A mixed $RT_0 - P_0$ Raviart-Thomas finite element implementation of Darcy Equation in GNU Octave

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Abstract

In this paper we shall describe mixed formulations -differential and variational- of Darcy's flow equation, an important model of elliptic problem. We describe * Galerkin method with finite dimensional spaces; * Local matrices and assembling; * Raviart-Thomas $RT_0 - P_0$ elements; * Edge basis and local matrices for $RT_0 - P_0$ FEM; * Model problem with corresponding local matrices, right hand side and treatment of boundary conditions. A simple demo written in GNU Octave is given.

Keywords: Darcy flow, Flow in porous media, Flux, Kekekalan Local, Metode Elemen Hingga Campuran

1. Introduction

This report describes basis of RT1 code, which can be characterized as a code for testing solvers and preconditioners for FEM systems arising from lowest order Raviart-Thomas discretization of Darcy flow problems, see also [2] [1]

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The code is characterized by simplicity and possibility of easy modifications,
• directly solving model problems on square domains (generalization possible),
• stochastic generation of heterogeneity,
• fast system assembling using vectorization and sparse reconstruction,
• possible testing of Krylov type solvers with both (block) matrix and matrix free (variable) preconditioners.

This report describes the finite element system generation, experiments are involved in papers, e.g. [3].

2. Problem formulation

Let us consider Darcy flow elliptic problem in the form

\[-\text{div}(k(-g + \text{grad } p)) = f \text{ in } \Omega\]
\[p = \hat{p} \text{ on } \Gamma_D\]
\[(-k \text{ grad } p) \cdot n = \hat{u} \text{ on } \Gamma_N\]

where \(g \neq 0\) if we consider elevation changes. It can be also written in a two field form with two basic variables \(p : \Omega \rightarrow \mathbb{R}^1\) and \(u : \Omega \rightarrow \mathbb{R}^n\),

\[
\begin{align*}
    k^{-1}u + \text{grad } p &= g \quad \text{in } \Omega \\
    \text{div}(u) &= f \\
    p &= \hat{p} \text{ on } \Gamma_D \\
    (-k \text{ grad } p) \cdot n &= \hat{u} \text{ on } \Gamma_N
\end{align*}
\]

The variational formulation uses test functions \(v\) and \(q\) to get

\[
\begin{align*}
    \int_{\Omega} k^{-1}u \cdot v \, dx + \int_{\Omega} \nabla p \cdot v \, dx &= \int_{\Omega} g \cdot v \, dx \\
    \int_{\Omega} \text{div}(u)q &= \int_{\Omega} f q \, dx
\end{align*}
\]

Transformation of one mixed term then provides

\[
\begin{align*}
    \int_{\Omega} \nabla p \cdot v &= \int_{\Omega} \sum_k \frac{\partial p}{\partial x_k} v_k \, dx = \sum_k \{\int_{\Omega} p v_k \cdot n_k - \int_{\Omega} p \frac{\partial v_k}{\partial x_k} \, dx\} \\
    &= \int_{\Omega} p(v \cdot n) - \int_{\Omega} p \text{div}(v) \, dx
\end{align*}
\]

Then the variational formulation gets the form

\[
\begin{align*}
    \int_{\Omega} k^{-1}u \cdot v - \int_{\Omega} \text{div}(v) \cdot p &= \int_{\Omega} g \cdot v \, dx - \int_{\Gamma_D} \hat{p}(v \cdot n) - \int_{\Gamma_N} p(v \cdot n) \\
    \int_{\Omega} \text{div}(u)q &= \int_{\Omega} f q
\end{align*}
\]

or in abstract form: find \((u, p) \in U_N \times P\)

\[
\begin{align*}
    m(u, v) + b(v, p) &= G(v) \quad \forall v \in U_0 \\
    b(u, q) &= F(v) \quad \forall q \in P
\end{align*}
\]

where

\[
\begin{align*}
    U &= \{v \in L_2(\Omega)^n : \text{div}(v) \in L_2(\Omega)\} \rightarrow H(\text{div}) \\
    U_0 &= \{v \in U : v \cdot n = 0 \text{ on } \Gamma_N\} \\
    U_N &= \{v \in U : v \cdot n = \hat{u} \text{ on } \Gamma_N\} \\
    P &= \{q \in L_2(\Omega)\}
\end{align*}
\]

Note that pressure BC enters \(G(v) = \ldots - \int_{\Gamma_N} \hat{p}(v \cdot n)\) whereas velocity BC are included in \(U_N\).
3. Galerkin method - Mixed FEM

We start with introducing FEM spaces $U_h \subset U, U_{N_h} \subset U_N, U_{0h} \subset U_0$ and $P_h \subset P$.

Then the Galerkin method is to find $(u_h, p_h) \in U_{hN} \times P_h$

$$m(u_h, v_h) + b(v_h, p_h) = G(v_h) \quad \forall v_h \in U_{0h}$$

$$b(u_h, q_h) = F(q_h) \quad \forall p_h \in P_h$$

After a choice of bases

$$U_h = \text{lin}\{\Phi_i, i \in I\}, P_h = \text{lin}\{\Phi_j : j \in J\}$$

$$U_{N_h} = u_N + u, u \in U_{0h}$$

$$U_{0h} = \text{lin}\{\Phi_i : i \in I_0\}$$

$$u_N \in \text{lin}\{\Phi_i : i \in I \setminus I_0\}, u_N = \sum (u \cdot n)(x_i) \Phi_i$$

the discrete mixed problem can be written as - find $(u_h, p_h) \in U_{hN} \times P_h, \ u_h = u_N + \sum_{i \in I_0} \alpha_i \Phi_i, \ p_h = \sum_{j \in J} \beta_j \Phi_j$

Rewriting to matrix form provides

$$B_\Omega + B^T \beta = G, \ \alpha \in R^{n_1}, \ n_1 = \# I_0$$

$$B^T \beta = F, \ \beta \in R^{n_2}, \ n_2 = \# J$$

where $M \in R^{n_1 \times n_1}, M_{ij} = m(\Phi_i, \Phi_j), B \in R^{n_2 \times n_1}, B_{ij} = b(\Phi_i, \Phi_j), B^T \in R^{n_1 \times n_2}, B^T_{ij} = b(\Phi_i, \Phi_j) = B_{ji}, G = (G_i), G_i = G(\Phi_i), F = (F_k), F_k = F(\Phi_k)$.

4. Lowest order Raviart-Thomas finite elements

Let $\Omega \in R^2$ be a 2D polygonal domain, $T_h$ be its triangulation, $E_h$ be set of edges of all elements $T \in T_h$ see the situation in the following Figure 1.

Then, we can define

$$RT_0(T) = \{v : T \rightarrow R^2, v(x) = \xi x_1 x_2^T + [\eta_1 \ \eta_2]^T, \xi, \eta_1, \eta_2 \in R\}$$

$$U_h = \{v : \Omega \in R^2, v|_T \in RT_0(T) \ \forall T \in T_h, v \cdot n_E \text{ is continuous over } E \in E_h\}$$

$$P_h = \{q : \Omega \in R^2, q|_T \text{ is constant } \forall T \in T_h\}.$$ 

Continuity of $v \cdot n_E$ guarantees $U_h \subset U, P_h \subset P$ is obvious. Note that $\forall E \in E_h$ we define $n_E$ (unit normal vector), independently of relation to triangles and consequently in possibly inner or outer direction, see Figure 2.

6. Local properties and local edge basis for RT(0) elements

**Lemma 4.1.** Let $T \in T_h, v \in RT_0(T)$. Then $\forall E \in E_h \cup \partial T : v \cdot n|E = \text{const.}$

**Proof.** Let $E \in E_h \cup \partial T, n_E$ be normal to $E$ (can be either outer or inner to $T$), $x^* \in E$ be arbitrary point at $E$. Then

$$x \in E \Rightarrow (x - x^*) \cdot n_E = 0, n_E = (n_1, n_2) \Rightarrow x_1 n_1 + x_2 n_2 = x^*_1 n_1 + x^*_2 n_2 = \text{const.} \Rightarrow$$

$$v(x) \cdot n = \xi x_1 n_1 + \xi x_2 n_2 + \eta_1 n_1 + \eta_2 n_2 = \xi (x^*_1 n_1 + x^*_2 n_2) + \eta_1 n_1 + \eta_2 n_2 = \text{const.}.$$
Gambar 1. \( \{ x^{(i)} \} \) set of centres of \( E_i \in \mathcal{E}_h \), \( \{ y^{(j)} \} \) barycentres of \( T_j \in \mathcal{T}_h \)

Gambar 2. Prescribed normal \( n_E \). Possible definition of \( n_E, E \in \mathcal{E}_h \).

Gambar 3. Triangle \( T \in \mathcal{T}_h \).
Lemma 4.2. (Expression for local basis functions.) Let

\[ \Phi_i(x) = \sigma_i \frac{E_i}{2|T|} (x - P_i), \sigma_i = n_{E_i} n^{(i)}, \]

where \( n_{E_i} \) are global prescribed normals and \( n^{(i)} \) are outer normals for \( T \in \mathcal{T}_h \), see Figure

3. Then

(i) \( \Phi_j(x) \cdot n_{E_i} = \delta_{ij} \),

(ii) \( \Phi_i \in RT_0(T) \),

(iii) \( \Phi_1, \Phi_2, \Phi_3 \) create a basis of \( RT_0(T) \),

(iv) \( \text{div} \Phi_i = \sigma_i \frac{E_i}{|T|} \).

Proof. (i) If \( i \neq j \), then \( P_i \in E_j \) and \( (x - P_i) \cdot n_{E_i} = 0 \) for \( x \in E_j \). If \( i = j \) then for \( x \in E_i \) the value \( (x - P_i) \cdot n_{E_i} \) appears in the projection of \( (x - P_i) \) to the height of \( T \) passing through \( P_i \) and therefore \( |(x - P_i) \cdot n_{E_i}| = h_i \). Moreover, \( \frac{1}{2} h_i |E_i| = |T| \) and \( h_i = 2|T|/|E_i|, (x - P_i) \cdot n^{(i)} \geq 0 \) - both vectors have outward direction w.r.t. \( T \). Finally

\[ (x - P_i) \cdot n_{E_i} = \sigma_i \frac{2|T|}{|E_i|} \]

(ii) obvious

(iii) \( u \in RT_0(T), w = u - \sum_i^3 (u \cdot n_{E_i}) \Phi_i \). Obviously \( w \cdot n_{E_i} = 0 \forall E_i \). Therefore \( \forall P_j \) :

\[ w(P_j) \cdot n_{E_i} = 0 \]

and because \( \forall E_i : P_j \in E_i \), it holds \( w(P_j) = 0 \forall j = 1, 2, 3 \). As \( w \) is linear polynomial, \( w = 0 \). Proof of uniqueness:

\[ w = \sum_1^3 \alpha_i \Phi_i = 0 \Rightarrow \forall E_i : \alpha_j \Phi_j n_{E_i} = \alpha_j \forall j \]

(iv) obvious

5. Local matrices and assembling

Assume that \( \Phi_i \) and \( \Psi_i \) are constructed as finite element basis functions above some triangulation \( \mathcal{T}_h \), i.e. \( T \in \mathcal{T}_h \)

\[ \Phi_i|T \in \{ \Phi_1, ..., \Phi_p, 0 = \Phi_0 \} \]

\[ \Psi_j|T \in \{ \Psi_1, ..., \Psi_s, 0 = \Psi_0 \} \]

Then

\[ m(\Phi_i, \Phi_k) = \int_T k^{-1} \Phi_i \Phi_k dx \]
\[ = \sum_{T \in \mathcal{T}_h} \int_T k^{-1} \Phi_i \Phi_k dx \]
\[ = \sum_{T \in \mathcal{T}_h} \int_T k^{-1} \Phi_{loc(i)}(\Phi_{loc(k)}) dx \]
\[ b(\Phi_i, \Psi_j) = \int_T (\text{div} \Phi_i) \Psi_j dx \]
\[ = \sum_{T \in \mathcal{T}_h} \int_T (\text{div} \Phi_{loc(i)}(\Psi_{loc(j)}) dx \]

where \( \text{loc}_k(i) = \text{loc}_k(i, T) \) is a transformation from global index to local index of basis function on \( T \). It can be also zero.

Vice versa, for \( T \in \mathcal{T}_h \), it is possible to construct local matrices

\[ M_{T,r} = \int_T k^{-1} \Phi_i \Phi_r dx \]
\[ B_{T,r} = -\int_T \text{div} \Phi_i \cdot \Psi_r dx \]

and then perform the assembling of local matrices to global \( M, B \)

\[ (M_{T,r} \rightarrow M_{\text{glob}(T,r)}\text{glob}(T,s) = +(M_{T,r}) \]
\[ (B_{T,r} \rightarrow B_{\text{glob}(T,r)}\text{glob}(T,s) = +(B_{T,r}) \]}
Note there are two sets of basis functions \( \{ \Phi_i \}, \{ \Psi_i \} \), two sets of local basis functions \( \{ \Phi_i \}, \{ \Psi_i \} \) and two mappings
\[
\text{loc}_1(i) = \text{loc}_1(i, T), \text{loc}_2
\]
\[
\text{glob}_1(r, T) = i, \text{glob}_2(s, T) = j.
\]

6. Local matrices

Let us consider the local basis on \( T \) created by \( \Phi_1, \Phi_2, \Phi_3 \in RT_0(T) \) and \( \Psi_1 = 1 \). Then
\[
B_T \in R^{3 \times 3},
\]
\[
(B_T)_{1s} = \int_T (\text{div} \Phi_s) \Psi_1 = \sigma_s |E_s| |T| = \sigma_s |E_s|,
\]
i.e. \( B_T = [\sigma_1 |E_1|, \sigma_2 |E_2|, \sigma_3 |E_3|] \in R^{3 \times 3} \). Further, \( M_T \in R^{3 \times 3} \),
\[
(M_T)_{rs} = \int_T k^{-1} \Phi_s \Phi_r dx = \sigma_r \sigma_s |E_r| |E_s| |T|^2 \int_T k^{-1}(x - P_s) \cdot (x - P_r) dx.
\]
To compute the integral \( \int_T k^{-1}(x - P_s) \cdot (x - P_r) dx \), we can use barycentric coordinates at \( T \),
\[
x = \lambda_1(x)P_1 + \lambda_2(x)P_2 + \lambda_3(x)P_3, \quad \lambda_1 + \lambda_2 + \lambda_3 = 1,
\]
thus
\[
x - P_r = \lambda_1(x)(P_1 - P_r) + \lambda_2(x)(P_2 - P_r) + \lambda_3(x)(P_3 - P_r)
\]
and
\[
(M_T)_{rs} = \sigma_r \sigma_s |E_r| |E_s| \sum_{a, \beta = 1}^3 \int_T \lambda_{a} \lambda_{\beta} k^{-1}(P_a - P_s) \cdot (P_{\beta} - P_r) dx.
\]
Assuming \( k \) constant on \( T \) and using the integration formula \( \int_T \lambda_{a} \lambda_{\beta} = |T| |\delta_{a\beta}| \),
which is a special case of
\[
\int_T \lambda_{a}^a \lambda_{b}^b \lambda_{c}^c dx = \frac{abc!}{(a+b+c+2)!} |T|
\]
\[
\int_T \lambda_{a}^a \lambda_{b}^b \lambda_{c}^c dx = \frac{abc!}{(a+b+c+d+3)!} |V|
\]
see e.g. [4, 5] the elements of \( M_T \) can be expressed as
\[
(M_T)_{rs} = \frac{1}{48 |T|} |E_r| \sum_{\alpha, \beta = 1}^3 (1 + \delta_{\alpha\beta}) k^{-1}(P_a - P_s) \cdot (P_{\beta} - P_r) \sigma_s |E_s|
\]
If we define vectors \( v_r, v_s \in R^{6 \times 1} \),
\[
v_r = \begin{bmatrix}
P_1 - P_r \\
P_2 - P_r \\
P_3 - P_r
\end{bmatrix}, v_s = \begin{bmatrix}
P_1 - P_s \\
P_2 - P_s \\
P_3 - P_s
\end{bmatrix}, p_i = \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
\]
Then
\[
(M_T)_{rs} = \frac{1}{48 |T|} |E_r| |v_r^T v_s| \begin{bmatrix}
2 & 0 & 1 & 0 & 1 & 0 \\
0 & 2 & 0 & 1 & 0 & 1 \\
1 & 0 & 2 & 0 & 1 & 0 \\
0 & 1 & 0 & 2 & 0 & 1 \\
1 & 0 & 1 & 2 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 2
\end{bmatrix} \begin{bmatrix}
k^{-1} \\
k^{-1}
\end{bmatrix} v_s \sigma_s |E_s|
\]
Note that the diagonal elements are equal to elements of $B_T$. If we denote $C \in \mathbb{R}^{6 \times 6}$ the matrix, which appeared in the expression above and

$$V = [v_1, v_2, v_3] = \begin{bmatrix} 0 & P_1 - P_2 & P_1 - P_3 \\ P_2 - P_1 & 0 & P_2 - P_3 \\ P_3 - P_1 & 0 & P_3 - P_2 \end{bmatrix}$$

then

$$(M_T) = \frac{1}{48|T|} \begin{bmatrix} \sigma_1|E_1| & 0 & 0 \\ 0 & \sigma_2|E_2| & 0 \\ 0 & 0 & \sigma_3|E_3| \end{bmatrix} V^T C \begin{bmatrix} k^{-1} & k^{-1} & k^{-1} \\ k^{-1} & k^{-1} & k^{-1} \end{bmatrix} V \begin{bmatrix} \sigma_1|E_1| & 0 & 0 \\ 0 & \sigma_2|E_2| & 0 \\ 0 & 0 & \sigma_3|E_3| \end{bmatrix}$$

where $S = \text{diag}[b_1E_1|, b_2E_2, b_3E_3]$, $V = \begin{bmatrix} 0 & P_1 - P_2 & P_1 - P_3 \\ P_2 - P_1 & 0 & P_2 - P_3 \\ P_3 - P_1 & 0 & P_3 - P_2 \end{bmatrix}$, $L = \begin{bmatrix} k^{-1} & k^{-1} \\ k^{-1} & k^{-1} \end{bmatrix}^{-1} = \frac{1}{k^2}I$, if we consider the isotropic environment, $k = k_T I$ on $T$. For comparison see [2] formula (4.6).

Note that we constructed velocity mass matrix $M$. In the case of time dependent problems, we also need the pressure mass matrix $(M_T)_{rs} = \int_T \Psi_r \Psi_s = \delta_{rs}|T|$

7. Model problem

We shall consider a model Darcy flow problems on a square domain with flow from left to right induced by the pressure gradient.

The problem domain is divided into rectangular elements with the size characterized by the parameter $ns$ = number of segments on the side.

**Heterogeneity.** We assume that each cell can possess a different permeability coefficient $k_i, i = 1, ..., nc = (ns)^2$. This can be produced by MATLAB using command sequence

```matlab
1) rng ('default');
2) RM = randn ( ns , ns ) ;
3) LK = exp ( 1 ) .^ ( sigma *RM ) ;
```

The first command initializes the random number generator to make the results in this example repeatable. The same sequence is generated as after restart of MATLAB. The second command generate a ns-by-ns matrix of normally distributed random numbers from $N(0, 1)$, i.e. with mean $\mu = 0$ and standard deviation 1. Then $s*RM$ is a matrix of normally distributed
random numbers with the mean $\mu = 0$ and standard deviation $\sigma^2$. Third command then creates a matrix of conductivities such that $\ln(LK)$ has normal distribution.

Orientation of (global) normals to element edges

Model problem -local matrices.

$$M_T = \frac{1}{24h^2} SV^T CLVS, L = \frac{1}{k_{cell}} I.$$

**Lower Triangle**

\[ B_T = [\sqrt{2}h, -h, -h], \quad S = h. \]

\[
\begin{pmatrix}
\sqrt{2} \\
1
\end{pmatrix}, \quad V = h
\]

\[
\begin{bmatrix}
0 & -1 & 0 \\
0 & 0 & -1 \\
1 & 0 & 1 \\
0 & 0 & -1 \\
0 & -1 & 0 \\
1 & 1 & 0
\end{bmatrix}.
\]

**Upper triangle**

\[ B_T = [-\sqrt{2}h, h, h], \quad S = h. \]

\[
\begin{pmatrix}
-\sqrt{2} \\
1
\end{pmatrix} = -S_{low}, \quad V_{upper} = h
\]

\[
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
-1 & 0 & 1 \\
0 & 0 & 1 \\
0 & 1 & 0 \\
-1 & -1 & 0
\end{bmatrix} = -V_{low}.
\]

As a conclusion - the matrices \( M_T = \frac{1}{2h^2} S V^T C L V S \) are the same for both lower and upper triangles.

**Right hand side and boundary conditions.** Consider the global system

\[
M\alpha + B_T^T\beta = G
\]

\[
B\alpha = E
\]

where

\[
G_i = -\int_{\Gamma_0} \hat{\rho}(p_i \cdot n) - \sum_{k \in \Gamma_1 \setminus \Gamma_0} \hat{u}_k m(\Phi_k, \Phi_i)
\]
Gambar 7. Pressure boundary conditions for the model problem.

Gambar 8. Treatment of velocity boundary conditions: a) exclude corresponding rows and columns and rhs entries, b) or put 1 on diagonal otherwise zeros in corresponding row, columns and rhs entries

\[ F_j = - \int_{\Omega} f \Psi_j - \sum_{k \in I \setminus I_0} \hat{u}_k \int_{\Omega} \text{div}(\Phi_k) \Psi_j \, dx = 0 \]

8. ASSEMBLING

Standard assembling

Algorithm 1 Standard assembling

\begin{verbatim}
define M = 0, B = 0
for 1:nt
  take M_T, B_T
  for r = 1,...,3
    for s = 1,2,3
      Mi(T,r) j(T,s) = (M_T)_{rs}
    end
  end
  B\kappa(T) i(T,r) = (B_T)_i r
end
\end{verbatim}

The standard assembling has two drawbacks: for cycles, which are not efficient in MATLAB, and dense matrix storage of the global matrix. Just replacing the global matrix declaration as sparse is not a good solution as it the sparse structure is not given apriori but must be constructed during the assembling process. This inefficiency can be removed by gradual recording the nonzero components and indices into one dimensional vectors X, I, J and constructing
the matrix through

\[
\text{sparse}(X, I, J, n, m).
\]

Further improvement and loop avoiding can be done by vectorization, see [6][6]. The resulting code is able fast assembly very large matrices.

Gambar 9. Transmissivity coefficient \( k \), blue color \( k = 1.0 \), red color \( k = 1.4 \)

9. Numerical Test

We test numerically an example from [7]. For simplicity, the exact \( u_\sigma \) in \( \Omega \) is set to be

\[
u_\sigma = \cos(x - 0.5) \ast \exp(y).
\]

\[
k = \sigma = \begin{cases} 
1.0, & 0 < x < 0.5, 0 < y < 1 \\
1.4, & 0.5 < x < 0, 0 < y < 1 
\end{cases}
\]

\[
\partial_n u = \begin{cases} 
\partial_x u = \sin(x - 0.5) \ast \exp(y), & 0 < y < 1, x = \{0, 1\} \\
\partial_y u = \cos(x - 0.5) \ast \exp(y), & 0 < x < 1, y = \{0, 1\}
\end{cases}
\]

The program implemented in GNU Octave run in octave-online.net, which is a web UI for GNU Octave.

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Daftar Pustaka


**Gambar 10. Displacement**

**Gambar 11. Flux**


**LAMPIRAN**

```matlab
% Program Darcy equation implementation based on
% EBFEM for 2D Raviart–Thomas mixed finite element method
% based on the edge-oriented basis function
% Agah D. Garnadi and C. Bahriawati
% File <Darcy_EBFem.m>
% M-files you need to run
% <stimaB.m>, <edge.m>, <f.m>, <u_D.m>, <u_N.m> (optional)
```
% Data (files) you need to prepare
% coordinate <coordinate.dat>,
% element <element.dat>,
% Dirichlet <Dirichlet.dat>,
% Neumann <Neumann.dat> (optional)

% This program and corresponding data-files is modified from
"Three Matlab Implementations of the Lowest-Order Raviart–Thomas
MFEM with a Posteriori Error Control" by C.Bahriawati and C. Carstensen

A1. The main program
load coordinate.dat;
coordinate = [0 0; 0.5 0; 1 0; 1 0.5; 1 1; 0 0.5; 0.5 0.5];
load element.dat;
element = [2 8 1; 2 9 8; 2 4 9; 2 3 4; 9 4 5; 9 5 6; 9 6 7; 9 7 8];
load k_element.dat;
k_element = [1 1 1.4; 1.4 1.4; 1.4 1; 1 1];
load dirichlet.dat;
dirichlet = [3 4; 4 5; 7 8; 8 1];
load Neumann.dat;
Neumann = [1 2; 2 3; 5 6; 6 7];

[nodes2element, nodes2edge, noedges, edge2element, interioredge] = edge(element, coordinate);

A2. EBmfem
function u=EBmfem(element, coordinate, dirichlet, Neumann, nodes2element, ... 
nodes2edge, noedges, edge2element);

Assembly matrices B and C
B=sparse(noedges, noedges);
C=sparse(noedges, size(element,1));
for j = 1:size(element,1)
    coord=coordinate(element(j,:),:);'
    I=diag(nelements(element(j,[2 3 1]),element(j,[3 1 2])));
    signum=ones(1,3);
    signum(find(j==edge2element(I,4)))=-1;
    B(element(j,:))= k_element(j)*diag(signum)*stimaB(coord)*diag(signum);
    n=coord(:,[3,1,2])-coord(:,[2,3,1]);
    B(I,I)= B(I,I) + B(element(j,:));
    C(I,j) = diag(signum)*[norm(n(:,1)) norm(n(:,2)) norm(n(:,3))];
end

Global stiffness matrix A
A = sparse(noedges+size(element,1), noedges+size(element,1));
A = [B , C, ];
C', sparse(size(C,2),size(C,2));

Volume force
b = sparse(noedges+size(element ,1),1);
for j = 1:size(element ,1)
b(noedges+j)= -det([1,1,1; coordinate(element(j,:),:)]) * ... 
f(sum(coordinate(element(j,:),:))/3)/6;
end

Dirichlet conditions
for k = 1:size(dirichlet,1)
b(necessaryelement(dirichlet(k,1),dirichlet(k,2)))= norm(coordinate(dirichlet(k,1),:))-... 
coordinate(dirichlet(k,2,:),:)+u.B(sum(coordinate(dirichlet(k,1),:))/3);
end

Neumann conditions
if ~isempty(Neumann)
    tmp=zeros(noedges+size(element,1),1);
    tmp(diag(nodes2edge(Neumann(:,1),Neumann(:,2))))=...
        ones(size(diag(nodes2edge(Neumann(:,1),Neumann(:,2))),1),1);
    FreeEdge=find(~tmp);
    x=zeros(noedges+size(element,1),1);
    CN=coordinate(Neumann(:,2),:)-coordinate(Neumann(:,1),:);
    for j=1:size(Neumann,1)
        x(nodes2edge(Neumann(j,1),Neumann(j,2)))=...
            g(sum(coordinate(Neumann(j,:),:))/2,CN(j,:)*[0,-1;1,0]/norm(CN(j,:)));
    end
    b=b-A*x;
    x(FreeEdge)=A(FreeEdge,FreeEdge)(FreeEdge);
else
    x = A\b;
end
figure(1)
ShowDisplacement(element,coordinate,x);
p=fluxEB(element,coordinate,x,noedges,nodes2edge,edge2element);
figure(2)
ShowFlux(element,coordinate,p);
pEval=fluxEBEval(element,coordinate,x,nodes2edge,edge2element);