{2} c_{ijrs}^{e} D_{k}^{e} G_{E_{1}^{e}}^{e} n_{s}^{e} - \frac{1}{2} c_{ijrs}^{e} D_{k}^{e} G_{E_{2}^{e}}^{e} n_{s}^{e} - \delta_{ij} \left( \phi_{ij}^{e} n_{j}^{e} - \phi_{ij}^{e} n_{j}^{e} \right) n_{j}. \quad \left(42\right) \]
Likewise, the displacement flux on the reference facet is
\[ \tilde{u}_{ij}^{e}(\chi) = \left\{ u_{i}, \phi_{ij}^{e}, \phi_{ij}^{e} \right\}. \quad \left(43\right) \]
The numerical fluxes on the fault, Dirichlet, or Neumann boundaries are derived similarly. We only note that on \( e \in \Gamma_{f} \) the flux is double valued with \( \tilde{u}_{ij}^{e} = \left\{ u_{i}, g_{f}^{e}, g_{f}^{e} \right\} / 2 \) and \( \tilde{u}_{ij}^{e} = \left\{ u_{i}, g_{f}^{e} \right\} / 2 \).

Next, we recover an expression for the residual \( r := \mathbf{A} u - b \). In (38), we move the force term to the left-hand side and plug in the basis expansion and the test function \( v = e_{n} \phi_{E}^{e} \):
\[ r_{(k u E)} = \mathbf{u}_{E}^{e} \int_{E} \frac{\partial \phi_{E}^{e}}{\partial x_{s}} \frac{\partial \phi_{E}^{e}}{\partial x_{s}} \, d\mathbf{X} + \sum_{e \in \partial E_{f}} c_{ijrs}^{e} \left( \tilde{u}_{ij}^{e} - \tilde{u}_{ij}^{e} \right) n_{s} \, ds - \int_{E} \mathbf{f}_{u} e_{n} \, d\mathbf{X}. \quad \left(44\right) \]

With the tools developed in Section 5.1 we map (44) to the reference element:
\[ r_{(k u E)} = \mathbf{u}_{E}^{e} \int_{E} D_{k}^{e} G_{E_{1}^{e}}^{e} c_{ijrs}^{e} D_{k}^{e} G_{E_{2}^{e}}^{e} j^{e} \, d\mathbf{X} + \sum_{f=0}^{D} \int_{E} D_{k}^{e} G_{E_{1}^{e}}^{f} c_{ijrs}^{f} \left( \tilde{u}_{ij}^{f} - \tilde{u}_{ij}^{f} \right) n_{s}^{f} \, d\mathbf{X} - \sum_{f=0}^{D} \int_{E} \phi_{E}^{f} \mathbf{f}_{u} \, d\mathbf{X}, \quad \left(45\right) \]
where \( e \) is the \( f \)-th facet of \( E \). Finally, we introduce the convention that an additional index \( q \) means evaluation at \( \xi_{q} \) or \( \chi_{q} \), e.g. \( G_{E_{q}^{e}} := G_{E_{q}^{e}}(\xi_{q}) \) or \( n_{q}^{e} := n_{q}^{e}(\chi_{q}) \). The fully discrete form is
\[ r_{(k u E)} \approx \sum_{q} W_{q} \phi_{E}^{e} D_{k}^{e} G_{E_{1}^{e}}^{e} c_{ijrs}^{e} D_{k}^{e} G_{E_{2}^{e}}^{e} j^{e} \]
\[ + \sum_{q} W_{q} \phi_{E}^{f} D_{k}^{e} G_{E_{1}^{e}}^{f} c_{ijrs}^{f} \left( \tilde{u}_{ij}^{f} - \tilde{u}_{ij}^{f} \right) n_{s}^{f} - \sum_{q} W_{q} \phi_{E}^{f} \mathbf{f}_{q} \mathbf{f}_{u} j^{e}. \quad \left(46\right) \]

The action \( \mathbf{A} u \) is readily obtained by setting \( g_{f}^{D} = 0, g_{f}^{F} = 0, \) and \( f_{1} = 0 \) in the residual. That is, setting all terms which would appear in \( b \) (in the discrete problem), or in \( L(v) \) (in the continuous problem), to zero.

5.3 Linear solvers

There are many direct and iterative methods to find the solution of the sparse linear system \( \mathbf{A} u = b \). We gain access to many of these methods by using the Portable, Extensible Toolkit for Scientific Computation (PETSc), which provides high-level data structures for distributed matrices and vectors and a unified interface to many Krylov methods, preconditioners, and sparse direct solvers (Balay et al. 1997, 2021a,b).
In this paper, we use two different kinds of solvers, LU factorisation and Krylov methods preconditioned with multigrid. LU factorisation is a good choice for small-scale problems, in particular in 2D. For SEAS problems, the operator $A$ is time-invariant, such that the expensive factorisation step is only required once. PETSc provides an interface to many state-of-the-art LU factorisation packages (Balay et al. 2021a). In this work, when we resort to using LU, we used the sparse direct solver MUMPS (Amestoy et al. 2001, 2019).

For large problems, in particular in 3D, iterative methods are used with a preconditioner that combines the p-multigrid and the algebraic multigrid methods. We define a sequence of coarse to fine grids $V_{h}^{0} \subset \ldots \subset V_{h}^{k}$ for the p-multigrid method, where $p_{i} = 1 < \ldots < p_{L} = N$. The spaces $V_{h}^{k}$ are obtained by replacing $N$ with $p_{i}$ in (16). As the spaces are nested and the same mesh is used, the interpolation operator $I_{i+1}^{i}$ for the transfer of the residual from grid $i$ to grid $i+1$ is simply a block diagonal matrix. Following the Galerkin approach, the restriction operator is set to $I_{i+1}^{i} = (I_{i+1}^{i})^\top$ and the coarse grid operators are given by $A_{i} = I_{i}^{0} A_{i-1}^{i+1} I_{i}^{0}$. On the coarsest grid, we apply the algebraic multigrid preconditioner GAMG (Balay et al. 2021a).

### 5.4 Implementation details

Our C++-implementation of the DG scheme is freely available under the BSD 3-Clause license (https://github.com/Tandem-ECC/tandem).

Tandem supports matrix assembly as well as the matrix-free approach. The matrix-free computation of the residual or operator application $Au$ consists solely of small tensor contractions. We use Yet Another Tensor Toolbox (Uphoff & Bader 2020) to generate high-performance kernels for (46).

Tandem is inherently parallel and designed for large scale distributed memory architectures. In addition to PETSc for parallel linear algebra, solvers, and preconditioner support, we also employ the graph partitioning libraries Metis and Parmetis (Karypis & Kumar 1998) to facilitate run-time mesh partitioning for a given MPI communicator size.

Tandem natively loads meshes generated with Gmsh in parallel. Once loaded, a new mesh decomposition (load balanced partition) is defined using Parmetis. Given the new partition, the elements and all associated attributes (e.g. cell tags / labels, vertex coordinates, vertex tags / labels) are scattered to the appropriate MPI communicator size.

In principle, given a DG function space of degree $N$, the error is expected to decrease with $O(h^{N})$, where $h$ is the maximum element diameter. In order to retain the desirable convergence behaviour in domains with curved boundary or heterogeneous material parameters, we approximate the geometry and material parameters with polynomials of the same maximum degree $N$ as the displacement.

### 6 COUPLING AND TIME-STEPPING

We complete the numerical scheme for SEAS model with the following four steps. First, we introduce trace spaces on $\Gamma^{F}$ in order to approximate the on-fault traction, slip, and state variable. Second, we discuss the coupling between on-fault and off-fault variables. Third, we show how (12) and (11) can be recast as an ordinary system of differential equations. Fourth, we discuss time-stepping.

#### 6.1 Trace spaces

The mapping $m_{e} : \bar{e} \to e$ from the reference facet $\bar{e}$ to $e \in \Gamma^{F}$ is in the notation of Section 5 given by

$$m_{e}(\bar{x}) = X_{\bar{x}}^{-1} \circ \mathcal{Y}^{\bar{F}} = X_{\bar{x}}^{-1} \circ \mathcal{Y}^{\bar{F}}.$$  (47)

Quantities $(T, S, \psi)$ on the physical element are related to quantities $(\bar{T}, \bar{S}, \bar{\psi})$ on the reference element via

$$T = \bar{T} \circ m_{e}^{-1}, \quad S = \bar{S} \circ m_{e}^{-1}, \quad \psi = \bar{\psi} \circ m_{e}^{-1}.$$  (48)

We introduce the finite element spaces

$$M_{h}^{d} = \{ \mu \in [L^{2}(\Gamma^{F})]^{d} : |\mu|_{e} \in [P_{N}(\bar{e})]^{d} \forall e \in \Gamma^{F} \}$$

for $T, S \in M_{h}^{D-1}$ and $\psi \in M_{h}^{1}$.

#### 6.2 Coupling

The change in traction $\delta T_{e} = T_{e} - T_{e}^{0} = B_{j} \sigma_{jk}(u)n_{k}$ couples the elasticity solver to the friction law, but $\sigma_{jk}$ is double-valued on $\Gamma^{F}$. Thus, we replace with $\sigma_{jk}$ with the numerical flux $\tilde{\sigma}_{jk}$, because the flux is single-valued and is designed to be a (stable) approximation of $\sigma_{jk}$ on $\Gamma^{F}$.

Moreover, $(B^{T} \tilde{\sigma} n)|_{\Gamma^{F}} \notin M_{h}$ for non-affine mappings. Hence, we define the projection operator $\pi_{d}^{(2)} : [L^{2}(\Gamma^{F})]^{d} \to M_{h}^{d}$

$$\forall \mu \in M_{h}^{d} : \int_{e} \mu \cdot \tau_{e}(\theta) \, ds = \int_{e} \mu \cdot \theta \, ds,$$

and let $\delta T_{e} = \pi_{d}^{(2)}(B^{T} \tilde{\sigma} n)$. In the implementation, we represent the change in traction using the nodal basis $\varphi_{k}(X)$ on the reference face $\bar{e}$. I.e., with $\delta T_{e} = m_{e}^{-1} = \bar{S}_{e}^{	op} \tilde{\sigma} \varphi_{k}$ we have

$$\delta T_{e} \int_{e} \varphi_{k} \varphi_{l} |n_{k}| \, dx = \int_{e} \varphi_{k} B_{j} \tilde{\sigma} \varphi_{l} n_{j} \, dx.$$  (51)

Likewise, the change in normal stress is with $\delta \sigma_{e} = \pi_{d}^{(1)}(n \tilde{\sigma} n)$. Slip is set as boundary condition in the elasticity solver, cf. (12). Using the nodal basis expansion $\bar{S}_{j} = \sum_{k} \bar{S}_{j}^{k} \varphi_{k}(X)$ we simply set $\bar{S}_{j} \circ m_{e} = B_{j} \pi_{d}^{(1)}(n \tilde{\sigma} n)$.

We note that in an early stage of developing the numerical scheme we defined on-fault traction as $\frac{N}{r_{c}+1} \partial_{n} \tilde{\sigma}_{i} \partial_{x_{j} n_{j}}$, instead of $\bar{S}_{i} \partial_{n}$. We found that this choice is unstable, i.e. the on-fault traction might become discontinuous along the fault and grow without bounds.

### 6.3 Ordinary differential equations

The Dirichlet and fault boundary conditions only appear in the right-hand side $L(u)$, cf. (22). Hence, the discrete elasticity problem in dependence on slip $S$ and time $t$ is given by

$$A u = b(S, t)$$  (52)

The above problem has a unique solution for a large enough penalty (cf. Section 4) formally given by $u = A^{-1} b(S, t)$. The change $\delta T$ in traction depends linearly on $u$, cf. Section 6.2, thus traction $T$ is a function of time and slip.

Slip-rate, state, and traction are linked via the friction law. Taking the magnitude $T = |T|$ reduces (10) to the single equation

$$\sigma_{e} F(V, \psi) + \eta V - T = 0,$$  (53)

where $V = |V|$. The derivative of (53) w.r.t. $V$ is $\sigma_{e} \frac{\partial F}{\partial V} + \eta$, where for the particular choice of $F$ from (6) we have $\partial F / \partial V > 0$. 

in the volume by the following we denote the number of displacement components solve the discrete elasticity problem millions of times, one solve percrete elasticity problem in Line 1. Given that SEAS problems typi-
we library PETSc/TS (Abhyankar et al. 2018). Experiments in
Equation (54) is the canonical formulation of a system of ordinary
where we show how to evaluate every node, i.e. the semi-discrete quasi-dynamic model is
Note that the root lies in the interval finding methods such as Brent’s method (Forsythe et al. 1977).
for

\begin{align*}
\delta T &= t Z_2 u, \\
\delta T &= t Z_2 A^{-1} b_D + \sum_{i=1}^{n_b} Z_2 A^{-1} Z_1 e_i S_i.
\end{align*}

Therefore, instead of solving a linear system the change in traction can be written in terms of the discrete Green’s function $g_D, g_1, \ldots, g_n$.
The matrices $A^{-1}, Z_1, Z_2$ are not required explicitly. Instead, we only need to compute the change in traction for the right-hand sides $b(0, 1)$ and $b(e_i, 0)$, $i = 1, \ldots, n_b$, in order to obtain the discrete Green’s function. We also note that the action of $A^{-1} y$ (for an arbitrary vector $y$) is defined via a solve. As an example we consider the first term on the right-hand side of Equation (57). This would be evaluated as $t Z_2 u^*$ where $u^*$ is the solution to $Au^* = b_D$. The solution strategies employed in Section 5.3 are also adopted for obtaining the auxiliary solutions required to define the discrete Green’s function. Since the operator $A$ is sparse and we have both optimal and parallel efficient solvers for this operator, we thus also have a highly scalable means to compute all of the required Green’s functions. The importance of this will be further elaborated upon in Section 10.2.

8 VERIFICATION

The presented SIPG method allows to model the slip boundary condition, curved boundaries, heterogeneous material parameters, and can be coupled to rate and state friction laws. In this section, we discuss and analyse various 2D and 3D model problems whereby each verifies one or multiple of these method capabilities.
First, we discuss static elasticity problems which verify that the SIPG method converges. An analytic solution is available for each of these problems, hence we determine the empirical convergence order and discuss the benefits and limitations of using a high-order method.
Second, we use the method of manufactured solutions to analyse whether convergence is achieved for a SEAS problem with a modified state equation. Due to manufacturing, the analytic solution is available such that the convergence behaviour of the method can be discussed.
Last, we verify the method in a community benchmark problem Erickson et al. (2020). While this problems resembles an actual SEAS model, no analytic solution is available, but we obtain a comparison to several independent methods and implementations.
We note that input files to reproduce all setups are openly available (see DATA AVAILABILITY).

8.1 Convergence of the elasticity solver

We analyse the convergence behaviour of the SIPG scheme for elastostatic problems. In our chosen verification problems we cover

\begin{algorithm}
\caption{Evaluation of right-hand side of quasi-dynamic model}
\begin{algorithmic}[1]
\Require Degrees of freedom $\psi, S$ and time $t$
\Ensure Computes $\delta S \leftarrow \mathbf{V}(\psi, S, t)$ and $\delta \psi \leftarrow \mathbf{G}(\psi, S, t)$
1: Solve $A_0 \mathbf{u} = b(S, t)$
2: Compute $\delta T$ and $\delta \sigma_n$
3: $T \leftarrow T + \delta T$
4: $\sigma_n \leftarrow \max(0, \sigma_n - \delta \sigma_n)$
5: for $e \in \Gamma^f$ do
6: \hspace{1em} for $k = 1, \ldots, (N + D - 1)$ do \Comment{For every node}
7: \hspace{2.5em} $T \leftarrow \sqrt{\sum (T_k^f)^2}$
8: \hspace{1em} if $T \neq 0$ then
9: \hspace{2.5em} $\mathbf{s}(V) \leftarrow \sigma_{n_e} F(V, \psi_{n_e}) + \eta V - T$
10: \hspace{2.5em} Solve $\mathbf{s}(V) = 0$ using Brent’s method
11: \hspace{2.5em} $V_i \leftarrow -\frac{V}{T} T_k^f$, for $i = 1, \ldots, D - 1$
12: \hspace{1em} else
13: \hspace{2.5em} $V \leftarrow 0$
14: \hspace{2.5em} $V_i \leftarrow 0$ for $i = 1, \ldots, D - 1$
15: end if
16: \hspace{1em} $\delta S_{n_e} \leftarrow V_i$, for $i = 1, \ldots, D - 1$
17: $\delta \psi_{n_e} \leftarrow \mathbf{G}(V, \psi_{n_e})$
18: end for
19: end for
\end{algorithmic}
\end{algorithm}

for $a > 0$ and $V_0 > 0$. With $\eta \geq 0$ it follows from the implicit function theorem that there exists a unique function $V(\psi, T, \sigma_n)$ that solves (53). It follows that $V$ is a function of $\psi, T, \sigma_n$.
In practice, the solution to (53) is found using standard root-finding methods such as Brent’s method (Forsythe et al. 1977). Note that the root lies in the interval $V \in [0, T/\eta]$.
So far we silently ignored that $S, \psi$ depend on time. With abuse of notation we understand that for fixed time $t$ we have $S(t) \in M_{N_h}^{D-1}$ and $\psi(t) \in M_1^h$. We solve for slip and state on every node, i.e. the semi-discrete quasi-dynamic model is
\begin{align*}
\frac{dS}{dt} &= \mathbf{V}(\psi, S, t), \\
\frac{d\psi}{dt} &= \mathbf{G}(\psi, S, t),
\end{align*}
where we show how to evaluate $\mathbf{V}$ and $\mathbf{G}$ in Algorithm 1.

6.4 Time-stepping

Equation (54) is the canonical formulation of a system of ordinary differential equations. Hence, time-stepping is handled by the software library PETSc/TS (Abhyankar et al. 2018). Experiments in this paper use explicit Runge-Kutta schemes with adaptive time-step control.

7 DISCRETE GREEN’S FUNCTION

The most expensive part in Algorithm 1 is the solution of the discrete elasticity problem in Line 1. Given that SEAS problems typically require hundreds of thousands of time-steps, we may need to solve the discrete elasticity problem millions of times, one solve per Runge-Kutta stage. Moreover, we solve for displacement in a D-dimensional domain whereas only the on-fault traction on $(D - 1)$-dimensional surfaces is required to evolve rate and state friction. In the following we denote the number of displacement components in the volume by $N_h$, and the number of slip components on the fault by $n_h$. By inspecting the linear form $L(v)$ in (23), and assuming zero body forces as well as a linear in time Dirichlet boundary condition, we can split up the discrete right-hand side as
\begin{align*}
b(S, t) &= t b_D + Z_1 S.
\end{align*}

two- and three-dimensional curvilinear meshes with heterogeneous material parameters as well as the slip boundary condition.

Errors in the displacement are measured with the norm

$$\|u\|_{0,h} = \left( \sum_{E \in T_h} \int_E u_i u_i \, dx \right)^{1/2}. \quad (58)$$

When coupling the elasticity solver to the non-linear rate and state friction law we need the Jacobian of the displacement. Therefore, we also measure the error in the semi-norm

$$|u|_{1,h} = \left( \sum_{E \in T_h} \int_E \left| \frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i} \right| \, dx \right)^{1/2}. \quad (59)$$

Given a sequence of meshes with mesh sizes $h_1 > \cdots > h_n$, respective discrete solutions $u_1, \ldots, u_n$, and the analytical solution $u^*$, the empirical convergence order is defined as

$$C_i = \frac{\log(E_{i-1}/E_i)}{\log(h_{i-1}/h_i)}, \quad (60)$$

where the error $E_i$ is either given by $\|u_i - u^*\|_{0,h}$ or $|u_i - u^*|_{1,h}$.

8.1.1 Slip boundary condition

In the first 2D verification setup, the slip boundary condition is evaluated. Our approach in constructing the test problem is to find solutions to the elasticity equation on wedge-shaped domains and then glue two wedge-shaped domains together, separated by a dipping fault.

Let a semi-infinite wedge be given by $\Omega^{A_1} := \{(x,y) \in \mathbb{R}^2 : \theta_1 < \theta(x,y) < \theta_2\}$, where $\theta \in [-\pi, \pi)$ is the polar angle, i.e. $x = r \cos(\theta)$ and $y = r \sin(\theta)$. Then we can construct a dipping fault by gluing domains $\Omega^\omega := \Omega^{A_1} \setminus \Gamma^\omega$ and $\Omega^{B_1} := \Omega^{A_2} \setminus \Gamma^{B_1}$, where the dip angle $\omega \in (-\pi, 0)$. Here, we set $\omega = -\pi/3$, as illustrated in Figure 3a. For our test problem we require the following boundary conditions to be satisfied:

$$\sigma_{\theta \theta}|_{\theta=\pi} = \sigma_{\theta \theta}|_{\theta=-\pi} = 0, \quad (61)$$

$$\sigma_{\theta \theta}|_{\theta=0} = \sigma_{\theta \theta}|_{\theta=\omega} = 0, \quad \text{(free surface)},$$

$$\sigma_{\theta r}|_{\theta=\omega} = 0, \quad \text{(continuity)},$$

$$u_\theta|_{\theta=\omega} = u_\theta|_{\theta=\omega} = 0, \quad \text{(no-opening)},$$

$$u_r|_{\theta=\omega} = u_r|_{\theta=\omega} = \frac{r^2}{2\mu} \quad \text{(slip)}.$$

It is convenient to express stress and displacement in terms of a biharmonic function $\phi$ and a harmonic function $\psi$. Following Williams (2021) we then have the plane strain solution

$$\sigma_{rr} = -\frac{\partial^2 \phi}{\partial r^2} + \frac{\nu}{\mu} \frac{\partial^2 \psi}{\partial r \partial \theta},$$

$$\sigma_{\theta \theta} = \frac{1}{r} \frac{\partial \phi}{\partial \theta} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2},$$

where $\mu$ is the shear modulus, $\nu$ is Poisson’s ratio, and $\phi$ and $\psi$ are related by

$$\nabla^2 \phi = \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial \psi}{\partial \theta} \right). \quad (63)$$

We use the ansatz (Williams 2021)

$$\phi^A = r^{\gamma+1} \left[ A_1 \sin((\gamma+1)\theta) + A_2 \cos((\gamma+1)\theta) \right] + A_3 \sin((\gamma-1)\theta) + A_4 \cos((\gamma-1)\theta),$$

$$\psi^A = r^{\gamma-1} \left[ -\frac{4A_3}{\gamma-1} \cos((\gamma-1)\theta) + \frac{4A_4}{\gamma-1} \sin((\gamma-1)\theta) \right], \quad (64)$$

on sub-domain $\Omega^A$ for constants $\gamma$ and $A_1, \ldots, A_4$. The ansatz $\phi^B, \psi^B$ for sub-domain $\Omega^B$ is defined likewise but with constants $B_1, \ldots, B_4$. The constants are determined as following:
(i) Use the free surface conditions to express \( A_1, A_2, B_1, B_2 \) in terms of \( A_3, A_4, B_3, B_4 \).
(ii) Add the slip equation and the continuity equation for \( \sigma_{\theta\theta} \).
Subtract the no-opening equation from the continuity equation for \( \sigma_{\theta\phi} \). Use the resulting two equations to express \( A_3 \) in terms of \( B_3 \) and \( A_4 \) in terms of \( B_4 \).
(iii) Find the remaining constants \( B_3 \) and \( B_4 \) from the continuity equations.

In practice, we use SymPy (Meurer et al. 2017) to execute the above steps. The full procedure is provided in an openly available Jupyter notebook (see DATA AVAILABILITY). Furthermore, to avoid having to deal with infinite domains, we limit the A and B domains to \( r \leq 1 \). At the “new” boundary at \( r = 1 \) we impose the solution as Dirichlet boundary condition.

At last, we discuss the choice of the smoothness parameter \( \gamma \). The term \( \sin(\pi \gamma) \) appears in the denominator of \( B_1 \) and \( B_4 \), thus we must choose \( \gamma \in \mathbb{R} \setminus \mathbb{Z} \). Moreover, the term \( r^{\gamma - 1} \) appears in the stress tensor. Hence, \( \gamma > 0 \) in order to bound the strain energy (Barber 2004, §11.2.1). More generally, the \( k \)-th derivative of the displacement contains the term \( r^{\gamma - k} \). The norm of this term on a wedge-shaped domain with maximum radius \( r_{\max} \), i.e. \( \left( \int_{\theta_0}^{2\pi} \int_{\theta}^{\theta_{\max}} r^{2(\gamma - k)} r \, dr \, d\theta \right)^{1/2} \), is only bounded if \( \gamma > k - 1 \). Consequently, \( \gamma \) controls the smoothness of the solution. E.g. for \( 0 < \gamma < 1 \) the solution cannot be element of \( H^k \) with \( k \geq 2 \) and hence one cannot achieve high-order convergence using the SIPG method (Riviére 2008).

Here, we analyse the cases \( \gamma = 0.5 \) and \( \gamma = 10.5 \). The constants of the respective solution are listed in Table 2.

**Table 2.** Numerical parameters for the dipping fault static test case.

<table>
<thead>
<tr>
<th>( \gamma )</th>
<th>( \gamma )</th>
<th>( A_1 )</th>
<th>( A_2 )</th>
<th>( A_3 )</th>
<th>( A_4 )</th>
<th>( B_1 )</th>
<th>( B_2 )</th>
<th>( B_3 )</th>
<th>( B_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>10.5</td>
<td>0.20833333333333</td>
<td>0.072168783648703</td>
<td>-0.20833333333333</td>
<td>0.21650635094611</td>
<td>-0.125</td>
<td>0.072168783648703</td>
<td>-0.375</td>
<td>-0.072168783648703</td>
</tr>
</tbody>
</table>

8.1.2 Circular hole

In the second 2D verification problem, we evaluate whether high-order convergence is achieved for a curved boundary. To this end, we let the domain be the unit disk with a small hole, that is, for constants \( S_0, A, B \) (Barber 2004, §8.3.2). From the traction free boundary conditions it follows that

\[
A = S_0 a^2, \quad B = -\frac{S_0 a^4}{2}.
\]
A harmonic function satisfying the compatibility condition (63) is
\[ \psi = -2Ar^{-2}\cos(2\theta). \]  
(68)

Hence, a displacement field satisfying the elasticity equations and (65) is given by
\[ 2\mu u = S_0 r^{-3}(r^4 + 4a^2r^2(1 - \nu) - a^4)\sin(2\theta), \]
\[ 2\mu u_\theta = S_0 r^{-3}(r^4 + 2a^2r^2(1 - 2\nu) + a^4)\cos(2\theta). \]  
(69)

We set \( S_0 = 1 \) and impose (69) on the outer circular boundary \((r = 1)\) as Dirichlet boundary condition.

### 8.1.3 Spherical hole

The spherical hole verification problem is the three-dimensional analogue of the circular hole test case. The domain is given by \( \Omega^a = \{ x \in \mathbb{R}^3 \mid a < |x| < 1, \text{cf. Figure 4b for hole radius} \ a = 0.5 \}. \) In spherical coordinates, that is,
\[ x = R\cos(\theta)\sin(\beta), \quad y = R\sin(\theta)\sin(\beta), \quad z = R\cos(\beta), \]  
(70)

where \( \theta \in [-\pi, \pi) \) and \( \beta \in [0, \pi), \) we look for a displacement field which satisfies
\[ \sigma_{RR} = \sigma_{R\theta} = \sigma_{R\beta} = 0, \quad \forall x \in \mathbb{R}^3 : |x| = a. \]  
(71)

Barber (2004, §24.1.2) gives potentials \( \phi, \omega \) from which stress fields satisfying (71) are derived. The displacement field can be computed from these potentials (Barber 2004, Table 19.2). After simplifying the resulting expressions, we have \( u_\theta = 0 \) and
\[ 2\mu u = S_0 \left( \frac{R(1 - \nu)}{2(1 + \nu)} + \frac{a^4(10\nu - 13)}{4R^2(5\nu - 7)} + \frac{3a^5}{4R^4(5\nu - 7)} \right) + \frac{R}{2} \left( \frac{5a^4(4\nu - 5)}{4R^2(5\nu - 7)} + \frac{9a^5}{4R^4(5\nu - 7)} \right) \cos(2\beta), \]
\[ 2\mu u_\beta = -\frac{R^2}{2} \left( \frac{5R^2a(1 - 2\nu) + 3a^5}{R^2(5\nu - 7)} + 1 \right) \sin(2\beta). \]  
(72)

We set \( S_0 = 1 \) and impose (72) on the outer sphere \((r = 1)\) as a Dirichlet boundary condition.

![Figure 5](image-url)  
**Figure 5.** Verification of heterogeneous material parameters. The empirical convergence order, printed in small text above points, mostly matches the theoretical expectations.

8.1.4 Heterogeneous material parameters

The material parameters are heterogeneous and set to
\[ \mu(x) = 1 + \sin(10\pi x_1)/4, \]
\[ \lambda(x) = 1 + \sin(10\pi x_2)/4. \]  
(73)

The solution is manufactured to be
\[ u^*(x) = \begin{pmatrix} \cos(\pi x_1) \cos(\pi x_2), & 0 \end{pmatrix}. \]  
(74)

We ensure that the manufactured solution is a solution of (12) by using only Dirichlet boundaries with \( g_D = u^* \) and
\[ f_i = -\frac{\partial \sigma_{ij}(u^*)}{\partial x_j}. \]  
(75)

The force term is computed using the symbolic math package SymPy (Meurer et al. 2017). The domain is chosen as the unit square as illustrated in Figure 5.

### 8.1.5 Results

We conduct a convergence study for the four described test problems for polynomial degrees \( N = 1, \ldots, 8. \) Again, mesh sequences are generated with Gmsh version 4.8.4 (Geuzaine &
Remacle 2009; Johnen et al. 2013). Based on a CAD model, an initial degree 8 coarse mesh is generated and successively refined such that the mesh size is roughly halved w.r.t. the previous mesh in the sequence. The heterogeneous material parameter verification setup is an exception, here we use an uniform mesh generator for rectangular domains. If the solution is smooth enough, the convergence order is expected to be $N + 1$ for displacement and $N$ for the first derivatives (Castillo et al. 2000; Rivière 2008).

We first discuss the numerical convergence results for the slip boundary condition. For the case $\gamma = 0.5$ we showed in Sec. 8.1.1 that only the first (weak) derivative of the displacement solution exists. Thus, a convergence order below one is expected independent of the polynomial degree $N$ (Rivière 2008). Empirically, we find that the convergence order is about $0.5$, cf. Figure 3b. Still, the error is reduced by a constant factor with increasing polynomial degree, for example the error for $N = 8$ is one tenth of the error for $N = 1$ and $N = 8$ is the finest mesh. For the case $\gamma = 10.5$ the empirical convergence order is close the expected order $N$ in the $||.||_{1,h}$ semi-norm, but we observe that the error is not reduced below $5 \cdot 10^{-10}$. The condition number of the operator $A$ is in $O(h^{-2})$ (Antonietti & Houston 2011), hence even when reducing the residual down to machine precision the error might be much larger than machine precision.

A posteriori error analysis of the linear solve is offered by the direct sparse solver MUMPS (Amestoy et al. 2001, 2019) using the method of Arioli et al. (1989). The error analysis returns a bound on $\|e u\|_{0,h}$, that is, a bound on the error introduced in the degrees of freedom by finite precision effects. The results of the error analysis for the dipping fault test case and the circular hole test case are shown in Figure 6. We see that an error in the degrees of freedom of at most $2 \cdot 10^{-7}$ in the infinity norm is possible. Therefore, the minimum achievable error in Figure 3b is explained by the amplification of the error in the residual due to conditioning of the linear system. Moreover, the error bound roughly follows the $h^{-2}$ scaling of the condition number.

We next discuss the numerical convergence results of the circular hole and spherical hole verification cases shown in Figure 4. For degrees $N = 1, \ldots, 6$ the empirical convergence orders match the theoretical expectation very well in both the $||.||_{0,h}$ norm and $||.||_{1,h}$ semi-norm. For degrees $N = 7, 8$ the empirical orders are slightly off but still increase with increasing $N$. We also repeated the test for an affine mesh (i.e. degree 1), as shown in Table 3. Due to the linear approximation of the boundary the convergence order in the $||.||_{0,h}$ norm is reduced to 2 (with a curious exception for $N = 2$ where it is slightly higher), which shows that the curvilinear geometry approximation is necessary to retain high-order convergence. In the spherical hole test case theoretical expectations are well met until $N = 3$. The empirical convergence orders for higher degrees are lower than expected but still increase with increasing $N$. The condition number of the circular hole test case increases with $h^{-2}$, as shown in Figure 6, such that the error is not reduced below $10^{-11}$, and, once the minimum attainable error is reached, gets slightly worse under mesh refinement.

Lastly, high-order convergence is also achieved when material parameters are smoothly varying in space, cf. Figure 5.

### Table 3. Circular hole verification problem on an affine mesh sequence. Shown are the errors and empirical convergence orders on the finest mesh ($h = 0.017$). The convergence order is limited due to linear approximation of the inner and outer circle.

| $N$ | $||u - u^*||_{1,h}$ | $C_{1,h}$ | $||u - u^*||_{0,h}$ | $C_{0,h}$ |
|-----|---------------------|-----------|---------------------|-----------|
| 1   | 2.23371e-02         | 1.0       | 1.89889e-04         | 2.0       |
| 2   | 8.23588e-04         | 2.0       | 3.02087e-06         | 2.4       |
| 3   | 5.33885e-05         | 2.2       | 2.62902e-06         | 2.0       |
| 4   | 7.07560e-05         | 1.6       | 2.62783e-06         | 2.0       |
| 5   | 9.18152e-05         | 1.5       | 2.62678e-06         | 2.0       |
| 6   | 1.16266e-04         | 1.5       | 2.62560e-06         | 2.0       |
| 7   | 1.43651e-04         | 1.5       | 2.62449e-06         | 2.0       |
| 8   | 1.69295e-04         | 1.5       | 2.62369e-06         | 2.0       |

### 8.2 Convergence of the SEAS solver

Establishing convergence of a SEAS model is difficult as there are no known analytic solutions. However, by adding a source term to the state equation one can manufacture a solution. Erickson & Dunham (2014) manufactured such a solution for a two-dimensional anti-plane shear problem (where the elasticity equations reduce to a Poisson equation). Here, we propose a manufactured verification problem for plane strain elasticity.

#### 8.2.1 Domain

Assume that the fault lies in the $x_1 = 0$ plane. The first step is to construct a displacement field possessing a tangential component which is discontinuous at the fault, but with a normal component and traction which are continuous. That is, $u_1$, $\sigma_{11}$, and $\sigma_{12}$ are continuous but $u_2$ is discontinuous. We find that the Airy stress function (Barber 2004)

$$\Phi^\pm(x) = x_1 K(t) \exp(-k^\pm x_1) \cos(k^\pm x_2)$$

leads to a displacement field with the desired properties. $K(t)$ is a time-dependent function and we define $k^+ = k$ and $k^- = -k$ for a constant $k$. Indeed, the stress and displacement components are

$$\sigma_{11}^\pm = -k^2 x_1 K \exp(-k^\pm x_1) \cos(k^\pm x_2),$$

$$\sigma_{22}^\pm = k^2 (k^\pm x_1 - 2) K \exp(-k^\pm x_1) \cos(k^\pm x_2),$$

$$\sigma_{12}^\pm = k^2 (k^\pm x_1) K \exp(-k^\pm x_1) \sin(k^\pm x_2),$$

$$u_1^\pm = \frac{k^\pm x_1 (\nu + 1) - 2\nu^2 - \nu + 1}{E} K \exp(-k^\pm x_1) \cos(k^\pm x_2),$$

$$u_2^\pm = \frac{k^\pm x_1 (\nu + 1) + 2\nu^2 - 2}{E} K \exp(-k^\pm x_1) \sin(k^\pm x_2),$$

where $E$ is Young’s modulus and $\nu$ is Poisson’s ratio. For $x_1 = 0$ we have $\sigma_{11}^\pm = \sigma_{11}^\mp$, $\sigma_{12}^\pm = \sigma_{12}^\mp$, $u_1^\pm = u_1^\mp$, but

$$[u_2^\pm] = u_2^\mp - u_2^\pm = 4\frac{1 - \nu^2}{E} K \sin(kx_2).$$

Following Erickson & Dunham (2014), we choose $K$ as

$$K(t) = \frac{2}{\pi} \left( \arctan \left( \frac{1 - t_0}{t_0} \right) + \frac{\pi}{2} \right),$$

such that there will be one “event” at time $t_0$ with a duration of about $t_0$. (cf. Table 4).

The domains are $\Omega^- = (-3/5, 0) \times (-4/5, -1/5)$ and $\Omega^+ = (0, 3/5) \times (-4/5, -1/5)$. On the internal fault boundary $\Gamma^F = \{0\} \times [-4/5, -1/5]$ we impose the slip $[u_2^\pm]$. On the remaining boundary, $\Gamma^{D,\pm} = \partial \Omega^\pm \setminus \Gamma^F$ we impose the displacement field from (77).
Table 4. Parameters for the 2D plane strain SEAS model.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t_w$</td>
<td>50 s</td>
<td>$a$</td>
<td>0.015</td>
</tr>
<tr>
<td>$\tau_w$</td>
<td>1 s</td>
<td>$b$</td>
<td>0.02</td>
</tr>
<tr>
<td>$\rho_0$</td>
<td>2670 kg/m³</td>
<td>$V_0$</td>
<td>$1 \times 10^{-6}$ m/s⁻¹</td>
</tr>
<tr>
<td>$c_s$</td>
<td>3.464 km/s⁻¹</td>
<td>$f_0$</td>
<td>0.6</td>
</tr>
<tr>
<td>$\nu$</td>
<td>0.25</td>
<td>$k$</td>
<td>$\pi$</td>
</tr>
<tr>
<td>$E$</td>
<td>2e5 [±(1 + $\nu$)] Pa</td>
<td>$\sigma_0^{\nu}$</td>
<td>25$</td>
</tr>
<tr>
<td>$\eta$</td>
<td>$\rho_0 c_s/2$ Pas m⁻¹</td>
<td>$T^0$</td>
<td>20$</td>
</tr>
</tbody>
</table>

8.2.2 Fault

We add a source term independent of state $\psi$ and slip $S$ to the state equation, that is,

$$\frac{d\psi}{dt} = G(V, \psi) + s(x, t).$$  \hspace{1cm} (80)

The source term is defined as

$$s(x, t) = -G(V^*, \psi^*) + \frac{dV^*}{dt},$$  \hspace{1cm} (81)

where $V^*(x, t)$ and $\psi^*(x, t)$ are the manufactured solutions. Given that $V(x, 0) = V^*(x, 0)$ and $\psi(x, 0) = \psi^*(x, 0)$ the fields $V^*, \psi^*$ are a solution for (80).

The manufactured slip $S^*$ needs to be compatible with the solution in the domain, in particular with (78). With the fault normal $(1, 0)$ and tangent $(0, -1)$ we have $S^* := -[u_2]$. It follows that the manufactured slip-rate is given by

$$V^* = -4 \frac{1 - \nu^2}{E} \frac{dK}{dt} \sin(kx_2).$$  \hspace{1cm} (82)

Likewise, from (77) the manufactured on-fault normal stress and shear stress are

$$\sigma_n^* = \sigma_n^0 - \sigma_{11}|_{x_1=0} = \sigma_n^0,$$
$$T^* = T^0 - \sigma_{12}|_{x_2=0} = T^0 - kK \sin(kx_2),$$  \hspace{1cm} (83)

with background stress $\sigma_n^0$, $T^0$.

Lastly, the manufactured quantities need to satisfy the friction law. From (6) and (10) we have

$$-T^* = \sigma_n^a \arcsin \left( \frac{V^*}{2V_0} \exp \left( \frac{\psi^*}{a} \right) \right) + \eta V^*.$$  \hspace{1cm} (84)

Solving the above equation for $\psi^*$ we obtain

$$\psi^* = a \ln \left( \frac{2V_0}{V^*} \sinh \left( - \frac{T^* + \eta V^*}{\sigma_n^a} \right) \right).$$ \hspace{1cm} (85)

The parameters $a, V_0, \eta$ as well the background stress and all other parameters are summarised in Table 4.

8.2.3 Results

We conduct a convergence study for polynomial degrees $N = 1, \ldots, 8$. The mesh sequence is generated with an uniform mesh generator for rectangular domains. The final simulation time is set to $t_{end} = 2t_e = 100$ s. We report the error at $t_{out}$, where the time-dependent error is given by

$$\left( \sum_{x \in T^e} \int_T (\psi - \psi^*)^2 + (S - S^*)^2 \, dx \right)^{1/2}.$$  \hspace{1cm} (86)

Time-stepping is handled by PETSc/TS (Abhyankar et al. 2018; Balay et al. 2021a). We use adaptive Runge-Kutta methods such as the third order Bogacki-Shampine scheme (3bs) with the second order embedded method or the fifth order Dormand-Prince scheme (5dp) with the fourth order embedded method. The local truncation error is estimated by the difference between the fifth (third) order method and the fourth (second) order embedded method. Time-step sizes are adapted automatically such that the estimated local truncation error stays below a given absolute tolerance. We set an absolute tolerance of $10^{-12}$ and find that the choice of time-stepping method has little impact when the error is above the error tolerance, cf. the solid and dotted lines in Figure 7. Therefore, we only discuss the results of method 5dp.

The effective frictional resistance is related to traction via the empirical rate and state friction law, thus, we expect a maximum convergence order of $N$. The results in Figure 7 show that order $N$ is reached, although the absolute error tolerance limits the minimum achievable error. Curiously, for $N = 1$ the empirical convergence order is close to two.

Plotting the error versus the solve time (Figure 7) shows that high-order clearly pays off. E.g. for $N = 1$ the error on the finest grid is $3.1 \times 10^{-3}$ and the solve time increases by about a factor of 4 when the mesh size is halved. Assuming second order con-

![Figure 7](image-url)

Figure 7. Results of convergence analysis for the manufactured SEAS problem. Left: The empirical convergence order, printed in small text above points, is close to the polynomial degree $N$ in almost all cases. Right: The error vs. time-to-solution plot shows that high-order pays off for this problem. The absolute tolerance of $10^{-12}$ used to select time-steps is indicated by the gray dashed line. The error reported on the y-axis is given by (86).

![Figure 8](image-url)

Figure 8. Convergence results for the manufactured 3D SEAS problem. The absolute tolerance of $10^{-10}$ used to select time-steps is indicated by the gray dashed line. The error reported on the y-axis is given by (86).
vergence, one needs about 8 refinement steps to reach an error of $1 \times 10^{-8}$. Therefore, an increase in solve time by $4^8$ is necessary, that is, about 24 days. In contrast, a solve time of about 45 s is required to fall below the same error threshold for the $N = 8$ scheme.

We extended the manufactured verification setup in a straightforward manner to three dimensions, essentially by solving the 2D problem with the 3D scheme. We run the 3D test case for polynomial degrees $N = 1, \ldots, 6$ setting an absolute tolerance of $1 \times 10^{-10}$. The results are shown in Figure 8. The empirical convergence order is close to $N$.

8.3 SEAS Benchmark Problem 1 (BP1)

Another means of verification is a code comparison in which a representative physical problem is defined and solved by different groups, with different numerical methods and implementations (Harris et al. 2018; Buiter et al. 2016, e.g.), and the results of all approaches are subsequently compared. While this approach does not reveal the “ground truth”, we can analyse the variation between codes. A community code comparison is organised by the SEAS working group (Erickson et al. 2020) of the Southern California Earthquake Center (SCEC). We use their Benchmark Problem 1 (BP1) to test several aspects of our method: First, we investigate whether local (statically adaptive) mesh refinement is effective in comparison to uniform mesh refinement. Second, we check whether p-refinement improves our results significantly for the BP1 problem where the regularity of the solution is unknown. Lastly, we verify that a high-resolution simulation with our method matches the results of other groups.

8.3.1 Local h-refinement

One of the issues raised by Erickson et al. (2020) when using volume based discretisations for BP1 is the unknown effect of the prescribed finite sized domain on the solution. Using an unstructured mesh, addressing this issue can be tackled efficiently. In the following, we chose a 2D volumetric domain as a $[0 \text{ km}, 400 \text{ km}] \times [-400 \text{ km}, 0 \text{ km}]$ rectangle. Four unstructured meshes were constructed with approximately the same number of elements, but with differing spatial resolution on-fault ($h_f$) and off-fault. The off-fault resolution is controlled by specifying the spatial resolution ($h_o$) on the exterior boundary not containing the fault. In practice we selected $h_f$ and adjusted $h_o$ to produce a mesh with approximately 15000 cells. The four meshes (shown in Figure 9) were defined using $h_f = \{5000, 1000, 100, 25\}$ m. The end-members ($T_4^A \cdot T_4^P$) have an approximately uniform spatial resolution of 5 km, c.f. an on-fault resolution of 25 m and an off-fault resolution of 130 km.

In Figure 10 we report the simulated shear stress over five seismic cycles using the four different meshes and a degree $N = 2$ polynomial. For cases when $h_f \geq 1$ km, our solutions dramatically deviate from the reference solution (grey line in Figure 10). In comparison, the higher resolution simulations using an on-fault resolution of 100 m (green line – $T_4^A$) and 25 m (black line – $T_4^P$) are seen to closely follow the reference in terms of both, phase and amplitude.

8.3.2 p-refinement

To highlight the benefits of using high polynomial degree function spaces, we conducted simulations using $T_4^P$ and vary the polynomial degree $N$. The rational for selecting one of the “low resolution” meshes (as inferred from the cell sizes $h_f$, $h_o$) is that with higher $N$, we expect that higher spatial resolution should be achieved even if the cell sizes $h$, where $h_f \leq h \leq h_o$, are in fact large. In Figure 11 this expectation is seen to be correct. For low degrees ($N = 2, 4$) the solutions obtained are inaccurate in terms of phase, amplitude and number of cycles. The degree $N = 6$ results (purple line) are very similar to the reference solution (grey line) with a slight phase shift. The highest degree used $N = 8$ is visually seen to yield the closest agreement with the reference solution. Despite the small phase shift associated with $N = 8$ simulations, the actual recurrence interval is very similar to the reference, and more accurate than that obtained from the lower degree experiments (see Table 5).

8.3.3 Code comparison

We compare a high-resolution run using our method ($N = 8$) with other codes. The domain is chosen as the $[0 \text{ km}, 400 \text{ km}] \times [-400 \text{ km}, 0 \text{ km}]$ rectangle. Note that only half of the domain needs to be modelled due to symmetry. The boundaries $y = -400 \text{ km}$ and $y = 0 \text{ km}$ are traction-free and we impose the displacement $V_p t/2, V_p = 1 \times 10^{-3} \text{ m/s}$, on the $x = 400 \text{ km}$ boundary. The on-fault mesh size is 250 m and the mesh size is gradually coarsened to 50 km towards the far boundaries for a total of 3648 triangles. We note that the on-fault mesh size is 10 times larger than the 25 m mesh size recommended for BP1.

The slip-rate at 7.5 km depth is compared to eight different reference solutions taken from Erickson et al. (2020) in Figure 12. Overall, we find good agreement in the long-term behaviour. The largest deviation is seen in the onset time of the later earthquake events, which might result from different choices of domain size, as discussed in Erickson et al. (2020). Yet, overlaying the eighth event and correcting for the time shift (Figure 12, lower panel) shows that the coseismic behaviour closely matches the other codes.

9 NUMERICAL EXPERIMENTS

We apply the SIPG method to two geometrically complex demonstration models and elucidate the importance of using curvilinear fault representations, compared with piece-wise linear approximations (affine), when accurate stresses (or displacement gradients) are required on-fault. The first demonstrator consists of a 2D SEAS model defined by a shallowly dipping normal fault interacting with four curved splay faults. In the second demonstrator, we consider a 3D elasto-static model defining a kinematic scenario of instantaneous deformation inspired by the 2019 Ridgecrest earthquake se-
C. Uphoff et al.

Figure 9. Family of unstructured meshes \(T^A_{h} \{ A,B,C,D \} \) used for BP1, each with approximately the same number of elements \( N_e \). Each mesh is constructed by specifying a target spatial resolution (triangle edge length) along the fault \( h_f \) and elsewhere along the exterior boundary \( h_b \). The degree of local refinement is emphasised by colouring cells according to their area \((\text{km}^2)\) where red hues indicates regions of high local spatial resolution and blue hues denote regions of low local spatial resolution.

9.1 A 2D SEAS multi-fault scenario on a shallowly dipping normal fault with four curved splay faults

SEAS simulations including more than one fault segment and complex fault geometries are methodologically challenging (Romanet et al. 2018; Galvez et al. 2020; Luo et al. 2020; Barbot 2021) (see also Sec. 10.2), but crucial to understand large earthquakes that often span several segments of natural regional-scale fault networks (Plesch et al. 2007; Ando & Kaneko 2018; Ulrich et al. 2019; Harris et al. 2021). The mechanics of splay fault systems are of specific interest to seismic and tsunami hazard assessment and to understand shallowly observed slow slip transients (Wendt et al. 2009; Ikari et al. 2013).

The mechanics of slip on detachment normal faults have been extensively debated. The geologic record shows that shallowly dipping (< 30°) crustal-scale detachment faults play a major role in accommodating extension at divergent plate boundaries (Collettini

Table 6. Parameters for the splay fault scenario.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_{\text{max}} )</td>
<td>0.025</td>
<td>( a_0 )</td>
<td>0.010</td>
</tr>
<tr>
<td>( L )</td>
<td>0.05 m</td>
<td>( b )</td>
<td>0.015</td>
</tr>
<tr>
<td>( f_0 )</td>
<td>0.6</td>
<td>( \nu )</td>
<td>0.25</td>
</tr>
<tr>
<td>( \rho_0 )</td>
<td>2670 kg/m³</td>
<td>( c_s )</td>
<td>3.464 m/s</td>
</tr>
<tr>
<td>( V_p )</td>
<td>( -1 \times 10^{-9} ) m/s⁻¹</td>
<td>( V_0 )</td>
<td>( 1 \times 10^{-6} ) m/s⁻¹</td>
</tr>
<tr>
<td>( \sigma_{\text{HS}} )</td>
<td>50 MPa</td>
<td>( T^0 )</td>
<td>26.5461 MPa</td>
</tr>
<tr>
<td>( \eta )</td>
<td>( \rho_0 c_s / 2 ) Pa s m⁻¹</td>
<td>( t_{\text{end}} )</td>
<td>1500 yr</td>
</tr>
</tbody>
</table>
2011, e.g.). However, such shallowly dipping normal faults appear to challenge the laws of classic static fault mechanical analysis (e.g., Fletcher et al. 2016; Anderlini et al. 2016; Webber et al. 2018; Biemiller et al. 2020, 2021). Furthermore, detachment systems can often be complex with more steeply dipping splay-faults rendering the hanging wall. Such splay faults may slip co-seismically posing seismic and tsunami hazard or relieve shallow interseismic stress on detachments by subsidiary creep. We demonstrate the potential of our method to model seismic and aseismic slip naturally interacting across a main fault with multiple, geometrically complex splay faults. Similar fault system geometries are also ubiquitous in subduction zone megathrust fault systems (Park et al. 2002; Waldhauser et al. 2012), where splay fault slip trading off with slip to the hanging wall. Such splay faults may slip co-seismically posing challenge the laws of classic static fault mechanical analysis (e.g., Fletcher et al. 2016; Anderlini et al. 2016; Webber et al. 2018; Biemiller et al. 2020, 2021).

We create a fault system model consisting of a planar, $\delta_0 = 20^\circ$ dipping main normal fault and four curved splay faults, as shown in the sketch in Figure 13a. For each splay fault three control points are defined that would model a $50^\circ$ and a $40^\circ$ dipping fault segment when connected by straight lines. Here, we define the splay faults as the spline interpolant of the three control points (using Gmsh’s Spline geometry object). The fault system is embedded in the $(X_0, X_1)$ × $(Y_0, Y_1)$ box, where $X_0 = -1000$ km, $X_1 = 1000$ km$(1 + \cos(\delta_0)/\sin(\delta_0))$, $Y_0 = -1000$ km, and $Y_1 = 0$ km. The origin of the main normal fault coincides with the origin of the coordinate system and the splays are labelled according to their offsets given by 30 km, 50 km, 70 km, and 90 km, respectively. With $V_p$ negative, cf. Table 6, the domain is under extension and we model normal faulting. Natural boundary conditions are imposed on the top $(y = Y_0)$ and bottom $(y = Y_1)$ boundary of the model domain. On the left $(x = X_0)$ boundary we impose $u_1 = V_p/2$, $u_2 = 0$ m and on the right $(x = X_1)$ boundary we impose $u_1 = -V_p/2$, $u_2 = 0$ m.

We adopt a piece-wise linear parametrisation of rate and state friction law parameters $a - b$ as proposed by Rice (e.g., 1993), resembling intact granite (Blanpied et al. 1991). We hold $b$ constant and let $a$ depend on depth by piece-wise linear interpolation of the points $(0$ km, $a_{\text{max}})$, $(4$ km, $a_0)$, $(15$ km, $a_0)$, and $(18$ km, $a_{\text{max}})$. We hold $a$ constant at $a_{\text{max}}$ below a depth of 18 km. Given $a_0$, $a_{\text{max}}$, and $b$ as in Table 6, we have velocity weakening behaviour in $(8/3$ km, $16$ km) and velocity strengthening above and beneath that depth range. All material parameters and other parameters of the rate and state model are constant and given in Table 6.

We follow a refinement strategy analogue to Section 8.3, and set the on-fault resolution to $h_f = 0.5$ km and the far boundary resolution to $h = 100$ km for a total of 46721 triangles. The polynomial degree is set to $N = 8$ and all five faults are also approximated with a degree 8 polynomial. We note that we ran the scenario for on-fault resolutions $h_f \in \{4$ km, $2$ km, $1$ km, $0.5$ km$\}$. Going from $h_f = 2$ km to 0.5 km there is little difference in the number of events, the timing of events, and the peak slip-rate.

Our exemplary results in Figure 13b and Figure 13c illustrate 1400 years of slip and slip rate evolution on the main low-angle normal fault and all four curved splay faults in response to steady tectonic loading. Note, that we here do not omit a potential “spin-up

---

**Figure 10.** Shear-stress obtained from BP1 on different meshes compared with the reference solution from C. Cattania (Erickson et al. 2020). All experiments used a polynomial degree of $N = 2$.

**Figure 11.** Shear-stress obtained from BP1 on mesh $TA^P(h_f = 1$ km) as a function of polynomial degree $N = 2, 4, 6, 8$ compared with the reference solution from C. Cattania (Erickson et al. 2020).

**Figure 12.** Comparison of the BP1 problem to Erickson et al. (2020, Figures 5 and 6). Top left: Evolution of slip rate over 800 years at 7.5 km depth. Top right: Detail view of the eighth event. The onset time of the event deviates largely between codes. Bottom: We shift the time-series by their onset time of the eighth event. The onset time is here defined as the first time when the slip-rate is greater or equal than 0.001 m s$^{-1}$.
lent strategy established for branching fault geometries in dynamic gap between the main fault and the splays following an equivalent cycle in low-angle normal fault as well as megathrust-splay due to the demonstration scope of this example, we note that future splay fault. While we refrain from further quantitative analysis here an earthquake rupturing the shortest splay after sampling (due to static and quasi-dynamic stress transfer). For example, pronounced at the free surface. Aseismic transients appear at emplary model setup, all splays are able to co-seismically slip up to magnitude appears to decrease over time. It is offset by roughly the same amount of slip as its shorter neighboring splay fault. In our example model setup, all splays are able to co-seismically slip up to the free surface. Aseismic transients appear at 90 km and less pronounced at 70 km distance only after initial earthquakes on those splays. Interestingly, we observe rather complex splay fault coupling (due to static and quasi-dynamic stress transfer). For example, an earthquake rupturing the shortest splay after ≈1000 years, triggers the highest co-seismic slip-rate on the longest, furthest away splay fault. While we refrain from further quantitative analysis here due to the demonstration scope of this example, we note that future work can readily use our setup for geophysical analysis of the seismic cycle in low-angle normal fault as well as megathrust-splay fault networks by adapting the loading conditions.

In the scenario presented here, we leave an arbitrarily small gap between the main fault and the splays following an equivalent strategy established for branching fault geometries in dynamic earthquake rupture code verification exercises (Pelties et al. 2014; Harris et al. 2018). We note that whilst intersecting faults can be handled by our elasticity solver, cf. Section 9.2, when running a SEAS model with intersecting faults ($h_f = 2 \text{ km}$), we observe that shortly after $t = 10 \text{ yr}$ the normal stress becomes tensile. Inspecting the results, see Figure 14a, reveals that strain becomes very large at the intersection between a splay and the main fault, and that the change in traction changes sign, indicated by red lines which show the fault system geometry warped by the change in shear traction in the direction of the normal of the main fault. We might overcome this issue by future implementation of appropriate jump conditions for tensile stresses (Day et al. 2005) or accounting for off-fault inelastic deformation (Templeton & Rice 2008; Gabriel et al. 2013; Erickson et al. 2017; Wollherr et al. 2018; Mia et al. 2021).

We repeated the scenario with non-intersecting faults on an affine mesh (degree 1) in which the splay faults are approximated by linear elements. Figure 14b shows artifacts in both the strain and traction (the latter indicated as red lines). These artifacts coincide with the vertices, indicating that they are caused by the abrupt change in angle of the fault normal. The rate and state simulation aborts after $t = 100 \text{ yr}$ due to unrealistically high stresses likely caused by the ODE solver becoming unstable. In comparison, when using a curvilinear mesh, we observe no artifacts in strain and traction at $t = 100 \text{ yr}$, cf. Figure 14c.

Lastly, we report the application run-times on a dual socket AMD EPYC 7662 server using 120 processes, the fifth order Dormand-Prince scheme with six stages, and the linear solver combining p-multigrid and algebraic multigrid. For the $h_f = 4 \text{ km}$ mesh, computing the discrete Green’s function took approximately 0.8 h and the time integration took 1.5 h. In comparison, the run without discrete Green’s function took 31 h showing that the pre-computation step clearly pays off for this application. Computation of the discrete Green’s function on the finest mesh ($h_f = 0.5 \text{ km}$),
Our demonstration model considers (simplified) distinct strike-slip displacements accumulated during both events across six geometrically complex and intersecting fault segments, inspired by data-constrained 3D dynamic rupture modeling of both events (Taufiqurrahman et al. 2019). The example geometry of the Ridgecrest fault system has been simplified from integrated geodetic InSAR data, satellite imagery, relocated seismicity, and selected focal mechanisms (Carena & Suppe 2002; Taufiqurrahman et al. 2019).

The model domain is $130 \text{ km} \times 135 \text{ km}$ in map view and $50 \text{ km}$ deep. The fault network is centrally located within the model domain. All faults are assumed to be purely vertical and extending to a depth of $11 \text{ km}$ (DuRoss et al. 2020). The domain size is chosen such that the faults are $\sim 50 \text{ km}$ from the side walls and $39 \text{ km}$ from the lower boundary. The geometry of each fault is represented by a Bezier spline provided by Gmsh’s $8\text{sp1}\text{ne}$ geometry object. We impose zero displacement (all components) on the lateral walls, and on the upper and lower boundaries we impose zero stress (in both tangential and normal directions). We drive displacement in the domain by imposing slip along each of the six fault segments. Along each fault, we prescribe constant slip to produce either pure left-lateral or right-lateral (depending on the fault) strike-slip motion. Faults with strike trending NW-SE are assumed to slip right-lateral, while those trending NE-SW are taken to be left-lateral (Barnhart et al. 2020). A schematic of the fault system with the imposed slip boundary conditions is shown in Figure 15. We further assume that the material is homogeneous with a Poisson ratio of $\nu = 0.25$ and shear modulus $\mu = 36 \text{ GPa}$.

The mesh is constructed such that the spatial resolution (element edge length) on the faults is $250 \text{ m}$. We further specified that the spatial resolution on the domain boundaries be $20 \text{ km}$. With these specifications, the mesh generated contained $421,154$ tetrahedra. A zoom (with cut-away) of the local mesh refinement near the fault system is shown in Figure 16. In all experiments, we used the multigrid preconditioner described in Section 5.3. For simulations employing $N = 1$ (the case where polynomial coarsening cannot be invoked), the multigrid preconditioner reduces to using the algebraic multigrid package GAMG from PETSc.

The instantaneous displacement obtained using a polynomial of degree $N = 3$ for the displacement unknown and a non-affine mesh geometry with polynomial degree $N = 2$ is shown in Figure 17. Due to the imposed values of slip with $S_1 > S_6$, $k = 2, \ldots, 6$, the displacement is primarily dominated by the slip from segment $S_1$ (see Figure 15). We note that despite the simplicity in terms of initial slip conditions of our demonstration model, the resulting displacements are in overall agreement (qualitatively) with more advanced inferences (e.g., Fialko & Jin 2021).

The discontinuous nature of the displacement approximation is apparent when inspecting the horizontal components $u_0, u_1$ (Figure 17 (a), (b)). This is expected and consistent with the imposed slip boundary conditions. What is note worthy is that no numerical artefacts appear at the intersecting points of segments $S_2-S_1$ and $S_2-S_3$. In contrast, the vertical component of displacement is nearly continuous due to the purely vertical fault orientation, homogeneous material properties and the pure strike-slip loading. Several components of the displacement gradient are shown in Figure 18. In the upper panels (a), a low degree polynomial was used for the displacement ($N = 1$) with an affine mesh. The lower panels (b) employed a displacement polynomial of $N = 3$ with a non-affine
geometry representation using $N = 2$. For the $N = 1$ scenario, the slip vector imposed is defined such that it is tangential to the affine cell facets which approximate the fault boundary. In the high-order scenario, since the fault is discretised via a curvilinear representation, the slip imposed is tangential to the Bezier spline line used to represent the fault geometry.

Comparing the gradient fields between (a) and (b) we note that while the discontinuities are captured by the low order displacement solution $(u_{0,x}, u_{0,y}, u_{0,z})$ significant numerical oscillations close to the fault (thin black line) are apparent. These oscillations vanish when using the higher degree displacement and geometry representation.

It is also striking that the low order discretisation is quite inaccurate in approximating the smooth field $u_{0,x}$. This is partially exaggerated by the fact that we constructed the mesh with coarse resolution off-fault; regardless, the benefits of using high-order approximations is evident: The high-order simulations resolve the singularity in the strain tensor which occurs at the end of each fault. While such singularities are the result of our idealized and somewhat unphysical model (i.e. the slip doesn’t taper off towards the end of the fault) our exemplary results demonstrate that the DG method is robust with respect to such loading conditions which might occur (and be physically meaningful): (i) when a fault intersects the free surface or terminates at a material interface with increased rigidity; (ii) when two faults intersect each other; (iii) at the location of a fault branch. Additionally we highlight that no special treatment was required to accommodate the fault intersections (i.e. between segments 1–3 and segments 2–3) – the DG solution, and its gradient, is well behaved at these locations.

10 DISCUSSION

In the following, we discuss the numerical results with respect to high-order convergence, give a qualitative comparison to the boundary element method, and describe how an extension to a fully-dynamic SEAS modelling method might be obtained.

10.1 A high-order method for seismic cycling problems

We measure high-order convergence rates for elastostatic problems with a slip boundary condition, curved boundary, heterogeneous material parameters, and for the time-dependent manufactured SEAS problem, cf. Sections 8.1 and 8.2. For smooth problems the high-order convergence pays off as the error decays much quicker in comparison to low-order rates, cf. the discussion in Section 8.2 and Figure 7.

A requirement for high-order convergence is that the solution is sufficiently smooth, meaning that higher-order (weak) derivatives exist. We see in Figure 3 that the convergence order is limited to 0.5 in the $|\cdot|_{L,0}$ semi-norm, due to a singularity at the origin in the second derivative for the case $\gamma = 0.5$. Therefore, the results in Figure 7 model only the case ideal for high-order methods. In actual SEAS problems, the smoothness of the solution is unknown.

For non-smooth solutions, we observe that the error is still reduced by a constant factor when comparing a higher-order method to a lower-order method (cf. Wollherr et al. (2018)’s discussion for dynamic rupture problems), thus the additional degrees of freedom per element of the higher-order method are not in vain. Moreover, in combination with unstructured meshes we can locally refine the mesh. That is, small mesh sizes can be used in non-smooth regions while large mesh sizes are sufficient in smooth regions, see Figure 9.

10.2 A qualitative comparison of DG versus BEM

The use of boundary integral equations is widespread in the SEAS community (Lapusta et al. 2000; Lapusta & Liu 2009; Liu & Rice 2005; Segall & Bradley 2012; Bradley 2014; Li & Liu 2016, 2017; Luo et al. 2017; Barbot 2019) and is also used in earlier work (e.g., Rice 1993). Here, we qualitatively compare the presented DG method to BEM for SEAS applications.
We first recall background and definitions of the BEM. For elastostatics in homogeneous and isotropic media, a necessary ingredient is the problem

\[
\frac{\partial^2 E_{ij}(x',x)}{\partial x_k \partial x_l} + (\lambda + \mu) \frac{\partial^2 E_{kl}(x',x)}{\partial x_i \partial x_j} = -\delta(x' - x) \delta_{ij}, \quad (87)
\]

The tensor \( E \) that solves (87) is called the fundamental solution or Kelvin’s solution and is well-known (Chen & Zhou 2010). The columns of \( E \) are interpreted as the displacement at \( x' \) due to a point force at \( x \). From the fundamental solution one obtains the Betti-Somigliana equation (Chen & Zhou 2010), i.e. for any interior point \( x' \in \Omega \) we have

\[
u_i(x') = \int_{\partial \Omega} E_{ij}(x',x) [t_j(u)](x) - T_{ij}(x',x) u_j(x) \, ds_x, \quad (88)
\]

where \( t \) is the traction vector given \( u \) and the \( j \)-th column of \( T \) is the traction vector given the \( j \)-th column of \( E \). Equation (88) is the basis for deriving BEMs with slip boundary condition (e.g., Portela et al. 1992), but the details are not important for this discussion.

Inspecting (88) reveals that we need both Dirichlet and Neumann data on the boundary in order to compute \( u \). As we typically know either one or the other, (88) can be slightly adapted such that it is valid for \( x' \in \partial \Omega \) (Portela et al. 1992; Chen & Zhou 2010). Then we can discretise the boundary and solve the boundary integral equations for the discrete traction given Dirichlet data (or vice versa when Neumann data is given).

Eventually, both the DG method and the BEM require the solution to a linear system \( A u = b \), although the matrices \( A \) are quite different. For DG, \( A \) is sparse and, assuming a uniform grid, the number of degrees of freedom scales with \( n_1^D \), where \( n_1 \) is the number of line elements. For BEM, \( A \) is dense and the number of degrees of freedom scales with \( n_1^{D-1} \), because only the boundary needs to be discretised. Recalling that the complexity of exactly solving (e.g by LU factorisation) an \( n \times n \) sparse linear system is \( \mathcal{O}(n^2) \) whilst the complexity for an \( n \times n \) dense linear system is \( \mathcal{O}(n^3) \), we can estimate a complexity of \( \mathcal{O}(n_1^{D-1}) \) for DG and a complexity of \( \mathcal{O}(n_1^D) \) for BEM. That is, we observe that BEM has a lower complexity in 2D, but actually the complexity is identical in 3D, at least using this simple estimate.

In the previous paragraph we provided pessimistic bounds which would result from straightforward implementations of the DG method and BEM. Specializations of specific components of each method exist which reduce the algorithmic complexity and potentially alter the cost of each approach. We outline several enhancements below:

- The complexity of solving the sparse DG system can be reduced to \( \mathcal{O}(n) \) when iterative solvers employing multigrid are used. However, in practice achieving perfect \( \mathcal{O}(n) \) scaling is somewhat problem dependent and hard to achieve in a consistent manner across a wide range of polynomial degrees (e.g., from \( N = 2 \)– 8).
- For BEM, a H-matrix approximation (Bradley 2014) might reduce the storage and solve cost (Hackbusch & Khoromska 2000).
- The cost of evaluating the integral operator (e.g. a matrix-vector product) required by integral methods can be reduced to \( \mathcal{O}(n) \) using the fast-multipole method (FMM) (Rokhlin 1985; Greengard & Rokhlin 1987; Carrier et al. 1988; Romanet et al. 2018). Additionally, large scale parallel implementations of FMM exist (Cruz et al. 2011; Yokota et al. 2011). We note that while FMM provides fast application of the kernels, it does not define a “solve” (e.g., \( A u = b \)). Hence when a FMM-BEM formulation requires the solution of a linear system, a preconditioner will be required to reduce the time complexity.
- Using the approach of the spectral boundary integral method, the cost of evaluating the integral operator can be reduced to \( \mathcal{O}(n \log n) \) per fault segment as stress interactions are evaluated in the Fourier domain and these computations can exploit a fast Fourier transform (FFT) (Geubelle & Rice 1995; Perrin et al. 1995; Bouchon & Streiff 1997; Lapusta & Liu 2009; Barbot 2021). Note that this specialization does limit the methods applicability to planar faults. Similarly to FMM (above), this fast integral operator evaluation does not replace the need for a solve.

In practice, one needs to consider many more aspects, e.g. which
Figure 18. Gradient of the $x$-component of displacement ($u_0$) ($x, y$ – horizontal directions, $z$ – vertical direction) in the vicinity of the fault network using different polynomial approximations. (a) $u_h$ approximated with polynomial degree $N = 1$ over an affine mesh (fault geometry given by $N = 1$). (b) $u_h$ approximated with polynomial degree $N = 3$ on a non-affine mesh with the fault geometry approximated by a polynomial of degree $N = 2$. The faults are indicated by the thin black lines.

The computational cost of a time-step using the discrete Green’s function, cf. Section 7, is comparable to the cost of a time-step using analytical Green’s function, because only matrix multiplication is needed in both cases. Clearly, the discrete Green’s function is expensive to obtain, as a linear system needs to be solved for every degree of freedom on-fault, but they may be computed for any fault geometry, fault network, and topography supported by our method. We note that computing the discrete Green’s function is an embarrassingly parallel problem.

10.3 Extension to a fully-dynamic method

In this work we adopt the quasi-dynamic approach and neglect inertia. We see this as the first step towards a future SIPG implementation of the fully-dynamic approach. In particular, the bilinear operator of the SIPG method can be directly employed in a wave-propagation – i.e. fully-dynamic – scheme (Mazzieri et al. 2013). Moreover, the linear solvers we develop for the quasi-dynamic approach can be readily adapted for implicit time-stepping in a fully-dynamic model.

The extension of the SIPG method for elastostatics to wave propagation problems is straightforward. The elastodynamic equations are given by

$$\rho \frac{\partial^2 u_i}{\partial t^2} - \frac{\partial \sigma_{ij}(u)}{\partial x_j} = f_i.$$  \hspace{1cm} (89)

The SIPG method for the above equation is formulated as following (Grote et al. 2006; Mazzieri et al. 2013): Find $u_h \in [0, t_{end}] \times$
DG for SEAS on unstructured grids

$$V_h \rightarrow \mathbb{R} \text{ such that}$$

$$\int_\Omega \rho v \cdot \frac{\partial^2 u_h}{\partial t^2} \, dx + a(u_h, v) = L(v),$$

(90)

for all \( v \in V_h, t \in [0, t_{\text{end}}] \), where the bilinear form \( a(\cdot, \cdot) \) and the linear form \( L(\cdot) \) are defined in (23). Written in terms of linear algebra, we have

$$M \frac{\partial^2 u}{\partial t^2} + A u = b,$$

(91)

where \( A \) and \( b \) are constructed identically as in Section 5, and \( M \) is the block-diagonal mass matrix weighted with the density \( \rho \). Low-storage schemes for the inverse mass matrix are discussed by Warburton (2013) and Chan et al. (2017) for curvilinear meshes.

Explicit time-stepping is subject to a CFL condition (Grote et al. 2006), rendering explicit time-stepping infeasible for the time spans considered in SEAS models. One might circumvent the time-step restrictions by switching between fully-dynamic solvers in the co-seismic phase and quasi-dynamic solvers in the aseismic phase (Lapusta et al. 2000; Duru & Dunham 2016; Romanet & Ozawa 2021). Using the same finite element space for both solvers is convenient as there is no need for interpolation.

Another possibility might be to treat the elastodynamic equation implicitly. For illustration, the classic Newmark scheme leads to the linear system that needs to be solved in every time step (Hughes 2000, Eq. 9.1.12)

$$(M + \beta \Delta t^2 A) u^{n+1} = b^{n+1} - A u^{n},$$

(92)

where the predictor \( u^{n+1} \) depends only on time-step \( n \). Here, linear solvers developed for the elastostatic problem can likely be reused, e.g. the multigrid solver, cf. Section 5.3. Whether such a scheme yields time-steps reasonable for SEAS models remains to be explored.

11 CONCLUSIONS

We present a symmetric interior penalty discontinuous Galerkin (SIPG) method to perform quasi-dynamic simulations of sequences of earthquakes and aseismic slip (SEAS). Through an extensive suite of analytic, manufactured, and code-verification examples for elastostatic and seismic cycle problems, we characterise the solution behaviour of the DG discretisation. Our verification suite demonstrates that high-order convergence of the discrete solution can be achieved in space (and time) for both elastostatic and SEAS problems when the polynomial degree of the displacement is increased. We demonstrate that high-order convergence is maintained when either the material properties vary within a cell, or the fault or exterior boundary is curved provided both the coefficients and geometry are discretised with the same polynomial degree as used for the displacement. We exploit the curvilinear geometry representation in two demonstrator models to further elucidate the importance of using curvilinear fault representations, compared with piece-wise linear approximations (affine), when accurate stresses (or displacement gradients) are required on-fault.

We provide an open-source reference implementation, Tandem, which supports both 2D and 3D simulations of the seismic cycle, accounting for geologic heterogeneities and geometric complexity and arbitrary polynomial degree for the displacement, material properties and geometry. Tandem natively supports node-level and distributed memory parallelism. With awareness of the high computational demands of SEAS type problems, our reference implementation exploits modern, efficient libraries for the DG kernels and linear algebra, solver and preconditioner support. Further flexibility and efficiency is provided by optionally allowing users to define the displacement evaluation via the discrete Green’s function, which are evaluated and checkpoint once in an embarrassingly parallel pre-computation step using algorithmically optimal and scalable sparse parallel solvers and preconditioners. This strategy appears to exploit advantages of both the boundary integral and volumetric methods and is an interesting avenue to pursue in the future for extreme scale 3D SEAS simulations.

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DATA AVAILABILITY

The Jupyter notebook, setups, and data are openly available in Zenodo at https://doi.org/10.5281/zenodo.5796104. Instructions on how to repeat the presented numerical experiments are also contained in the Zenodo dataset. The code is available in GitHub at https://github.com/TEAR-ERC/tandem. Instructions on how to build and run the code are available at https://tandem.readthedocs.io. The setups in Section 8 are alternatively available from https://github.com/TEAR-ERC/tandem/blob/master/examples/. The Jupyter notebook which details the steps to construct the solution in Section 8.1.1 is alternatively available at https://github.com/TEAR-ERC/tandem/blob/main/notebooks/wedge.ipynb.

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**APPENDIX A: PROOF OF COERCIVITY AND BOUNDEDNESS**

In the following we prove Theorem 4.1. Several definitions are
required before we proceed. For matrices $A, B$ let $A : B = A_{ij}B_{ij}$. Given a vector-valued function $m$ and a matrix-valued function $M$, on a facet $e$ we will denote inner products via $\langle m e \rangle = \int_E m \cdot m \, ds$ and $\| M e \| = \int_E M : M \, ds$. Inner prod-
cuts over a volume $E$ will be denoted as $\| m \|_M^2 = \int_E m \cdot m \, dx$ and $\| M \|_M^2 = \int_E M : M \, dx$.

For improved readability of this section, we abbreviate the ten-
sor contraction $c_{ijkl} \partial u_k / \partial x_i$ with $c \nabla u$. Using $\sigma = c \nabla u$ the bilinear form is

$$a(u, u) = \int_{E \in T_h} \int_E \sigma : \nabla u \, dx + \sum_{e \in T_h^{\partial \Omega} \cup T_h} \int_e \delta [u] \cdot [u] \, ds - 2 \sum_{e \in T_h^{\partial \Omega} \cup T_h} \int_e \langle |\sigma n| \rangle \cdot [u] \, ds \quad (A.1)$$

Note that the first two terms are equivalent to $\| u \|_0^2$, hence it is
sufficient to find a lower bound for the last term.

We bound the last term in (A.1) using the Cauchy-Schwarz in-
quality:

$$\int_e \langle \sigma n \rangle \cdot [u] \, ds \leq \| \sigma n \|_1 \| [u] \|_1 \quad (A.2)$$

Using the definition of the average, the triangle inequality, and re-
calling that $|n| = 1$ we have
\[ \| \sigma_n \|_\infty \leq \frac{1}{2} \| c \nabla u \|_{E^1_1} + \frac{1}{2} \| c \nabla u \|_{E^2_2}. \] (A.3)

We denote the square root of stiffness tensor $c$ with $c^{1/2}$. The square root has the property $c^{1/2} c^{1/2} = c$. For an isotropic stiffness tensor the square root can be shown to be given by $c^{1/2}_{ijkl} = a \delta_{ij} \delta_{kl} + b (\delta_{il} \delta_{jk} + \delta_{ik} \delta_{jl})$, where $\lambda = Da^2 + 4ab$ and $\mu = 2b^2$.

Using the square root $c^{1/2}$, $c \nabla u = c \varepsilon(u)$, and property (27) we have on element $E$
\[ c \nabla u : c \nabla u \leq (c^{E})^2 \varepsilon(u) : \varepsilon(u). \] (A.4)

Conversely, we have
\[ c_0 \varepsilon(u) : \varepsilon(u) \leq \nabla u : c \nabla u = c^{1/2} \nabla u : c^{1/2} \nabla u. \] (A.5)

We proceed with bounding $\| \sigma_n \|_\infty$ using (A.4). As $X_E$ is assumed to be affine we have $\varepsilon(u) \in [P_{N-1}(E)]^{D \times D}$ and (24) can be applied:
\[ \| \sigma_n \|_\infty \leq \frac{E^2_1}{2} \| \varepsilon(u) \|_{E^1_1} + \frac{E^2_2}{2} \| \varepsilon(u) \|_{E^2_2} \]
\[ \leq \frac{E^2_1}{2} \frac{\sqrt{\alpha_{e,1}}}{2} \| \varepsilon(u) \|_{E^1_1} + \frac{E^2_2}{2} \frac{\sqrt{\alpha_{e,2}}}{2} \| \varepsilon(u) \|_{E^2_2}, \] (A.6)

where $\alpha_{e,i} := (N(N-1+D)/D)(c_i/|E_i^c|)$. With (A.5) we have
\[ \| \sigma_n \|_\infty \leq \frac{\sqrt{\beta_{e,1}}}{2} \| c^{1/2} \nabla u \|_{E^1_1} + \frac{\sqrt{\beta_{e,2}}}{2} \| c^{1/2} \nabla u \|_{E^2_2}, \] (A.7)

where $\beta_{e,i} := \alpha_{e,i} (c^{E_i^c})^2/c_0$. Similarly, we get for boundary facets
\[ \int_{\partial E} \| \sigma_n \| \cdot |u| \, ds \leq \frac{\sqrt{\beta_{e,1}}}{2} \| c^{1/2} \nabla u \|_{E^1_1} \| |u| \|_{E_1^1}. \] (A.8)

Summing over all edges and applying the Cauchy-Schwarz inequality we obtain
\[ \sum_{e \in \Gamma \cup \Gamma D} \int_{\partial e} \| \sigma_n \| \cdot |u| \, ds \leq \left( (D + 1) \sum_E \left\| c \nabla u \right\|_{E_1^1}^2 \right)^{1/2} \]
\[ \times \left( \sum_{e \in \Gamma \cup \Gamma D} \frac{\beta_{e,1}}{4} \| |u| \|_{E_1^1}^2 + \sum_{e \in \Gamma \cup \Gamma D} \frac{\beta_{e,2}}{4} \| |u| \|_{E_1^1}^2 + \sum_{e \in \Gamma \cup \Gamma D} \beta_{e,1} \| |u| \|_{E_1^1}^2 \right)^{1/2}. \] (A.9)

The inequality $ab \leq \zeta a^2/2 + b^2/(2\zeta)$ with $\zeta > 0$ (which follows from Young’s inequality) is used to obtain the final bound
\[ \sum_{e \in \Gamma \cup \Gamma D} \int_{\partial e} \| \sigma_n \| \cdot |u| \, ds \leq \frac{\zeta}{2} \sum_{E} \left\| c^{1/2} \nabla u \right\|_{E_1^1}^2 \]
\[ + \frac{D + 1}{2\zeta} \left( \sum_{e \in \Gamma \cup \Gamma D} \frac{\beta_{e,1}}{4} \| |u| \|_{E_1^1}^2 + \sum_{e \in \Gamma \cup \Gamma D} \frac{\beta_{e,2}}{4} \| |u| \|_{E_1^1}^2 + \sum_{e \in \Gamma \cup \Gamma D} \beta_{e,1} \| |u| \|_{E_1^1}^2 \right). \] (A.10)