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A mixed $RT_0 - P_0$ Raviart-Thomas finite element implementation of Darcy Equation in GNU Octave

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Abstrak
*In this paper we shall describe mixed formulations -differential and variational- of Darcys flow equation, an important model of elliptic problem. We describe * Galerkin method with finite dimensional spaces; * Local matrices and assembling; * Raviart-Thomas $RT_0 - P_0$ elements; * Edge basis and local matrices for $RT_0 - P_0$ FEM; * Model problem with corresponding local matrices, right hand side and treatment of boundary conditions. A simple demo written in GNU Octave is given.*
Kata kunci: Persamaan Darcy, Aliran di bahan berpori, Flux, Kekekalan Local, Metode Elemen Hingga Campuran

Abstract
*In this paper we shall describe mixed formulations -differential and variational- of Darcys flow equation, an important model of elliptic problem. We describe * Galerkin method with finite dimensional spaces; * Local matrices and assembling; * Raviart-Thomas $RT_0 - P_0$ elements; * Edge basis and local matrices for $RT_0 - P_0$ FEM; * Model problem with corresponding local matrices, right hand side and treatment of boundary conditions. A simple demo written in GNU Octave is given.*
Keywords: Darcy flow, Flow in porous media, Flux, Local conservation, Mixed finite element methods

Contents

1. Introduction

This report describes basis of RT1 code, which can be characterized as a code for testing solvers and preconditioners for FEM systems arising from lowest order Raviart-Thomas discretization of Darcy flow problems, see also [2] [1]

The code is characterized by simplicity and possibility of easy modifications,

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- directly solving model problems on square domains (generalization possible),
- stochastic generation of heterogeneity,
- fast system assembling using vectorization and sparse reconstruction,
- possible testing of Krylov type solvers with both (block) matrix and matrix free (variable) preconditioners.

This report describes the finite element system generation, experiments are involved in papers, e.g. [3].

2. Problem formulation

Let us consider Darcy flow elliptic problem in the form

$$\begin{array}{rcl} -\mathrm{div}(k(-g + \text{ grad } p) & = & f \text{ in } \Omega \\ p & = & \hat{p} \text{ on } \Gamma_D \\ (-k \text{ grad } p)\cdot n & = & \hat{u} \text{ on } \Gamma_N \end{array}$$

where $g \neq 0$ if we consider elevation changes. It can be also written in a two field form with two basic variables $p : \Omega \to R^1$ and $u : \Omega \to R^n$,

$$\left.\begin{array}{lcl} k^{-1}u + \text{ grad } p & = & g \\ \mathrm{div}(u) & = & f \end{array}\right\} \text{ in } \Omega$$

$$\begin{array}{lcl} p & = & \hat{p} \text{ on } \Gamma_D \\ (-k \text{ grad } p)\cdot n & = & \hat{u} \text{ on } \Gamma_N \end{array}$$

The variational formulation uses test functions v and q to get

$$\begin{array}{lcl} \int_\Omega k^{-1}u\cdot v dx + \int_\Omega \nabla p \cdot v dx & = & \int_\Omega g\cdot v dx \\ \int_\Omega \mathrm{div}(u) q & = & \int_\Omega f q dx \end{array}$$

Transformation of one mixed term then provides

$$\begin{array}{rcl} \int_\Omega \nabla p \cdot v & = & \int_\Omega \sum_k \frac{\partial p}{\partial x_k} v_k dx = \sum_k \{\int_{\partial\Omega} p v_k \cdot n_k - \int_\Omega p \frac{\partial v_k}{\partial x_k} dx\} \\ & = & \int_{\partial\Omega} p(v\cdot n) - \int_\Omega p \mathrm{div}(v) dx \end{array}$$

Then the variational formulation gets the form

$$\begin{array}{lcll} \int_\Omega k^{-1}u\cdot v - \int_\Omega \mathrm{div}(v)\cdot p & = & \int_\Omega g\cdot v dx - \int_{\Gamma_D} \hat{p}(v\cdot n) - \int_{\Gamma_N} p(v\cdot n) & \forall v \\ \int_\Omega \mathrm{div}(u) q & = & \int_\Omega f q & \forall q \end{array}$$

–or in abstract form: find $(u, p) \in U_N \times P$

$$\begin{array}{lcll} m(u,v) + b(v,p) & = & G(v) & \forall v \in U_0 \\ b(u,q) & = & F(v) & \forall q \in P \end{array}$$

where

$$\begin{array}{lcl} U & = & \{v \in L_2(\Omega)^n : \mathrm{div}(v) \in L_2(\Omega)\} \to H(\mathrm{div}) \\ U_0 & = & \{v \in U : v\cdot n = 0 \text{ on } \Gamma_N\} \\ U_N & = & \{v \in U : v\cdot n = \hat{u} \text{ on } \Gamma_N\} \\ P & = & \{q \in L_2(\Omega)\} \end{array}$$

Note that pressure BC enters $G(v) = ... - \int_{\Gamma_0} \hat{p}(v\cdot n)$ whereas velocity BC are included in U_N.

3. Galerkin method -Mixed FEM

We start with introducing FEM spaces $U_h \subset U, U_{N_h} \subset U_N, U_{0h} \subset U_0$ and $P_h \subset P$.

Then the Galerkin method is to find $(u_h, p_h) \subset U_{hN} \times P_h$

$$\begin{array}{lcll} m(u_h, v_h) + b(v_h, p_h) & = & G(v_h) & \forall v_h \in U_{0h} \\ b(u_h, q_h) & = & F(q_h) & \forall p_h \in P_h \end{array}$$

After a choice of bases

$$\begin{array}{rcl} U_h & = & \text{lin}\{\Phi_i, i \in I\}, P_h = \text{lin}\{\Psi_j : j \in J\} \\ U_{N_h} & = & u_N + u, u \in U_{0h} \\ U_{0h} & = & \text{lin}\{\Phi_i : i \in I_0\} \\ u_N & \in & \text{lin}\{\Phi_i : i \in I \backslash I_0\}, u_N = \sum (\hat{u} \cdot n)(x_i)\Phi_i \end{array}$$

the discrete mixed problem can be written as -find $(u_h, p_h) \in U_{hN} \times P_h, \quad u_h = u_N + \sum_{i \in I_0} \alpha_i \Phi_i, \quad p_h = \sum_{j \in J} \beta_j \Psi_j$

$$\begin{array}{lcll} \sum_{i \in I_0} \alpha_i m(\Phi_i, \Phi_k) + \sum_{j \in J} \beta_j b(\Phi_k, \Psi_j) & = & G(\Phi_k) - m(u_N, \Phi_k) & \forall k \in I_0 \\ \sum_{i \in I_0} \alpha_i b(\Phi_i, \Psi_l) & = & F(\Psi_l) - b(u_N, \Psi_l) & \forall l \in J \end{array}$$

Rewriting to matrix form provides

$$\begin{array}{lcll} B\underline{\alpha} + B^T \underline{\beta} & = & G, & \underline{\alpha} \in R^{n_1}, \quad n_1 = \# I_0 \\ B\underline{\alpha} & = & F, & \underline{\beta} \in R^{n_2}, \quad n_2 = \# J \end{array}$$

where $M \in R^{n_1 \times n_1}, M_{ij} = m(\Phi_j, \Phi_i), B \in R^{n_2 \times n_1}, B_{ij} = b(\Phi_j, \Psi_i), B^T \in R^{n_1 \times n_2}, B^T_{ij} = b(\Phi_i, \Psi_j) = B_{ji}, G = (G_i), G_i = G(\Phi_i), F = (F_k), F_k = F(\Psi_k)$.

4. Lowest order Raviart-Thomas finite elements

Let $\Omega \in R^2$ be a 2D polygonal domain, $\mathcal{T}_h$ be its triangulation, $\mathcal{E}_h$ be set of edges of all elements $T \in \mathcal{T}_h$ see the situation in the following Figure 1.

Then, we can define

$$RT_0(T) = \{v : T \to R^2, v(x) = \xi [x_1 \; x_2]^T + [\eta_1 \; \eta_2]^T, \xi, \eta_1, \eta_2 \in R\}$$

$$\begin{array}{rcl} U_h & = & \{v : \Omega \in R^2, v|_T \in RT_0(T) \quad \forall T \in \mathcal{T}_h, v \cdot n_E \text{ is continuous over } E \in \mathcal{E}_h\} \\ P_h & = & \{q : \Omega \in R^1, q|_T \text{ is constant} \quad \forall T \in \mathcal{T}_h\}. \end{array}$$

Continuity of $\nu \cdot n_E$ guarantees $U_h \in U, P_h \in P$ is obvious. Note that $\forall E \in \mathcal{E}_h$ we define n_E (unit normal vector), independently of relation to triangles and consequently in possibly inner or outer direction, see Figure 2.

6 Local properties and local edge basis for RT(0) elements

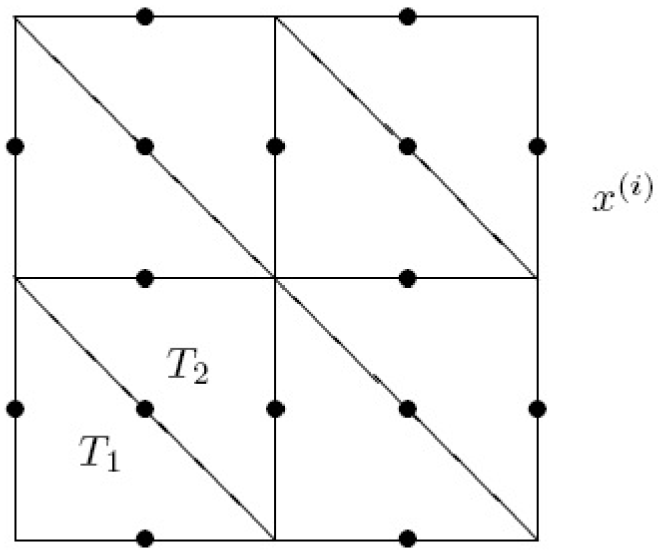
Lemma 4.1. *Let $T \in \mathcal{T}_h, v \in RT_0(T)$. Then $\forall E \in \mathcal{E}_h \cup \partial T : v \cdot n|E = const.$*

Proof. Let $E \in \mathcal{E}_h \cup \partial T, n_E$ be normal to E (can be either outer or inner to T), $x^* \in E$ be arbitrary point at E. Then

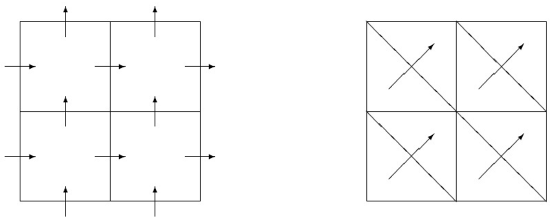
$$x \in E \Rightarrow (x - x^*) \cdot n_E = 0, n_E = (n_1, n_2) \Rightarrow x_1 n_1 + x_2 n_2 = x^*_1 n_1 + x^*_2 n_2 = const. \Rightarrow$$

$$v(x) \cdot n = \xi x_1 n_1 + \xi x_2 n_2 + \eta_1 n_1 + \eta_2 n_2 = \xi(x^*_1 n_1 + x^*_2 n_2) + \eta_1 n_1 + \eta_2 n_2 = \text{const} .$$

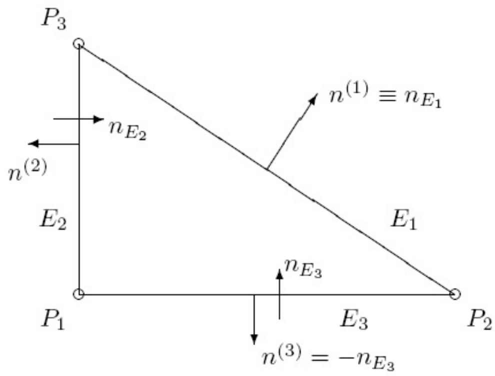
□



GAMBAR 1. $\{x^{(i)}\}$ set of centres of $E_i \in \mathcal{E}_h$, $\{y^{(j)}\}$ barycentres of $T_j \in \mathcal{T}_h$



GAMBAR 2. Prescribed normal n_E. Possible definition of $n_E, E \in \mathcal{E}_h$.



GAMBAR 3. Triangle $T \in \mathcal{T}_h$.

Lemma 4.2. *(Expression for local basis functions.) Let*

$$\Phi_i(x) = \sigma_i \frac{E_i}{2|T|}(x - P_i), \sigma_i = n_{E_i} n^{(i)},$$

where n_{E_i} are global prescribed normals and $n^{(i)}$ are outer normals for $T \in \mathcal{T}_h$, see Figure 3. Then

(i) $\Phi_j(x) \cdot n_{E_i} = \delta_{ij}$,
(ii) $\Phi_i \in RT_0(T)$,
(iii) Φ_1, Φ_2, Φ_3 *create a basis of* $RT_0(T)$,
(iv) $div\Phi_i = \sigma_i \frac{E_i}{|T|}$.

Proof. (i) If $i \neq j$, then $P_i \in E_j$ and $(x - P_i) \cdot n_{E_j} = 0$ for $x \in E_j$. If $i = j$ then for $x \in E_i$ the value $(x - P_i) \cdot n_{E_i}$ appears in the projection of $(x - P_i)$ to the height of T passing through P_i and therefore $|(x - P_i) \cdot n_{E_i}| = h_i$. Moreover, $\frac{1}{2} h_i |E_i| = |T|$ and $h_i = 2|T|/|E_i|, (x - P_i) \cdot n^{(i)} \geq 0$ -both vectors have outward direction w.r.t. T. Finally

$$(x - P_i) \cdot n_{E_i} = \sigma_i \frac{2|T|}{|E_i|}$$

(ii) obvious
(iii) $u \in RT_0(T), w = u - \sum_1^3 (u \cdot n_{E_i})\Phi_i$. Obviously $w \cdot n_{E_i} = 0 \forall E_i$. Therefore $\forall P_j$: $w(P_j) \cdot n_{E_i} = 0$ and because $\forall E_i : P_j \in E_i$, it holds $w(P_j) = 0 \forall j = 1, 2, 3$. As w is linear polynomial, $w = 0$. Proof of uniqueness:

$$w = \sum_1^3 \alpha_i \Phi_i = 0 \Rightarrow w n_{E_j} = \alpha_j \Phi_j n_{E_j} = \alpha_j = 0 \forall j.$$

(iv) obvious

□

5. Local matrices and assembling

Assume that Φ_i and Ψ_i are constructed as finite element basis functions above some triangulation $\mathcal{T}_h, i.e. T \in \mathcal{T}_h$

$$\Phi_i|_T \in \{\Phi_1, ..., \Phi_\rho, 0 = \Phi_0\}$$
$$\Psi_j|_T \in \{\Psi_1, ...\Psi_s, 0 = \Psi_0\}.$$

Then

$$\begin{aligned} m(\Phi_i, \Phi_k) &= \int_\Omega k^{-1} \Phi_i \Phi_k dx \\ &= \sum_{T \in \mathcal{T}_h} \int_T k^{-1} \Phi_i \Phi_k dx \\ &= \sum_{T \in \mathcal{T}_h} \int_T k^{-1} \Phi_{\text{loc}(i)} \Phi_{\text{loc}(k)} dx \\ b(\Phi_i, \Psi_j) &= \int_\Omega (\text{div}\Phi_i) \Psi_j dx \\ &= \sum_{T \in \mathcal{T}_h} \int_T (\text{div}\Phi_{\text{loc}(i)}) \Psi_{\text{loc}(j)} dx \end{aligned}$$

where $\text{loc}_k(i) = \text{loc}_k(i, T)$ is a transformation from global index to local index of basis function on T. It can be also zero.

Vice versa, for $T \in \mathcal{T}_h$, it is possible to construct local matrices

$$\begin{aligned} M_T, (M_T)_{rs} &= \int_T k^{-1} \Phi_s \Phi_r dx \\ B_T, (B_T)_{rs} &= -\int_T \text{div}\Phi_s \cdot \Psi_r dx \end{aligned}$$

and then perform the assembling of local matrices to global M, B

$$(M_T)_{rs} \to M_{\text{glob}(T,r)\text{glob}(T,s)} = +(M_T)_{rs}$$

$$(B_T)_{rs} \to B_{\text{glob}_1(T,r)\text{glob}_2(T,s)} = +(B_T)_{rs}$$

Note there are two sets of basis functions $\{\Phi_i\}, \{\Psi_i\}$, two sets of local basis functions $\{\Phi_i\}, \{\Psi_i\}$ and two mappings

$$\mathrm{loc}_1(i) = \mathrm{loc}_1(i,T), \mathrm{loc}_2$$
$$\mathrm{glob}_1(r,T) = i, \mathrm{glob}_2(s,T) = j.$$

6. Local matrices

Let us consider the local basis on T created by $\Phi_1, \Phi_2, \Phi_3 \in RT_0(T)$ and $\Psi_1 = 1$. Then $B_T \in R^{1\times 3}$,

$$(B_T)_{1s} = \int_T (\mathrm{div}\Phi_s)\Psi_1 = \sigma_s \frac{|E_s|}{|T|}|T| = \sigma_s|E_s|,$$

i.e. $B_T = [\sigma_1|E_1|, \sigma_2|E_2|, \sigma_3|E_3|] \in R^{1\times 3}$. Further, $M_T \in R^{3\times 3}$,

$$(M_T)_{rs} = \int_T k^{-1}\Phi_s\Phi_r dx = \sigma_r\sigma_s\frac{|E_r||E_s|}{4|T|^2}\int_T k^{-1}(x-P_s)\cdot(x-P_r)dx.$$

To compute the integral $\int_T k^{-1}(x-P_s)\cdot(x-P_r)dx$, we can use barycentric coordinates at T,

$$x = \lambda_1(x)P_1 + \lambda_2(x)P_2 + \lambda_3(x)P_3, \qquad \lambda_1 + \lambda_2 + \lambda_3 = 1,$$

thus

$$x - P_r = \lambda_1(x)(P_1 - P_r) + \lambda_2(x)(P_2 - P_r) + \lambda_3(x)(P_3 - P_r)$$

and

$$(M_T)_{rs} = \sigma_r\sigma_s\frac{|E_r||E_s|}{4|T|^2}\sum_{\alpha,\beta=1}^{3}\int_T \lambda_\alpha\lambda_\beta k^{-1}(P_\alpha - P_s)\cdot(P_\beta - P_r)dx.$$

Assuming k constant on T and using the integration formula $\int_T \lambda_\alpha\lambda_\beta = \frac{|T|}{12}(1+\delta_{\alpha\beta})$, which is a special case of

$$\int_T \lambda_1^a\lambda_2^b\lambda_3^c dx = \frac{a!b!c!}{(a+b+c+2)!}2|T|$$
$$\int_V \lambda_1^a\lambda_2^b\lambda_3^c\lambda_4^d dx = \frac{a!b!c!d!}{(a+b+c+d+3)!}6|V|$$

see e.g. [4, 5]

the elements of M_T can be expressed as

$$(M_T)_{rs} = \frac{1}{48|T|}\sigma_r|E_r|\sum_{\alpha,\beta=1}^{3}(1+\delta_{\alpha,\beta})k^{-1}(P_\alpha - P_s)\cdot(P_\beta - P_r)\sigma_s|E_s|$$

If we define vectors $v_r, v_s \in R^{6\times 1}$,

$$v_r = \begin{bmatrix} P_1 - P_r \\ P_2 - P_r \\ P_3 - P_r \end{bmatrix}, v_s = \begin{bmatrix} P_1 - P_s \\ P_2 - P_s \\ P_3 - P_s \end{bmatrix}, p_i = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Then

$$(M_T)_{rs} = \frac{1}{48|T|}\sigma_r|E_r|v_r^T \begin{bmatrix} 2 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 & 1 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 \\ 1 & 0 & 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} k^{-1} & & \\ & k^{-1} & \\ & & k^{-1} \end{bmatrix} v_s\sigma_s|E_s|$$

$$C := (\mathrm{df}) \begin{bmatrix} 2 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 & 1 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 \\ 1 & 0 & 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 2 \end{bmatrix}$$

Note that the diagonal elements are equal to elements of B_T. If we denote $C \in R^{6\times 6}$ the matrix, which appeared in the expression above and

$$V = [v_1, v_2, v_3] = \begin{bmatrix} 0 & P_1 - P_2 & P_1 - P_3 \\ P_2 - P_1 & 0 & P_2 - P_3 \\ P_3 - P_1 & 0 & P_3 - P_2 \end{bmatrix}$$

`\in R^{6\times 3}`

then

$$(M_T) = \frac{1}{48|T|} \begin{bmatrix} \sigma_1|E_1| & 0 & 0 \\ 0 & \sigma_2|E_2| & 0 \\ 0 & 0 & \sigma_3|E_3| \end{bmatrix} V^T C \begin{bmatrix} k^{-1} & & \\ & k^{-1} & \\ & & k^{-1} \end{bmatrix} V \begin{bmatrix} \sigma_1|E_1| & 0 & 0 \\ 0 & \sigma_2|E_2| & 0 \\ 0 & 0 & \sigma_3|E_3| \end{bmatrix}$$

`S \in R^{3\times 3}`
`L \in R^{6\times 6}`
`S`

i.e.

$$(M_T) = \frac{1}{48|T|} S V^T C L V S$$

where $S = \mathrm{diag}[b_1 E_1|, b_2 E_2, b_3 E_3], V = \begin{bmatrix} 0 & P_1 - P_2 & P_1 - P_3 \\ P_2 - P_1 & 0 & P_2 - P_3 \\ P_3 - P_1 & P_3 - P_2 & 0 \end{bmatrix}$,

$L = \begin{bmatrix} k^{-1} & & \\ & k^{-1} & \\ & & k^{-1} \end{bmatrix}^{-1} = \frac{1}{k_T} I$, if we consider the isotropic environment, $k = k_T I$ on T. For comparison see [2] formula (4.6).

Note that we constructed velocity mass matrix M. In the case of time dependent problems, we also need the pressure mass matrix $(M_T)_{rs} = \int_T \Psi_r \Psi_s = \delta_{rs}|T|$

7. Model problem

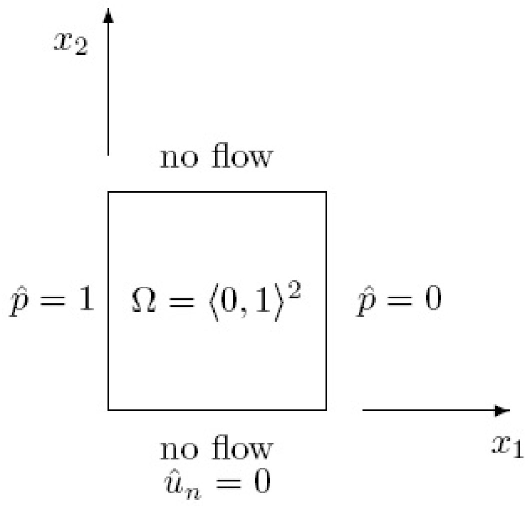
We shall consider a model Darcy flow problems on a square domain with flow from left to right induced by the pressure gradient.

The problem domain is divided into rectangular elements with the size characterized by the parameter ns = number of segments on the side.

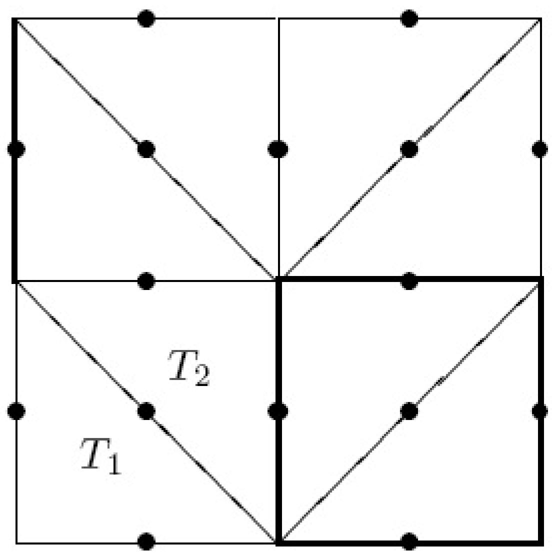
Heterogeneity. We assume that each cell can possess a different permeability coefficient $k_i, i = 1, ..., nc = (ns)^2$. This can be produced by MATLAB using command sequence

```
1 ) rng ( ' d e f a u l t ' ) ;
2 ) RM = randn ( ns , ns ) ;
3 ) LK = ( exp ( 1 ) . ^ ( sigma *RM) ) ;
```

The first command initializes the random number generator to make the results in this example repeatable. The same sequence is generated as after restart of MATLAB. The second command generate a ns-by-ns matrix of normally distributed random numbers from $N(0, 1)$, i.e. with mean $\mu = 0$ and standard deviation 1. Then $s{*}RM$ is a matrix of normally distributed



Gambar 4. Model problem



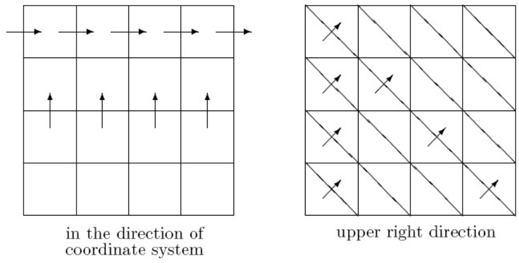
Gambar 5. Discretization of the model problem.

random numbers with the mean $\mu = 0$ and standard deviation σ^2. Third command then creates matrix of conductivities such that $\ln(LK)$ has normal distribution.

Orientation of (global) normals to element edges

Model problem -local matrices.

$$M_T = \frac{1}{24h^2} S V^T C L V S, L = \frac{1}{k_{\text{cell}}} I.$$



Gambar 6. Global normals.

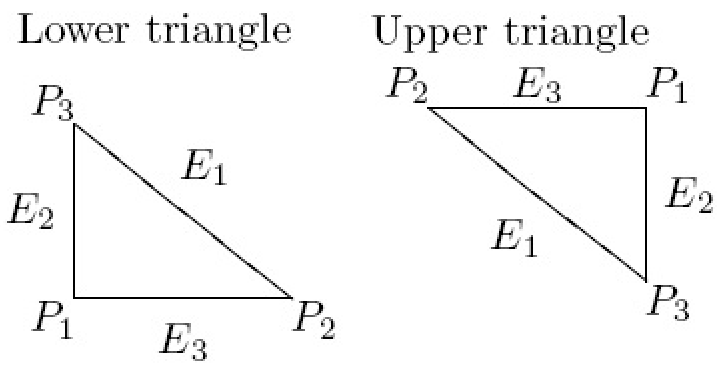
Lower Triangle

$$B_T = [\sqrt{2}h, -h, -h], S = h.\begin{bmatrix} \sqrt{2} & & \\ & -1 & \\ & & -1 \end{bmatrix}, V = h\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$

Upper triangle

$$B_T = [-\sqrt{2}h, h, h], S = h.\begin{bmatrix} -\sqrt{2} & & \\ & 1 & \\ & & 1 \end{bmatrix} = -S_{low}, V_{upper} = h\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & -1 & 0 \end{bmatrix} = -V_{low}.$$

As a conclusion -the matrices $M_T = \frac{1}{24h^2}SV^TCLVS$ are the same for both lower and upper



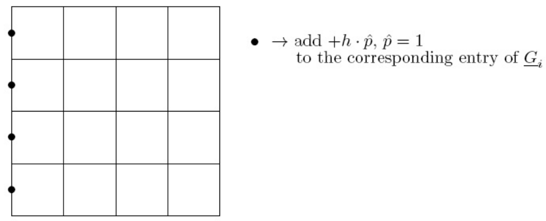
triangles.

Right hand side and boundary conditions. Consider the global system

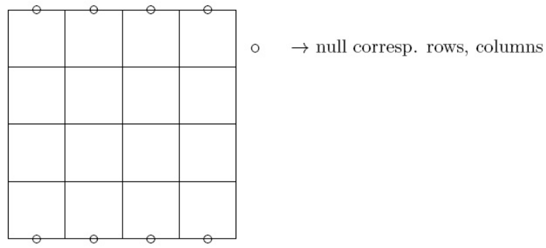
$$\begin{aligned} M\underline{\alpha} + B_T\underline{\beta} &= \underline{G} \\ B\underline{\alpha} &= \underline{F} \end{aligned}$$

where

$$G_i = -\int_{\Gamma_0} \hat{p}(p_i \cdot n) - \sum_{k\in I\setminus I_0} \hat{u}_k m(\Phi_k, \Phi_i)$$



Gambar 7. Pressure boundary conditions for the model problem.



Gambar 8. Treatment of velocity boundary conditions: a) exclude corresponding rows and columns and rhs entries, b) or put 1 on diagonal otherwise zeros in corresponding row, columns and rhs entries

```
r.h.s. contribution
l.h.s., in our case
```

$$F_j = -\int_\Omega f\Psi_j - \sum_{k \in I \setminus I_0} \hat{u}_k \int_\Omega \mathrm{div}(\Phi_k)\Psi_j dx = 0$$

```
= \int_{T_j} f =0 in our case
\int_{T_j} \mbox{div} (\Phi_k );  u_k are zero in our case
```

8. Assembling

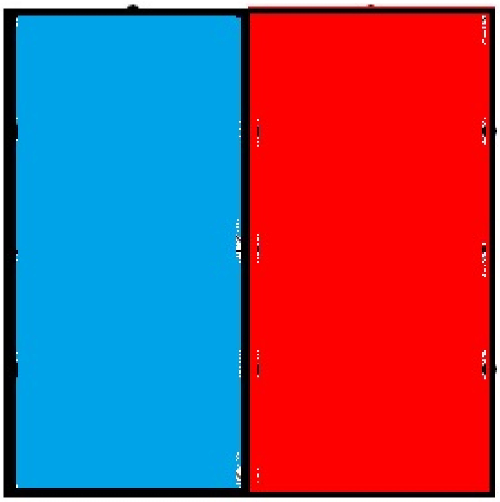
Standard assembling

```
Algorithm 1 Standard assembling
define M = 0,B = 0
for 1:nt
    take M_T ,B_T
    for r = 1,...,3
        for s = 1,2,3
            Mi(T,r) j(T,s) = (M_T)_{rs}
            B\kappa(T ) i(T,r) = (B_T )1r
        end
    end
end
```

The standard assembing has two drawbacks: **for** cycles, which are not efficient in `MATLAB`, and *dense* matrix storage of the global matrix. Just replacing the global matrix declaration as sparse is not a good solution as it the *sparse* structure is not given apriori but must be constructed during the assembling process. This inefficiency can be removed by gradual recording the nonzero components and indices into one dimensional vectors `X, I, J` and constructing the matrix through

$$\texttt{sparse}(\texttt{X}, \texttt{I}, \texttt{J}, \texttt{n}, \texttt{m}).$$

Further improvement and loop avoiding can be done by vectorization, see [6][6]. The resulting code is able fast assembly very large matrices.



GAMBAR 9. Transmissivity coefficient k, blue color $k = 1.0$, red color $k = 1.4$

9. Numerical Test

We test numerically an example from [7]. For simplicity, the exact u_σ in Ω is set to be $u_\sigma = cos(x - 0.5) * exp(y)$.

$$k = \sigma = \begin{cases} 1.0, & 0 < x < 0.5, 0 < y < 1 \\ 1.4, & 0.5 < x < 0, 0 < y < 1 \end{cases}$$

$$\partial_n u = \begin{cases} \partial_x u & = & sin(x - 0.5) * exp(y), 0 < y < 1, x = \{0, 1\} \\ \partial_y u & = & cos(x - 0.5) * exp(y), 0 < x < 1, y = \{0, 1\} \end{cases}$$

The program implemented in *GNU Octave* run in `octave-online.net,` which is a web UI for GNU Octave.

Acknowledgement.

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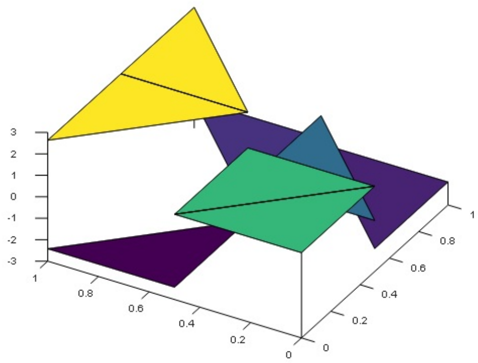
Daftar Pustaka

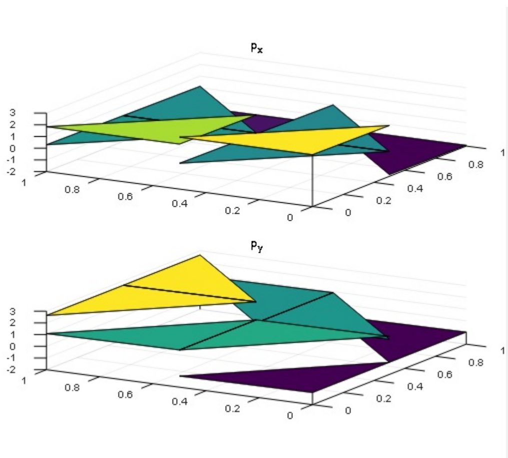
[1] Carstensen, C., 2009, Lectures on Adaptive Mixed Finite Element Methods. In: C. Carstensen, P. Wriggers: Mixed finite element technologies, CISM Udine, Courses and Lectures No.509, Springer, Wien

[2] Bahriawati, C. Carstensen C., 2005, Three MATLAB implementations of the lowest-order Raviart-Thomas MFEM with a posteriori error control. *Computational Methods in Applied Mathematics*, 5, pages 333-361.

[3] Axelsson, O, Blaheta, R., Byczanski, P., Kar atson J., and Ahmad, B,. 2015, Preconditioners for regularized saddle point operators with an application for heterogeneous Darcy flow and transport problems. *Journal of Computational and Applied Mathematics*, vol. 280, pages 141–157.



GAMBAR 10. Displacement



GAMBAR 11. Flux

[4] Akin, J.E., 2005, *Finite Element Analysis with Errror Estimators*, Wiley
[5] Braess, D. 2007., *Finite Elements: Theory, Fast Solvers, and Applications in Solid Mechanics.* Third ed., Cambridge University Press
[6] Chen, L., Programming of finite element methods in MATLAB, *http : www.math.uci.edu/ chenlong/226/Ch3FEMCode.pdf* on January 31,2020
[7] Faouzi Triki and Tao Yin., 2020,Inverse conductivity equation with internal data,*arXiv*, eprint= 2003.13638

LAMPIRAN

```
%    Program Darcy equation implementation based on
%    EBMFEM for 2D Raviart-Thomas mixed finite element method
%    based on the edge-oriented basis function
%
%    Agah D. Garnadi and C. Bahriawati
%
%    File <Darcy_EBmfem.m>
%
%    M-files you need to run
%       <stimaB.m>, <edge.m>, <f.m>, <u_D.m>, <u_N.m> (optional)
%
```

```
%    Data-(files) you need to prepare
%       coordinate <coordinate.dat>,
%       element    <element.dat>,
%       Dirichlet  <Dirichlet.dat>,
%       Neumann    <Neumann.dat> (optional)
%
%    This program and corresponding data-files is modified from
%    "Three Matlab Implementations of the Lowest-Order Raviart-Thomas
%     MFEM with a Posteriori Error Control" by C.Bahriawati and C. Carstensen

%
% A.1. The main program
% load coordinate.dat;
coordinate = [ 0  0; 0.5 0; 1  0; 1 0.5 ; 1 1; 0.5 1; 0 1; 0 0.5; 0.5 0.5];%
% load element.dat;
element = [2 8 1; 2 9 8 ; 2 4 9; 2 3 4; 9 4 5; 9 5 6; 9 6 7; 9 7 8];%
% load k_element.dat;
k_element =  [1 ; 1 ; 1.4 ; 1.4 ; 1.4 ; 1.4 ; 1 ; 1 ];%
% load dirichlet.dat;
dirichlet = [ 3 4; 4 5; 7 8 ; 8 1];
%load Neumann.dat;
Neumann = [1 2; 2 3; 5 6; 6 7];
%
[nodes2element,nodes2edge,noedges,edge2element,interioredge]=edge(element,coordinate);

% A.2. EBmfem
%function u=EBmfem(element,coordinate,dirichlet,Neumann,nodes2element,...
%         nodes2edge,noedges,edge2element);

% Assemble matrices B and C
 B=sparse(noedges, noedges);
 C=sparse(noedges,size(element,1));
for j = 1:size(element,1)
   coord=coordinate(element(j,:),:)';
   I=diag(nodes2edge(element(j,[2 3 1]),element(j,[3 1 2])));
   signum=ones(1,3);
   signum(find(j==edge2element(I,4)))=-1;
   B_element = k_element(j)*diag(signum)*stimaB(coord)*diag(signum);
   n=coord(:,[3,1,2])-coord(:,[2,3,1]);
   B(I,I)= B(I,I) + B_element ;
   C(I,j) = diag(signum)*[norm(n(:,1)) norm(n(:,2)) norm(n(:,3))]';
end
% Global stiffness matrix A
  A = sparse(noedges+size(element,1), noedges+size(element,1));
  A = [B ,          C,       ;
       C', sparse(size(C,2),size(C,2))];
% Volume force
b = sparse(noedges+size(element ,1),1);
for j = 1:size(element ,1)
  b(noedges+j)= -det([1,1,1; coordinate(element(j,:),:)']) * ...
                f(sum(coordinate(element(j,:),:))/3)/6;
end
% Dirichlet conditions
for k = 1:size(dirichlet,1)
  b(nodes2edge(dirichlet(k,1),dirichlet(k,2)))= norm(coordinate(dirichlet(k,1),:)-...
       coordinate(dirichlet(k,2),:))*u_D(sum(coordinate(dirichlet(k,:),:))/2);
end
% Neumann conditions
```

```
if ¬isempty(Neumann)
tmp=zeros(noedges+size(element,1),1);
tmp(diag(nodes2edge(Neumann(:,1),Neumann(:,2))))=...
   ones(size(diag(nodes2edge(Neumann(:,1),Neumann(:,2))),1),1);
FreeEdge=find(¬tmp);
x=zeros(noedges+size(element,1),1);
CN=coordinate(Neumann(:,2),:)-coordinate(Neumann(:,1),:);
for j=1:size(Neumann,1)
 x(nodes2edge(Neumann(j,1),Neumann(j,2)))=...
 g(sum(coordinate(Neumann(j,:),:))/2,CN(j,:)*[0,-1;1,0]/norm(CN(j,:)));
end
b=b-A*x;
x(FreeEdge)=A(FreeEdge,FreeEdge)\b(FreeEdge);
else
    x = A\b;
end
figure(1)
ShowDisplacement(element,coordinate,x);
p=fluxEB(element,coordinate,x,noedges,nodes2edge,edge2element);
figure(2)
ShowFlux(element,coordinate,p);
pEval=fluxEBEval(element,coordinate,x,nodes2edge,edge2element);
```

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