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Efficient Estimation of Climate State and Its Uncertainty Using Kalman Filtering with Application to Policy Thresholds and Volcanism

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ABSTRACT

We present the Energy Balance Model – Kalman Filter (EBM-KF), a hybrid model of the global mean surface temperature (GMST) and ocean heat content anomaly (OHCA). It combines an annual energy balance model (difference equations) with 17 parameters drawn from the literature and a statistical Extended Kalman Filter assimilating GMST and OHCA, either observed timeseries or simulated by earth system models. Our motivation is to create an efficient and natural estimator of the climate state and its uncertainty, which we believe to be Gaussian at a global scale. We illustrate four applications: 1) EBM-KF generates a similar estimate to the 30-year time-averaged climate state 15 years sooner, or a model-simulated hindcasts' annual ensemble average, depending on the preparation of volcanic forcing. 2) EBM-KF conveniently assesses annually likelihoods of crossing a policy threshold, e.g., based on temperature records up to the end of 2023, p=0.0017 that the climate state was 1.5°C over preindustrial, but a 16% likelihood that the GMST in 2023 itself could have been over that threshold. 3) The EBM-KF also approximates the spread of an entire climate model large ensemble using only one or a few ensemble members. 4) The EBM-KF is sufficiently fast to allow thorough sampling from non-Gaussian probabilistic futures, e.g., the impact of rare but significant volcanic eruptions. This sampling with the EBM-KF better determines how future volcanism may affect when policy thresholds will be crossed and what an ensemble with thousands of members exploring future intermittent volcanism reveals.

SIGNIFICANCE STATEMENT

The global average of the Earth's historical climate over the past 150 years can be explained by a thermal/radiation physics equation involving a small number of constants (17), atmospheric CO_2 concentration, human-produced cloud-seeding aerosols, and dust from volcanic eruptions. Global mean surface temperature measurements vary around this climate state within a consistent normal distribution. This physics equation and statistical depiction allowed us to construct a simple model that can rapidly estimate the uncertainty in Earth's current climate, aid in policy discussions, and reduce ensemble modeling costs.

1. Introduction

What is the uncertainty in Earth's climate? From a measurement standpoint, this issue was resolved many decades ago. The instantaneous measurement of global mean surface temperature (GMST) is currently performed with average accuracy of 0.05°C (max 0.10°C) via arrays of infrared-sensing satellites and ground stations (Susskind, Schmidt et al. 2019). Both satellite and ground datasets extend back to 1981 (Merchant, Embury et al. 2019), and the yearly seasonal fluctuation is easy to smooth with a running annual average. However, this GMST still has significant dynamical and random stochasticity, from processes like the 2-7 year quasi-periodic El Nino events (Hu and Fedorov 2017) and volcanic eruptions that intermittently affect climate for 1-2 years (Soden, Wetherald et al. 2002). Measurement errors also arise from sparse or inconsistently calibrated historical data and paleoproxies (Carré, Sachs et al. 2012; Emile-Geay, McKay et al. 2017; Kaufman, McKay et al. 2020; McClelland, Halevy et al. 2021). Internal variability dominates over climate-forced variability in most short-term signals, both in climate simulations and reality (Kirtman, Power et al. 2013; Marotzke and Forster 2015; Gulev, Thorne et al. 2021; Lee, Marotzke et al. 2021). By "simulations", we refer to computationally expensive global coupled models (and occasionally to numerical weather model predictions). Other climate variables reveal warming that is steadier than GMST (less "noisy" annual variability). One such steady climate variable is the Ocean Heat Content Anomaly (OHCA), where >90% of the anthropogenic energy anomaly is found (Cheng, Trenberth et al. 2017; Fox-Kemper, Hewitt et al. 2021; Gulev, Thorne et al. 2021; Cheng, von Schuckmann et al. 2022). Even radical reductions in global CO_2 emissions may not show an identifiable impact on GMST over a time scale of a few years (Szopa et al. 2021), posing a challenge for policy and assessment.

In 1935 the World Meteorological Association began reporting the "standard climate normal" as surface temperature averages over an interval of 30 years ($\overline{{}_{30}Y_t}$ in this paper's notation, starting with 1901-1930). A 30-year window was chosen to minimize most internal fluctuations (such as El Nino) and short-term forcings such as single volcanoes (Guttman 1989). Fig. 1 shows this metric and emphasizes the 30-year span over which the average is taken. To generate continuous estimates of the climate, this 30-year average can be updated annually rather than decadally, forming a running mean (Supp. Fig. 4b). While standard climate normals and running means are straightforward and widely accepted definitions of climate, they involve lag: the most current 30-year unweighted average necessarily describes the average climate state of Earth over a window centered on 15 years ago. Weighted moving averaging can shift the center of this window closer toward the current year but some lag always remains. Moreover, anthropogenic climate change distorts standard statistical metrics: most of the variance in recent 30-year periods derives from the trend rather than internal variability (Fig. 1). Averaging filters (such as a running mean) remove high-frequency signals that reflect year-to-year variations in global weather, as do other statistical approaches better-suited to removing frequencies above a particular cutoff (Smith 2003). The anthropogenic change in Fig. 1 is gradual enough to be preserved by moving averages (running mean) or any lowpass filter / smoother. But this is not true in general: in a hypothetical (or extraterrestrial) climate where forcings undergo an impulse change, such as a quadrupling of CO_2 within 1 year, the 30-year running mean inadequately represents the climate state (see supplemental Section B, Supp. Fig. 3). To directly fit the effect of forcings to the climate (incorporating relaxation time), the multi-pattern fingerprint method was developed, (Hasselmann 1997) leading to "attributable anthropogenic warming" (Otto, Frame et al. 2015) and a "real-time Global Warming Index" (Haustein, Allen et al. 2017). This methodology is statistically conservative, generating a wide 5-95% confidence interval spanning ±0.1°C from 1980-2010, which then expands out to ±0.15°C. Other example applications to Earth's recent GMST of statistical, as opposed to physical, filters used in climate analysis are shown in supplemental Section B (Supp. Figs. 4c,d & 5).

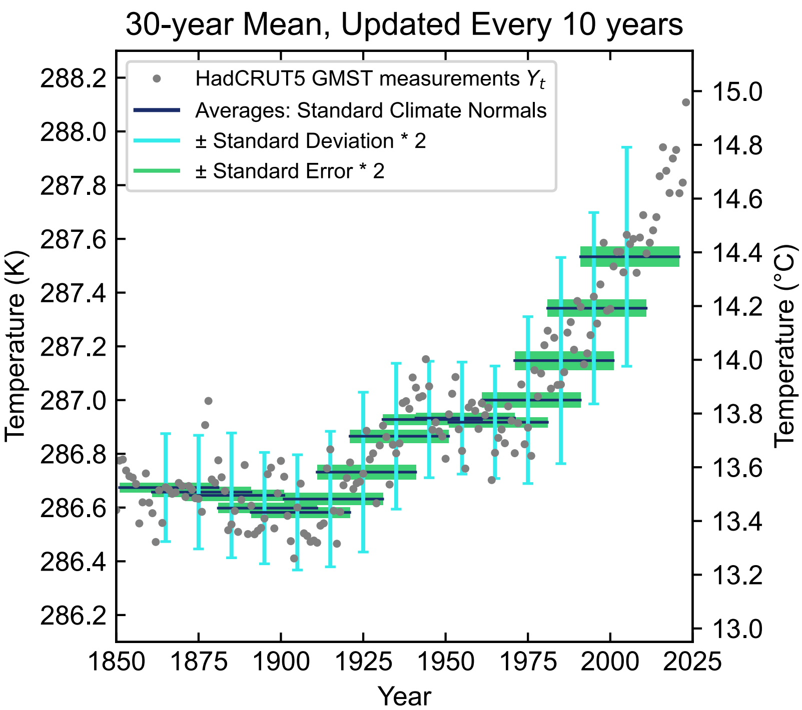

Fig. 1: Illustration of Standard Climate Normals $\overline{{}_{30}Y_t}$ (blue horizontal lines in 10-year overlapping bins) as applied to the HadCRUT5 GMST dataset (grey dots) (Morice, Kennedy

et al. 2021). Twice the population standard deviation (±2σ) is plotted above and below (cyan error bars), and two standard errors are also plotted (green rectangles). Note how standard deviations widen in recent decades due to the anthropogenic trend, and the most recent standard climate normal is much cooler than recent GMST measurements.

Policy goals often are framed via climate change staying below a particular policy threshold (e.g., 1.5°C or 2°C above pre-industrial conditions as in the Paris Agreement). Using a 30-year mean brings difficulty in determining exactly when or if a policy threshold is crossed (Lee, Marotzke et al. 2021). Policy thresholds are not system thresholds — temperature "tipping" points when the dynamics of the climate system are reorganized, often occurring abruptly or irreversibly — and so they are subject to definitional uncertainty. Relatedly, magnitudes and uncertainty ranges are meaningful only under specific averaging windows, e.g., "GMST increased by 0.85 (0.69 - 0.95) °C between 1850–1900 and 1995–2014 and by 1.09 (0.95 - 1.20)°C between 1850–1900 and 2011–2020." (Gulev, Thorne et al. 2021). Tools for assessing if a policy threshold has been crossed yet will be useful as these policy targets approach. Throughout this paper we use both $\mu \pm 2\sigma$ and μ (a - b) notation to refer to 95% confidence intervals (95% CI), in contrast to [a-b] notation which refers to finite or closed ranges. In this notation, μ is a point estimate, σ is a standard deviation, a is the minimum of the interval or range, and b is the maximum.

To overcome limited sampling of the real world, many climate studies instead investigate the climate system within globally coupled climate simulations ("coupled" refers to interacting sub-models, such as atmosphere/ocean/land/ice components), also known as earth system models or ESMs (Meehl, Moss et al. 2014). Typically, these simulations are forced using historical records and a range of scenarios for future projections including CO_2 emissions, other pollutants, land use, and volcanic eruptions (Lee, Marotzke et al. 2021). The chaotic nature of weather and varying initial conditions produce an ensemble of identically-forced simulations that explore the span of outcomes consistent with forcing, such as for the CESM2 Large Ensemble (Rodgers, Lee et al. 2021), abbreviated here as LENS2 (Supp. Fig. 6). Unfortunately, each ensemble member simulation is computationally expensive and does not accurately or transparently reflect the real climate system, but only one realization of it including model errors. Combining such ensembles with real observations yields improvements, such as a more realistic possible spread (due to internal variability) of winter temperatures in North America from 1966-2015. (McKinnon, Poppick et al. 2017) Betts, Belcher et al. (2023) proposed combining 10 years of previous observations with a subsequent 10 years forecasted by several ESMs, an approach named the “current global warming level”. While useful, this technique cannot completely solve all issues inherent to ESMs, such as whether some predictions should be weighted over others (Lehner, Deser et al. 2020; Sherwood, Webb et al. 2020), or how an ensemble of short-term projections should be initialized.

We sought an efficient and natural estimator of the climate state and its uncertainty: the EBM-KF. We combined a nonlinear energy-balance difference equation (EBM) and a statistical observation equation (KF) that brings in the available measured GMST and OHCA data, yielding a hybrid physical model – statistical filter. This data-driven climate emulator (Forster, Storelvmo et al. 2021) is vastly more computationally efficient than ensembles of ESMs that provide similar information about GMST. Our emulator is interpretable as a global energy budget (and so assimilates OHCA as well as GMST), benefits from the mathematical similarities between an energy balance model and a Kalman Filter, and allows access to proven methodologies for parameter estimation (Chen, Heckman et al. 2018; Zhang and Atia 2020) and uncertainty quantification (Sætrom and Omre 2013). We did not empirically fit this emulator to the climate record: 12 of the 17 parameters within the energy-balance equation were directly obtained from literature estimates, whereas the remaining 5 parameters are inferred indirectly from assumed pre-industrial climate equilibrium and literature estimates of climate sensitivities. Thus, while some of these parameters were calibrated to the historical climate record independently by other researchers, they were not adjusted to suit this novel EBM combination. Our simple iterative EBM has good skill at predicting the GMST and OHCA despite being by itself "blind" to all measurements (i.e., it’s a “forward” model in numerical weather prediction terminology). The statistical component is an Extended Kalman Filter, which allows for incorporation of current measurements to "course-correct" under a well-understood mathematical framework, with time-varying “weather” and “climate state” uncertainty. Other noise covariance matrices had to be fixed a priori within the statistical Kalman Filter framework to incorporate observational uncertainty. Part of this noise was due to time-varying uncertainty provided with the observations of GMST and OHCA. Another part of the noise covariance was chosen such that the variability in “climate state” most closely resembles the 30-year running mean of GMST and OHCA. While perhaps unconventional in data assimilation, this approach is directly analogous to the inference of some of our parameters: a handful of numbers were abstracted from the historical climate record using established statistical methods. Hybridizing the EBM with the Extended Kalman

Filter yields statistical distributions of internal variability and a physical rationale for the filtered current climate state.

First, the EBM-KF is introduced within Section 2 in phases: the EBM in Section 2a and the structure of the Extended Kalman Filter in Section 2b. An elaboration beyond fixed assumed measurement uncertainty is detailed in Section 2c. The scope of EBM-KF is expanded to future projections including volcanic eruptions in Section 2d. Then in Section 3, EBM-KF is illustrated on four applications to historical and future climate. Section 3a shows that it estimates the 30-year mean climate normal every year, including the latest observations and without lag. Section 3b shows how it can be used to assess the probability that a policy threshold has been crossed in any particular year. Section 3c shows how it can be used to estimate the ensemble mean of an ESM Large Ensemble from only one ensemble member. Section 3d shows that the EBM-KF is sufficiently fast to allow high-density sampling of non-Gaussian probabilistic futures, e.g., directly sampling over highly intermittent distributions of future volcanic eruptions. Section 4 discusses these results, some cautionary remarks, opportunities for extension, and application to policymaking. Section 5 concludes. Extensive appendices and supplementary material convey additional detail. Throughout, a 2σ or approximately 95% confidence interval is used, indicating the *extremely likely* range in IPCC terminology.

2. Methods

a. Energy-Balance Model

We constructed the energy-balance model (Fig. 2) by envisioning a uniform planet and capturing the principal atmospheric and surface energy fluxes (Budyko 1969; Sellers 1969). This model is "blind" with respect to observations and is inspired by other energy-budget models illustrating quantitative skill (Hu and Fedorov 2017; Kravitz, Rasch et al. 2018) at approximating both GMST and the 30-year running mean. The model includes two idealized layers, with each layer having homogenous temperature: a surface layer including thermally active soil and 86m of ocean water depth (with temperature approximating GMST), and a deep ocean layer reaching (1141+86)m depth that exchanges energy (part of OHCA) with the surface layer (Gregory 2000; Held, Winton et al. 2010; Geoffroy, Saint-Martin et al. 2013). These depths are chosen to select a two-state system that best represents the heat capacities of spatially complex heat uptake patterns in total into the global oceans (Newsom, Zanna et al. 2023), rather than representing the heat uptake relative to depths associated with observational oceanographic traditions (e.g. 700m, 2000m). As this EBM does not directly incorporate any spatial dimensions, it should be considered 0-dimensional in the context of other ESMs with spatial gradients. Closely related variables to GMST, such as Global Surface Air Temperature (GSAT), differ only from GMST by measurement and slightly in uncertainty (by less than our confidence intervals) but not systematically (Gulev et al. 2021). For example, GMST is easier to measure in the past, while GSAT is a standard output variable in future model projections, so here we do not distinguish between them.

The overall energy fluxes into the model layers are as follows:

$$\mathrm{C_{surf}}\ \frac{dT}{dt} = \mathcal{F}_{sw}(T,t) - \phi_{\mathrm{LW}}(T,t) - \gamma * (T - \theta - \zeta_0) \tag{1}$$

$$\mathrm{C_{deepO}}\ \frac{d\theta}{dt} = \gamma * (T - \theta - \zeta_0) \tag{2}$$

$$H = (T - T_{1850}) * \mathrm{C_{upperO}} + (\theta - \theta_{1850}) * \mathrm{C_{deepO}} \tag{3}$$

$$\mathcal{F}_{\mathrm{sw}}(T,t) = \frac{1}{4} G_{SC}(t) * \tilde{\mathrm{d}}(t) * f_{\alpha A}(T,t) * f_{\alpha S}(T) \tag{4}$$

$$\phi_{\mathrm{LW}}(T,t) = \sigma_{sf} T^4 * \tilde{\mathrm{g}}(t) * f_{H2O}(T) \tag{5}$$

T is GMST, whereas *θ* is the potential (or conservative) temperature of the deep ocean in that same year, and *H* is OHCA including both that deep ocean layer and the surface ocean (McDougall, Barker et al. 2021). The time variable *t* is the calendar year index, and often used as a subscript (e.g. *t*=2000, T_{2000} is the modeled GMST in the year 2000). On the right side of the equation, both the shortwave radiative flux ($\mathcal{F}_{\mathrm{sw}}(T,t)$) and longwave radiative flux ($\phi_{\mathrm{LW}}(T,t)$) take the same form: (source $\frac{G_{SC}}{4}$, $\sigma_{sf}T^4$) * (prescribed attenuation from forcing: $\tilde{\mathrm{d}}(t)$, $\tilde{\mathrm{g}}(t)$)* (attenuation functions with feedback: $f_{?}(T)$). The attenuation function of clouds on shortwave light $f_{\alpha A}(T,t)$, contains both prescribed forcing and feedback. The overall surface heat capacity, $\mathrm{C_{surf}}$, is 17 ± 7 W (year) m^{-2} K^{-1}, obtained from modeling / timeseries analysis (Schwartz, 2007), including 11.7 W (year) m^{-2} K^{-1} or 86m of upper surface ocean $\mathrm{C_{upperO}}$, while there is a separate deep ocean heat sink with capacity 155.7 W (year) m^{-2} K^{-1} or 1141m $\mathrm{C_{deepO}}$ (Geoffroy, Saint-Martin et al. 2013; Hall and Fox-Kemper 2023). $\frac{G_{SC}}{4}$ is the total solar irradiance (TSI) normalized to the Earth's surface area at ~340.2 W/m^2; we incorporated its record $\frac{1}{4}G_{SC}(t)$ of [340.06 – 340.48] from Coddington (2017) but these variations were found to be insignificant to the climate. $\tilde{\mathrm{d}}(t)$ is the prescribed record of shortwave radiation attenuation due to volcanic dust (values of aerosol optical depth AOD_t

from Sato (1993), Vernier (2011), and NASA (2018)), $f_{\alpha A}(T,t)$ is the additional atmospheric shortwave attenuation due to cloud albedo incorporating both feedback and anthropogenic cloud-nucleating aerosols AC_t, while $f_{\alpha S}(T)$ is the surface shortwave attenuation due to ground albedo. Infrared heat emitted from the surface is $j^\star = \sigma_{sf}T^4$, the ideal Planck black body radiation. $\tilde{g}(t)$ is the prescribed record of longwave attenuation due to CO_2 and other greenhouse gasses combined as effective carbon dioxide concentration $[eCO_2]_t$, and $f_{H2O}(T)$ is the additional atmospheric longwave attenuation due to water vapor parameterized as a function of GMST.

We can choose to pre-filter the input forcings because we are most interested in the slowly evolving climate state. This pre-filtering is inconsequential for greenhouse gasses which are slowly evolving anyway, but it is very consequential for aerosol optical depth which undergoes impulse changes during volcanic eruptions. In the discussion section, we will return to the consequences of pre-filtering.

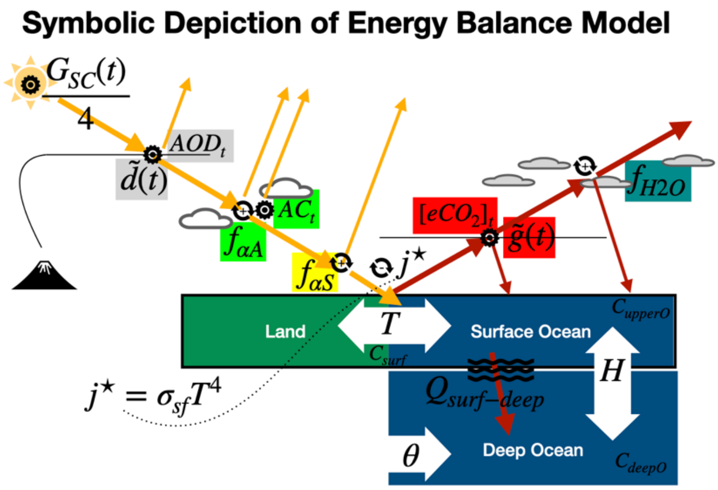

Fig. 2: Diagram of the Energy Balance Model, with all major forcing functions, corresponding climate driver datasets, and feedback functions ($j^\star$ or $f_?$) represented. All feedback functions are dependent on surface temperature T, but this is not written on the diagram above for simplicity. Shortwave incoming solar radiation is successively fractionated by various forcing and feedback functions, as is outgoing longwave radiation. These in net warm the surface layer, which in turn warms the deep ocean.

Both AC_t and $\tilde{g}(t)$ are taken from Forster et. al. (2023). Several of the coefficients within the feedback functions $f_?$ are defined to satisfy the constraints of the climate feedbacks presented in the IPCC AR6 (Forster et al. 2021; particularly Table 7.10), and all coefficients are based on observational and modeling literature values, typically with energy fluxes measured from satellites and temperature feedback coefficients determined from model results (full derivation in Appendix A and Supplemental). Because the Planck radiation requires absolute temperatures, we use degrees Kelvin in model calculations and convert to °C. OHCA is also approximately convertible to thermosteric sea level rise, via the 0.0121 cm/ZJ factor from analysis of 1995 to 2014 (Fox-Kemper, Hewitt et al. 2021). With this factor, the estimated thermosteric sea level rises we find are consistent with observations and projections. Thus, the EBM also estimates sea level rise in this manner (Fox-Kemper, Hewitt et al. 2021). The two negative albedo attenuations $f_{\alpha A}(T,t)$ * $f_{\alpha S}(T)$ are expressed relative to Y_{2002} =287.55K (14.40°C), the inferred (see below) GMST measurement in 2002 (Jones and Harpham 2013; Morice, Kennedy et al. 2021).

$\zeta_0 = 10°C$ is an equilibrium temperature difference between the surface layer (including the upper ocean) and the deep ocean, arising because the global ocean is thermally stratified. This realistic choice of ζ_0, explained below, does not affect either T or H, provided that θ and T are in equilibrium at the model's preindustrial initialization (and thus ζ_0 is often abstracted away in similar 2-layer energy-balance models) (Gregory 2000; Held, Winton et al. 2010; Geoffroy, Saint-Martin et al. 2013). γ is the thermal conductivity or "efficiency" between layers of the ocean, taken from Geoffroy et al. Part II (2013) to be 0.67 W/m^2/K, the average from the CMIP5 models. The form of this parameterization of deep ocean temperature exchange follows recent work in emulating ocean heat uptake, ignoring "efficacy factor" heat loss (Gregory 2000; Winton, Takahashi et al. 2010; Geoffroy, Saint-Martin et al. 2013; Emile-Geay, McKay et al. 2017; Palmer, Harris et al. 2018).

We first obtained the baseline ζ_1=287.01K of the HadCRUT5 GMST anomaly (Morice, Kennedy et al. 2021) to place the 1960-1989 "standard climate normal" of absolute GMST HadCRUT5 measurements to fall at 13.85°C, the center of the range (13.7°C - 14°C) given by Jones and Harpham (2013). Measurements of surface temperature will later be assimilated as absolute temperatures: $Y_t = \zeta_1 + (\text{HadCRUT5})_t$. Our model assumes energy fluxes were balanced before industrial forcings, which requires an equilibrium temperature. We set this preindustrial equilibrium temperature to the 1850-1879 "standard climate normal" of 286.67K (13.52°C) = T_{1850}, and initialized the 1850 climate state to this temperature. This choice is important regarding the determination of many nonlinear feedback functions and coefficients affecting the surface layer (Eq. 7 below), particularly with respect to the Planck feedback. Inconsequentially to the EBM dynamics, the deep potential ocean temperature θ was initialized to be 276.67K (3.52°C) = θ_{1850}, such that current deep ocean potential temperatures are ~3.8°C, choices consistent with both recent and historical measurements of the globally averaged ocean temperature (Robinson and Stommel 1959; Abraham, Baringer et al. 2013). So $\zeta_0 = T_{1850} - \theta_{1850}$ = 13.55°C - 3.55°C = 10°C. Initializing the deep ocean potential temperature θ to another value would change ζ_0 correspondingly, such that the modeled heat flow into the deep ocean would be unchanged.

$$\theta_t = \left(H_t - (T_t - T_{1850}) * \mathrm{C}_{\mathrm{upper0}}\right)/\mathrm{C}_{\mathrm{deep0}} + \theta_{1850} \tag{6}$$

$$T_{t+1} = T_t + \frac{(\frac{G_{SC}}{4})_t * c_2}{(AOD_t + c_4)}\left(1+\beta_2(T_t\text{-}Y_{2002}) + \frac{AC_t - AC_{2002}}{c_3}\right)\left(1+\beta_3(T_t\text{-}Y_{2002})\right)$$

$$- c_1 * (T_t)^{4-\eta}\left(1\text{-}\beta_0 \log_{10}([\mathrm{eCO_2}]_t)\right) - \frac{\gamma}{\mathrm{C_{surf}}}(T_t - \theta_t - \zeta_0) \tag{7}$$

$$H_{t+1} = H_t + (T_{t+1} - T_t) * \mathrm{C}_{\mathrm{upper0}} + \gamma * (T_t - \theta_t - \zeta_0) \tag{8}$$

Symbol	Value (Range)	Derivation or Def.	Source
ζ_1	287.01 K (13.86°C)	$13.85°C = \zeta_1 + \frac{1}{30} * \sum_{t=1960}^{1989}(\mathrm{HadCRUT5})_t$	(Jones and Harpham 2013)
Y_{2002}	287.55 K (14.40°C)	$\zeta_1 + (\mathrm{HadCRUT5})_{2002}$	(Morice, Kennedy et al. 2021)
T_{1850}	286.67K (13.52°C)	$\zeta_1 + \frac{1}{30} * \sum_{t=1850}^{1879}(\mathrm{HadCRUT5})_t$	(Morice, Kennedy et al. 2021)
θ_{1850}	276.67K (3.52°C)	approx.	(Abraham, Baringer et al. 2013)
ζ_0	10 K (10°C)	$\mathrm{T}_{1850} - \theta_{1850}$	(Abraham, Baringer et al. 2013)
γ	$0.67\ \frac{W}{K\,m^2}$	Ocean heat conductivity/year	(Geoffroy, Saint-Martin et al. 2013)
$\mathrm{C_{surf}}$	$17\ \frac{W}{K\,m^2}$	Heat capacity/year, Earth surface	(Schwartz 2007)
$\mathrm{C_{upper0}}$	$11.7\frac{W}{K\,m^2}$ (86m H_2O)	Heat capacity/year, upper ocean	(Geoffroy, Saint-Martin et al. 2013)
$\mathrm{C_{deep0}}$	$155.7\frac{W}{K\,m^2}$ (1141m)	Heat capacity/year, deep ocean	(Geoffroy, Saint-Martin et al. 2013)
η	1.615	Degree of H_2O feedback	Eq. A26 & A27
β_0	0.04660	Infrared reflect per $\log_{10}$ ppm CO_2	Eq. A20, A21, & A35
c_1	$2.1989\ 10^{-5}\ K^{-3+\eta}$	$\frac{\sigma_{sf}\beta_1}{\mathrm{C_{surf}}}$	Eq. A22 & A35
β_2	$0.00136\ K^{-1}$	Atm. albedo temp. feedback	Eq. A13 & A28
β_3	$0.00163\ K^{-1}$	Ground albedo temp. feedback	Eq. A14 & A29
c_2	$0.4044\ \frac{K\,m^2}{W}$	$0.834 * 0.909 * 9.068/\mathrm{C_{surf}}$	Eq. A11, A23, A24
c_3	$264.377\ \frac{W}{m^2}$	$\frac{G_{SC}}{4}\ \widetilde{\mathrm{d}_{2002}}\ 0.834$	Eq A23
c_4	9.73 (unitless)	$\frac{2q'}{1-g}$	Eq. A11 (Notation is eq9 of (Harshvardhan and King 1993)
AC_{2002}	$-0.988\ \frac{W}{m^2}$	Anthro. cloud rad. forcing, 2002	Eq A23
$(\frac{G_{SC}}{4})_t$	$[340.06 – 340.48]\ \frac{W}{m^2}$	Top of atm. total solar irradiation	(Coddington, Lean et al. 2017)

AOD_t	[0.2 – 142.9]	Aerosol optical depth	(Miller, Schmidt et al. 2014; Nasa/Larc/Sd/Asdc 2018)
AC_t	[-1.09 – -0.06] $\frac{W}{m^2}$	Anthro. cloud radiative forcing	(Forster, Smith et al. 2023)
$[eCO_2]_t$	[287.9 – 563.4]	Effective CO_2 concentration, ppm	(Forster, Smith et al. 2023)

Table 1: Constants and climate driver datasets referenced in Eqs. 6-8, in addition to temperature baselines. Equations referenced in "Source" column are found in Appendix A and Supplement A1&A2.

Future projections along the shared socioeconomic pathways (SSPs) for the EBM-KF also require the concentrations of greenhouse gasses including carbon dioxide ($[eCO_2]_t$), stratospheric aerosol optical depth due to volcanic dust and human emissions (AOD_t), and reflective flux from anthropogenic clouds (AC_t). ESMs simulate the carbon cycle and thus find an equivalent of $[eCO_2]_t$ from specified CO_2 and greenhouse gas emissions, but our EBM-KF does not have this capability. Future greenhouse gas concentrations and anthropogenic cloud forcings are instead taken from a conversion of anthropogenic fluxes by the MAGICC7.0 carbon cycle emulator (Meinshausen, Nicholls et al. 2020), as reported by (Smith, Forster et al. 2021). For instance, SSP1-2.6 and SSP3-7.0 are shown in Figs. 9 & 10, which flank the most likely result of current environmental policies (Pielke Jr, Burgess et al. 2022). Projection of anthropogenic forcings from Nazarenko et. al. (2022) using the NASA GISS ESM yield very similar future curves (not shown).

Overall, the blind (forward) energy-balance model (orange dashed line in Fig. 2) has 4 yearly forcing inputs ($[eCO_2]_t$, AOD_t, AC_t, $(\frac{G_{SC}}{4})_t$) and 17 irreducible parameters (including 1 inferred exponent η, 4 inferred β coefficients, 3 heat capacities, and 3 reference temperatures). The time step of this iterative difference equation model is 1 year chosen arbitrarily to coincide with the calendar year. The deep ocean potential temperature θ_t is recalculated at each time step from the GMST (T_t) and the OHCA (H_t) by (6), and then these two terms are updated with (7)-(8). The measured temperature in the year 2002 (Y_{2002}) appears prominently in this model because that was the midpoint of the measurement window of the CERES satellite (Wielicki, Barkstrom et al. 1996; Loeb, Wielicki et al. 2009), and all albedo-related feedbacks are expressed relative to these measurements. For this model, the OHCA (H_t) is calculated in units of W*year/m^2 on an average of the Earth's surface, and then converted to ZJ within the ocean by multiplying by a factor of 11.42 = 3.154e7 s/year * 5.101e7 m^2 / Earth surface * 0.71 ocean/surface. This time-step function (6-8) and its partial derivative (see Appendix A4) will become critical parts of our Kalman Filter: (9, 10) below.

This blind EBM model had good skill at predicting the GMST with r^2=0.902 when compared to the HadCRUT5 GMST timeseries (Morice, Kennedy et al. 2021), and OHCA with r^2=0.907 when compared with the inferred history (Zanna, Khatiwala et al. 2019), as is demonstrated by the dashed orange lines in Fig. 3. The blind EBM has a comparably high correlation (r^2=0.890) with the 30-year running mean (i.e., the climate normal) of the HadCRUT5 GMST, indicating that this forward energy balance model also has skill in reproducing the climate state as determined by standard approaches, with departures due to volcanic eruptions. Thus, most observed climate change can be explained by the literature-based blind, forward EBM and records of anthropogenic emissions (greenhouse gasses, cloud-seeding tropospheric aerosols) and records of top-of-atmosphere aerosol optical depth (primarily altered by stratospheric aerosol concentrations), consistent with recent forward-EBM applications (Hu and Fedorov 2017; Kravitz, Rasch et al. 2018). The distribution of residuals in the GMST record from either the 30-year running mean or the EBM has small bias and skewness (see Supp. Fig. 9). These residuals' kurtosis is slightly less than Gaussian to accommodate measurement uncertainty, as discussed in Section 3a in relation to Figs. 4 & 5. So, the 30-year running mean's "weather" or "noise" empirical probability density function combining residuals and measurement uncertainty is very nearly Gaussian, and thus amenable to treatment by a Kalman filter framework (see section 2b). The Fig. 3 forward model comparisons were made without any assimilated data, illustrating that the EBM physics alone has skill in reproducing aspects of the GMST and OHCA records. Tuning the EBM parameters may further improve skill, but the EBM is only the forward projection component of the data assimilating Kalman Filter hybrid model described in the next section. The combined system is the focus of this paper.

b. EBM-Kalman Filter: A Weighted Average of Energy Balance and Measurements

While similar algorithms were developed in the 1880s by Thorvald Nicolai Thiele (Lauritzen 1981; Lauritzen and Thiele 2002), Kalman filtering rose to prominence due to its use in the Apollo navigation computer as proposed by Ruslan Stratonovich (1959; 1960), Peter Swerling (1959), Rudolf E. Kálmán (1960), Richard S. Bucy (1961), and implemented by Stanley Schmidt (1981). Versions of this statistical filter are universally used in aerospace guidance systems (Grewal and Andrews 2001), aspects of numerical weather prediction (Houtekamer and Mitchell 1998; Kalnay 2002), and recently popularly in climate science as Ensemble Kalman filters (which use a Monte Carlo approximation via simulations in highdimensional space, see below). Ensemble Kalman filters (not to be confused with Extended Kalman filters, the local linearization extension method of this paper described below) have been instrumental to 20th century reanalysis (Compo, Whitaker et al. 2011) and last millennium reanalysis projects (Hakim, Emile-Geay et al. 2016) of global atmospheric circulation. In the Ensemble Kalman Filter, observations sample the full gridded weather patterns (a space with hundreds to millions of dimensions) within an ensemble of ESMs. Despite the success of Ensemble Kalman filters, *Extended* Kalman filters are computationally intractable as the sole data assimilation tool for atmospheric weather patterns (Bouttier 1996) because many local weather processes do not sample from a Gaussian distribution. However, the central limit theorem states that taking the average of many independent non-Gaussian samples will produce a mean that approximates a Gaussian distribution. This is the case for annual GMST (Montgomery and Runger 2013), which is the average of many non-Gaussian regional and daily weather patterns (Quevedo and Gonzalez 2017). Likewise, while annual OHCA is largely constrained by the subtropical pycnocline depth (Newsom, Zanna et al. 2023), it too is comprised of numerous regional and seasonal patterns (Hummels, Dengler et al. 2013; Cheng, Trenberth et al. 2017; Huguenin, Holmes et al. 2022). In this case of *global* GMST and OHCA, an Extended Kalman Filter works because both measurement and dynamical noise are approximately Gaussian (by the Central Limit central limit theorem, to be verified in Section 3), and the energy-balance equation (Section 2a) has a continuous and bounded gradient (see Appendix A4), so it can be locally linearized. Careful construction of the EBM with T^2 in the shortwave term and $T^{2.39}$ in the counteracting longwave term in (Eqs. 1 & 5) ensures the derivative (Eqs. A37-41) does not change significantly over the relevant range of temperatures [286 – 291]K, $[eCO_2]_t$ effective CO_2 concentrations [278 – 2000] ppm, AOD_t aerosol optical depths [0 - 0.15], and AC_t anthropogenic cloud forcing [-1 – 0] W/m^2. This approximate linearity means that more complex realizations of the Kalman filter, particularly the Unscented Kalman Filter (Julier and Uhlmann 1997; Wan and Van Der Merwe 2000) are not necessary (see Supplement Section D). This approach has already proven successful using a 1-(spatial)-dimensional (north-south) energy balance model, with time-steps of decades (or longer), and optimized for use in paleoclimate research (García-Pintado and Paul 2018). Thus, for a variety of reasons an EBM-Kalman Filter (EBM-KF) can be built from an Extended Kalman Filter combined with an (annual, 0-spatial-dimensional) Energy Balance Model.

In-depth derivations and tutorials for constructing Kalman filters have been published elsewhere (Miller 1996; Lacey 1998; Särkkä 2013; Benhamou 2018; Youngjoo and Hyochoong 2018; Ogorek 2019). Here we describe enough for basic intuition, and we refer readers to Kalnay (2002), page 281, for a more detailed explanation with alternative notation. We use the term "forecast" where other authors use "prior", and we avoid use of "measurement error" in a manner that would be ambiguous and confusing in this application. Initially, there is some estimated *state vector* (GMST and OHCA within this paper) $\hat{\mathbf{x}}_{\mathbf{t-1}}$ and a Gaussian uncertainty envelope around this vector defined by a *state covariance matrix* $\mathbf{P}_{\mathbf{t-1}}$. This *state vector* is transformed (or projected) one year into the future using a *dynamic model Jacobian matrix* $\mathbf{\Phi}$ into a *forecast state* $\hat{\mathbf{x}}_{\mathbf{t|t-1}} = \mathbf{\Phi}\hat{\mathbf{x}}_{\mathbf{t-1}}$, a transformation that may depend on time-varying control parameters u_t. For our climate system this linear projection is extended to the nonlinear function $\hat{\mathbf{x}}_{\mathbf{t|t-1}} = \mathbf{F}(\hat{\mathbf{x}}_{\mathbf{t-1}}; u_t)$ in (11), which is just the forward energy balance model equations (6)-(8), where u_t represents the collection of climate forcings ($[eCO_2]_t$, AOD_t, AC_t, $(\frac{G_{SC}}{4})_t$).This simple extension to nonlinearity is the meaning of "Extended" Kalman Filter. The *state covariance* is projected to the next year using a local linear approximation $\boldsymbol{\Phi_t}$ (Eq. 9) and enlarges by an additional *assumed model error covariance* $\mathbf{Q}$, yielding $\mathbf{P}_{\mathbf{t|t-1}}$ the *forecast covariance* (Eq. 12). To arrive at a *posterior* (including observations) information a *measurement vector* $\mathbf{y}_{\mathbf{t}}$ is considered (Eq. 13). The probabilistic range of discrepancies between $\hat{\mathbf{x}}_{\mathbf{t|t-1}}$ and $\mathbf{y}_{\mathbf{t}}$ is given by the *innovation covariance matrix* $\mathbf{S}_{\mathbf{t}}$, which is the sum of $\mathbf{P}_{\mathbf{t|t-1}}$ and an *assumed measurement covariance* $\mathbf{R}$ (14). The *posterior estimate* of the state $\hat{\mathbf{x}}_{\mathbf{t}}$ is found by taking a weighted average of $\hat{\mathbf{x}}_{\mathbf{t|t-1}}$ and $\mathbf{y}_{\mathbf{t}}$ (16), with the weight on $\mathbf{y}_{\mathbf{t}}$ given by $\mathbf{P}_{\mathbf{t|t-1}}(\mathbf{S}_{\mathbf{t}})^{-1}$, a product known as the *Kalman gain* (15). To reflect the greater certainty in the state vector because of this correction, $\mathbf{P}_{\mathbf{t}}$, the *posterior covariance matrix*, is $\mathbf{P}_{\mathbf{t|t-1}}$ shrunk by a factor of ($\mathbf{I}$ minus the Kalman gain) per (17). Within the context of climate modeling, this "posterior state estimate" is somewhat analogous to a climate reanalysis product, as both combine observations and models. Within the context of Bayesian probability, the *prior (forecast) distribution* is given by projecting $N(\hat{\mathbf{x}}_{\mathbf{t-1}}, \mathbf{P}_{\mathbf{t-1}})$ into the future using the Jacobian matrix $\mathbf{\Phi}$, which is multiplied by the marginalized likelihood of $\mathbf{y}_{\mathbf{t}}$ to give a *posterior distribution* $N(\hat{\mathbf{x}}_{\mathbf{t}}, \mathbf{P}_{\mathbf{t}})$. Note that $(\Phi_t)^*$ in (12) below indicates matrix transposition.

$$\Phi_t = \frac{\partial \mathbf{F}(x;u_t)}{\partial x}\Big|_{x=\hat{x}_{t-1}} \qquad \text{linearization at timepoint t} \qquad (9)$$

$$\begin{cases} \boldsymbol{x}_t = \mathbf{F}(x_{t\text{-}1}; u_t) + \boldsymbol{w}_t & \text{dynamic model, error:} \quad \text{Q=Cov}(\boldsymbol{w}_t) \\ \boldsymbol{y}_t = \boldsymbol{x}_t + \boldsymbol{v}_t & \text{weather variability, error:} \quad \text{R=Cov}(\boldsymbol{v}_t) \end{cases} \tag{10}$$

$$\hat{\boldsymbol{x}}_{t|t\text{-}1} = \mathbf{F}(\hat{\boldsymbol{x}}_{t\text{-}1}; u_t) \qquad \text{forecast ("prior") state estimate} \tag{11}$$

$$\mathrm{P}_{t|t\text{-}1} = \Phi_t \mathrm{P}_{t\text{-}1} (\Phi_t)^* + \mathrm{Q} \qquad \text{forecast ("prior") covariance} \tag{12}$$

$$\mathbf{z}_t = \boldsymbol{y}_t - \hat{\boldsymbol{x}}_{t|t\text{-}1} \qquad \text{innovation residual} \tag{13}$$

$$\mathrm{S}_t = \mathrm{P}_{t|t\text{-}1} + \mathrm{R}_t \qquad \text{innovation covariance} \tag{14}$$

$$\mathrm{K}_t = \mathrm{P}_{t|t\text{-}1} (\mathrm{S}_t)^{-1} \qquad \text{Kalman gain} \tag{15}$$

$$\hat{\boldsymbol{x}}_t = \hat{\boldsymbol{x}}_{t|t\text{-}1} + \mathrm{K}_t \mathbf{z}_t \qquad \text{posterior state estimate} \tag{16}$$

$$\mathrm{P}_t = (\mathbf{I} - \mathrm{K}_t) \mathrm{P}_{t|t\text{-}1} \qquad \text{posterior state covariance} \tag{17}$$

If $\mathbf{y}_t$ is an indirect measurement of the state vector $\mathbf{x}_t$, an observation (or emission) matrix **H** further complicates the procedure (details in the references above). Here we consider only direct "observations" of GMST and OHCA making mapping and interpolation errors implicit and the observation matrix **H** = **I**.

The true climate state $\boldsymbol{x}_t$ is the 2-entry vector underlying GMST and OHCA, filtering out weather and internal variability: $\boldsymbol{x}_t = [T_t, H_t]$. Throughout this paper, [a, b] indicates a 2-dimensional vector with components a and b, and we use italic symbols to indicate the climate state vector containing temperature and heat content. The noisy measurements $\boldsymbol{y}_t = [Y_t, \psi_t]$ are the yearly time series of GMST and OHCA, and $\hat{\boldsymbol{x}}_t = [\hat{T}_t, \hat{H}_t]$ is the estimate of the unknown 2-dimensional climate state, expressed in degrees Kelvin and $\frac{W\,yr}{m^2}$ (convertible to ZJ by the factor 5.1006*3.154*0.71 = 11.42). The energy-balance model F (10) governing $\hat{T}_t$ is nonlinear (with T^2 and $T^{2.385}$ terms due to albedo and Planck feedbacks) (Friedrich, Timmermann et al. 2016), which necessitates some way to handle the nonlinearity: an Extended Kalman filter, Ensemble Kalman filter, or Unscented Kalman filter are among the potential nonlinear methods. In our Extended Kalman Filter, the *forecast* state $\hat{\boldsymbol{x}}_{t/t\text{-}1}$ (11) is given by (6)-(8) above and Φ_t and the *forecast* covariance projection (12) is a time-varying linearization (Eqs. A21-A25). This energy-conserving difference equation thus resembles a first-order Taylor series approximation of a differential energy-balance model (if discretization errors are considered part of the tendency), or the integral form of a conservative discretization in time (if shortwave and longwave fluxes are taken as a model for their time-integrated value), and the Kalman Filter re-approximates a GMST and OHCA climate state every year. The initial estimated state uncertainty is intentionally overestimated at $P_{1850} = \begin{bmatrix} 1K^2 & 1\,K\frac{W\,yr}{m^2} \\ 1\,K\frac{W\,yr}{m^2} & 20\,\left(\frac{W\,yr}{m^2}\right)^2 \end{bmatrix}$ and then P_t rapidly converges (within 15 years) in the EBM-KF to $P_{1865} = \begin{bmatrix} 0.0017\,K^2 & 0.035\,K\frac{W\,yr}{m^2} \\ 0.035\,K\frac{W\,yr}{m^2} & 4.0\,\left(\frac{W\,yr}{m^2}\right)^2 \end{bmatrix}$, and then continues to slowly shrink with time as more accurate measurements are made. For convenience we form confidence intervals for GMST and OHCA climate state by taking twice the square root of the respective diagonal elements of P_t. For clarity, we give both diagonal elements their own symbols, and similarly for S_t, noting that here T as a superscript just labels a scalar value.

$$[\hat{p}_t^T, \hat{p}_t^H] = \mathrm{diag}(P_t) \quad (18a) \qquad\qquad [\hat{s}_t^T, \hat{s}_t^H] = \mathrm{diag}(S_t) \quad (18b)$$

$$\text{95\% CI GMST state, 1865: } \hat{T}_{1865} \pm 2\sqrt{\hat{p}_{1865}^T} = 286.66\mathrm{K} \pm 2\sqrt{0.0017\mathrm{K}^2} = 286.66 \pm 0.07\mathrm{K} \quad (19)$$

Similarly, we use the diagonal elements of S_t to form confidence intervals of next-year measurements about $\hat{\boldsymbol{x}}_{t|t-1}$ (18b). The Extended Kalman Filter implicitly assumes that Gaussian “model” noise is added to this climate state at each time step (10a), and additionally the climate state emits annual “weather variability” from a yet wider Gaussian noise distribution (10b). Whereas we interpret global annual weather to be the noisy measurements $\boldsymbol{y}_t = [Y_t, \psi_t]$, “weather variability” is observed via *innovation residuals* $\boldsymbol{z}_t$, These *innovation residuals* have components $[z_t^T, z_t^H] = \boldsymbol{z}_t$, and the Kalman Filter expects them to come from an unbiased Gaussian noise distribution with covariance S_t (not R_t because the Kalman Filter does not have knowledge of the true climate state $\boldsymbol{x}_t$.)

$$z_t^T = Y_t - \hat{T}_{t|t-1} \quad (20a) \qquad\qquad z_t^H = \psi_t - \hat{H}_{t|t-1} \quad (20b)$$

The EBM-KF climate state $\hat{\boldsymbol{x}}_t$ and state covariance P_t only access information from the measurements taken prior to and at year t: $\{\boldsymbol{y}_{1850}, \boldsymbol{y}_{1851}, \ldots \boldsymbol{y}_t\}$. This past-to-present Kalman Filter (Eqs. 9-17) can be further extended into a RTS smoother (Rauch, Tung et al. 1965) by additional steps (see Supp. Section A3), which meld information from all measurements in the time window $\{\boldsymbol{y}_{1850}, \boldsymbol{y}_{1851}, \ldots \boldsymbol{y}_{2023}\}$ into each re-estimated posterior state $\hat{\hat{x}}_t$ and posterior state covariance $\hat{\hat{P}}_t$ by running backward from the latest EBM-KF state estimates ($\hat{\boldsymbol{x}}_{2023}$ and P_{2023}). In the 1850 to present application, this extension has little effect on $\hat{\hat{x}}_t$ (Supp. Fig. 2), but there is greater certainty in the smoothed state. Defining diag($\hat{\hat{P}}_t$)=[$\hat{\hat{p}}_t^T$, $\hat{\hat{p}}_t^H$], for the GMST uncertainty $\hat{p}_t^T \approx 2.25 * \hat{\hat{p}}_t^T$, and for the OHCA uncertainty $\hat{p}_t^H \approx 2.84 * \hat{\hat{p}}_t^H$.

In summary, the Extended Kalman Filter projects forward one year into the future based on the unbalanced fluxes of the energy balance model equation, and then takes a weighted average of this projection with the annual GMST measurement (the data assimilation increment). Thus, even though the EBM conserves energy (by construction), the combined EBM-KF does not, unlike other alternative data assimilation approaches (Wunsch and Heimbach 2007). The state estimates from this EBM-KF (in navy blue in Fig. 3) often lie between the blind EBM (in dashed orange in Fig. 3) and the annual temperature measurements (scattered gray dots in Fig. 3), a corrective effect that can be seen most clearly within the GMST measurements in Fig. 3a from 1900 to 1945 and within the OHCA measurements in Fig. 3b from 1940 to 1970. It is possible for the EBM-KF state estimates to escape these bounds for a short time, for instance from 1945 to 1950 in Fig. 3a or after 2007 in Fig 3b. These "escape periods" may reflect bias in the measurements, such as warm-biased WWII-era measurements of (sea) surface temperature (Chan and Huybers 2021), or the bias of the Zanna et al. (2019) OHCA product to indicate less heat uptake since 2005 than all but 1 of 19 other OHCA estimates (Gulev, Thorne et al. 2021). Both the "blind" EBM predictions $[\tilde{T}_{t+1}, \tilde{H}_{t+1}] = \boldsymbol{F}(\tilde{T}_t, \tilde{H}_t;\ u_t)$ where u_t represents the 4 climate forcings, and EBM-KF state estimates $\hat{\boldsymbol{x}}_t = [\hat{T}_t, \hat{H}_t]$ dip down with each major volcanic eruption within the AOD record (see Fig. 11 in the Discussion, Section 4). These volcanic dips are far more pronounced for the GMST component than for OHCA (see Fig. 3) and are present only as flat spots in the deep ocean potential temperature curve (see Supp. Fig. 11).

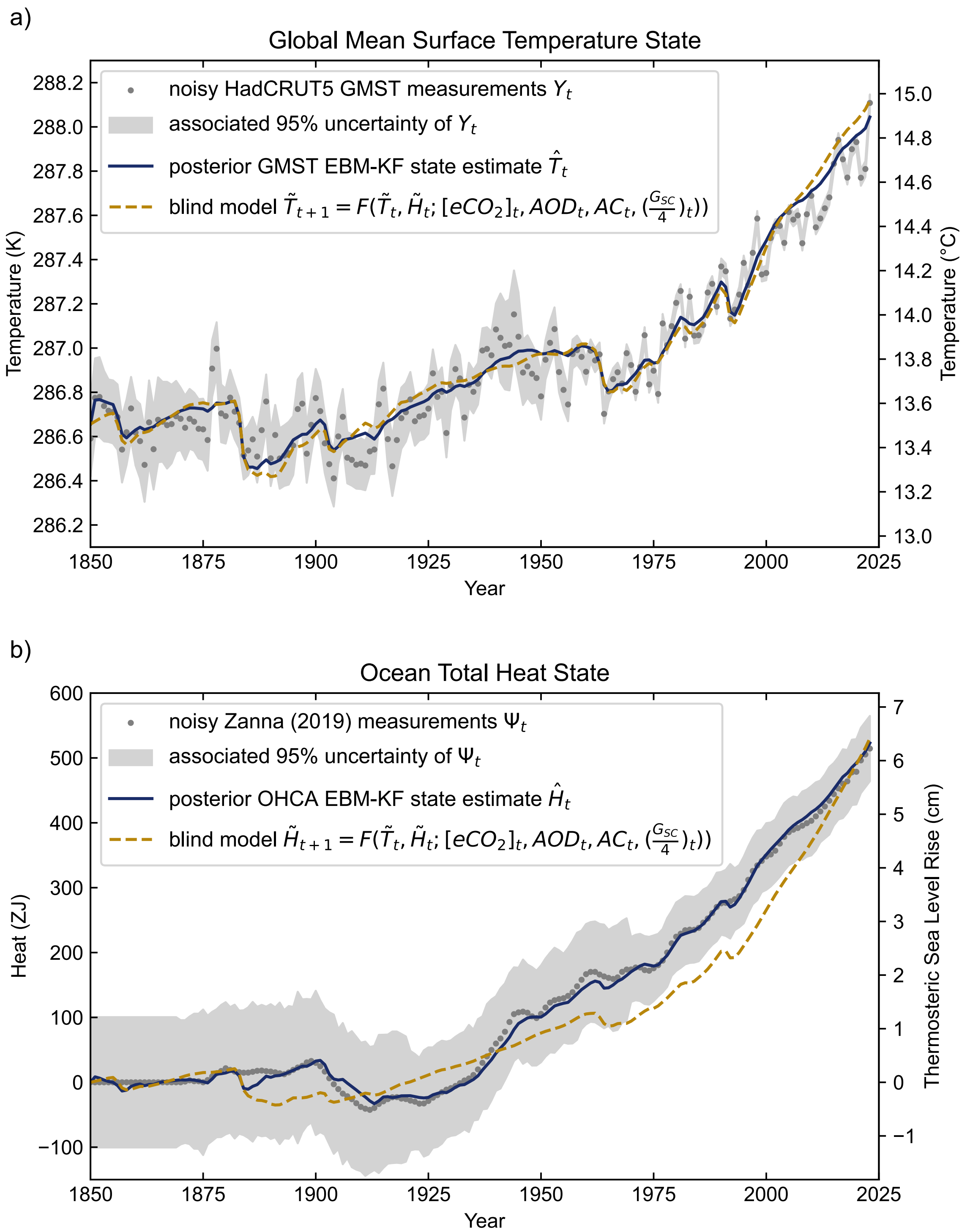

Fig. 3: Behavior of the EBM-KF state in relation to blind EBM projections and the stochastic measurements of GMST and OHCA. Panel a) shows GMST prediction and b) the OHCA prediction. The blind model (dashed orange) and Kalman Filter state estimate (navy blue) use EBM dynamics to project from the previous state to the current state, but the state estimate also assimilates observations (grey dots; GMST from HadCRUT5 (Morice, Kennedy et al. 2021) and OHCA from Zanna et al. (2019)). Incorporation of these observations makes only small modifications to the EBM-KF's GMST state in a), whereas in b) there is an impressive difference between the blind EBM's OHCA projections and the EBM-KF's OHCA state - the latter sticks close to observations.

c. Selection of Model Uncertainty and Time-Varying Measurement Uncertainty

Fig. 3 also demonstrates the accuracy associated with each of the temperature measurements. The uncertainty in the climate state P_n automatically responds to unexpected values of the measured temperature (Wunsch 2020). The HadCRUT5 GMST decreases in reported measurement standard deviation from 0.079K in the 1850-1879 window to 0.017K in the 1990-2019 window (Morice, Kennedy et al. 2021), a 78% reduction primarily reflecting a lack of observations in the Southern hemisphere before the satellite age. The inferred deep ocean heat content taken primarily from a hybrid model-observation reconstruction (Zanna, Khatiwala et al. 2019) has a very wide confidence interval before the introduction of modern sampling methods in the 1970s. We use the Zanna et al. (2019) hybrid product due to its long duration of OHCA estimates (based on surface forcing in early years) rather than the shorter direct measurement products (Ishii, Fukuda et al. 2017), although both could be assimilated simultaneously within EBM-KF if desired (as discussed in Section 4c). The additional increase in OHCA after 2018 was provided from a separate NCEI dataset (Levitus, Antonov et al. 2017). Our EBM-KF incorporates these known physical measurement uncertainties in the HadCRUT5 measurements of GMST and the OHCA reconstruction as R_t^{var}. The total assumed measurement covariance R_t (in Eq. 14) is composed of two components: the time-varying physical measurement uncertainty R_t^{var}, and the constant uncertainty R^{const} reflecting internal variability due to dynamical chaos: primarily ENSO and other climate oscillations with limited predictability. Both forms of measurement noise are added onto the underlying climate signal via the random vector $\mathbf{v_t}$ to generate annual observations (see Eq. 10). We assume that R_t^{var} is diagonal and simply sum the two variance matrices to obtain a time-varying value:

$$R_t = R_t^{var} + R^{const} \tag{21}$$

The realization of the EBM-KF shown in Fig. 2 also has a measurement uncertainty R^{const} that is constant in time and based on the [HadCRUT5's GMST, Zanna (2019) OHCA] residual co-variance with respect to their 30-year running means. In other words, we computed:

$$R^{const} = \mathrm{Cov}(\boldsymbol{y_t} - {}_{30}\overline{\boldsymbol{y}}_{\boldsymbol{t}}) = \begin{bmatrix} 0.01107\,K^2 & 0.04627\,K\,\frac{W\,yr}{m^2} \\ 0.04627\,K\,\frac{W\,yr}{m^2} & 1.17278\,\left(\frac{W\,yr}{m^2}\right)^2 \end{bmatrix} = 30{*}Q \tag{22}$$

The assumed model covariance, Q (see Eq. 12), is set to $R^{const}/30$ to emulate the 30-year running average definition of climate state (Guttman 1989), thus we assume that the random noise contained within the climate model has a variance that is $1/30^{th}$ as large as the variance in the "weather" measurements. By this simple method, the data-assimilating EBM-KF is tuned to match the "standard climate normal", as any 30-member sample average has a variance $1/30^{th}$ as large as the annual measurements' variance (assuming yearly anomalies are uncorrelated). Variance in these annual measurements arises due to chaos within the climate system, so this R^{const} contribution to the model and measurement uncertainty would exist even if all measurements could be made with arbitrary accuracy.

d. Non-Gaussian Future Projection and Sampling of Volcanic Activity

The EBM-KF can project one year into the future, given greenhouse gas and aerosol concentrations, without any new measurements using only the forward model to obtain *forecast* estimates (11)-(12). To project farther into the future, the *posterior* state and *posterior* covariance are set equal to the *forecast* state and *forecast* covariance, i.e., a *posterior* unaffected by any new observations: $\hat{\mathbf{x}}_{\mathbf{t}}=\mathbf{F}(\hat{\mathbf{x}}_{\mathbf{t-1}})$ and $P_t=\Phi_t P_{t-1}(\Phi_t)^T+Q$. While these far-future state estimates behave equivalently to a blind model, the covariance grows over time, either sub-linearly or exponentially (see Section 3d).

While the SSPs are used for most forcing variables, future volcanic eruptions require modeling as well. Volcanic eruptions determining AOD_t are inherently stochastic, but the time intervals between eruptions can be approximated using exponential distributions (Papale 2018). In standard ESM SSPs, future volcanism is usually included by a steady "background" volcanism, neglecting volcanism's intermittency and the associated exponential distributions. Even though the EBM-KF assumes Gaussian error and thus cannot include exponential distributions in the same way as measurement and internal chaotic variability, it is so computationally inexpensive that it can be rerun to sample over complex non-Gaussian distributions. This ability to include future volcanoes illustrates a major advantage of this system: thousands of future scenario inputs can be generated and utilized within minutes on a laptop, while each ESM of the LENS2 ensemble took over a week to run on a supercomputer (roughly a billion times more effort in core-hours per ensemble member) which limits the ensemble size and thus motivates using only a background constant level of volcanism. No single exponential distribution fits well to the observed series of volcano eruption intervals, so an exponential mixture with two components was found to be the best fit to the data using the decomposed normalized maximum likelihood (Okada, Yamanishi et al. 2020). See Appendix B for further details.

3. Results

a. Examination of the EBM-KF Climate State (1850-Present)

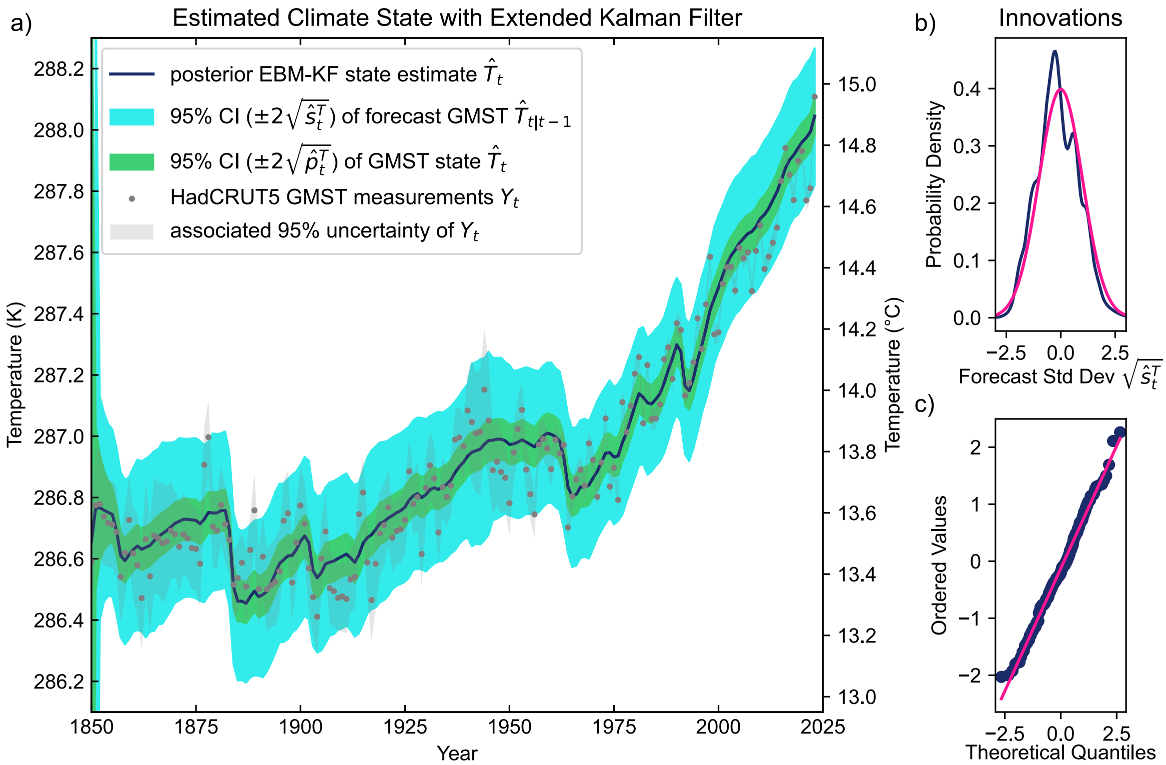

Fig. 4: EBM-KF and associated uncertainties. a) The EBM-KF climate state estimate (navy blue line) is drawn with a 95% or *extremely likely* confidence interval (light green area) of its posterior uncertainty $\pm 2\sqrt{\hat{s}_t^T}$. Annual-mean HadCRUT5 GMST measurements are assimilated (gray dots and gray area mostly within the light blue). A 95% confidence interval (CI) in light blue indicates the forecast uncertainty $\pm 2\sqrt{\hat{s}_t^T}$. b) The Gaussian mixture of innovations z_t^T (deviations between measurements minus Y_t the projected climate state $\hat{T}_{t|t-1}$) with each year's associated measurement uncertainty (navy blue), normalized onto a horizontal axis labeled with standard deviations $\sqrt{\hat{s}_t^T}$ of the ideal forecast covariance (pink). c) Quantile-quantile plot of these normalized innovations ($z_t^T \,/\, \sqrt{\hat{s}_t^T}$). All panels demonstrate that the gray HadCRUT5 GMST observations appropriately fill out the 95% CI of the forecast uncertainty (light blue) around the EBM-KF state estimate (navy blue).

A primary product of this paper is the EBM-KF climate state, spanning from 1850 to present. Recall that the forward EBM uses published literature values: this is not an empirical fit to GMST and OHCA data, but rather the EBM-KF assimilates these data. We first examine the GMST component $\hat{T}_n$ of the Kalman-Filtered climate state $\hat{\boldsymbol{x}}_{\rm n}$. There are two distinct Gaussian distributions relevant to understanding our method: the uncertainty in the current GMST *climate state*, as graphed in narrow green envelope in Fig. 4a, and the uncertainty window of possible next-year (forecast) GMST measurements, as graphed in the light blue wider envelope in Fig. 4a. Further examination of the "*update*" difference (from $K_t \boldsymbol{z}_t$ in Eq. 16) of the posterior estimated states minus forecast states ($\hat{T}_t - \hat{T}_{t|t-1}$) reveals that in any individual year after 1855, assimilation of the GMST measurement only shifts the forecast GMST state projection $\hat{T}_{t|t-1}$ by -0.001±0.009K/yr (± standard deviation), range [-0.020 – 0.022]K/yr. This *update* value is miniscule compared with the GMST adjustment in $\tilde{T}_n$ from the blind, forward EBM contribution of forced climate state change of +0.025±0.027K/yr since 1975, and +0.002±0.027K/yr from 1850 to 1975, while the forecast change can be as large as [-0.191 - 0.053]K in a single year.

However, as shown in Fig. 3, repeated small *updates* in the same direction (due to repeatedly lower or higher than expected GMST measurements) can drift $\hat{T}_t$ away from the blind model estimate $\tilde{T}_t$. This "*accumulated correction*" ($\hat{T}_t - \tilde{T}_t$) is +0.004K on average, and as much as [-0.086 – 0.096]K (after 1885: +0.02K range [-0.086 – 0.062]K). *Accumulated corrections* are 3-4 times larger in extreme than the most extreme *updates*, indicating that these *updates* had accumulated over >4 years prior to 1886 and 2022 (~5 and ~8 years respectively, see Fig. 3a). Note the mean *accumulated correction* is slightly positive while the mean *update* is slightly negative because of the influence of OHCA corrections (see below and Fig. 3b). The EBM-KF state $\hat{T}_t$ is still very highly correlated with the blind, forward EBM $\tilde{T}_n$ (r^2=0.992). Measurements Y_t have nearly equal warming and cooling contributions to the underlying $\hat{T}_t$ climate state, forming the expected Gaussian distribution of normalized *innovations* ($z_t^T \,/\, \sqrt{\hat{s}_t^T}$) as demonstrated over the entire timeseries in Fig 4b and in every full 50-year period in Supp. Fig. 12. The GMST observations since 2000 slightly cool the EBM (Supp. Fig. 12d,h) indicating that the EBM may have oversized positive climate feedbacks, an issue which could be rectified with parameter adjustment (Section 4c).

After an initial convergence period of about a decade, the green 95% CI of the GMST state uncertainty $2\sqrt{\hat{p}_t^T}$ slightly shrinks from ±0.067K in the 1870s to ±0.062K since 1980, a reduction of -7.5%. The 95% CI of forecast uncertainty, from $2\sqrt{\hat{s}_t^T}$, is drawn in light blue around the forecast estimated GMST state projection $\hat{T}_{t|t-1}$, showing where the Kalman Filter expects the subsequent year's temperature measurement to be. This forecast uncertainty converges from roughly ±0.26K in the 1870s to ±0.223K since 1980. Both reductions reflect the improvement in the GMST component of the time-varying measurement uncertainty, R_t^{var}, with modern satellite observations. But these reductions are modest compared to the 76% reduction in time-varying HadCRUT5 measurement uncertainty over the same period because the EBM-KF is also assuming time-invariant climate process uncertainty, Q and the associated $\mathrm{R}^{\mathrm{const}}$. The empirical projection probability distribution (a Gaussian mixture of all measurement uncertainties relative to the EBM-KF forecast distribution) and an ideal Gaussian distribution closely match (Fig. 3b), confirming that the annual measurements of GMST can be interpreted as Gaussian noise around an underlying climate state. The quantile-quantile plot (Fig. 3c) demonstrates this same finding, just using gray points of innovations (z_t^T the difference between EMB-KF forecasts $\hat{T}_{t|t-1}$ and measurements Y_t) rather than each innovation being a distribution (with variance from R_t^{var}) as in Fig. 3b. Each innovation point is normalized to the forecast uncertainty ($z_t^T \,/\, \sqrt{\hat{\mathrm{s}}_t^{\mathrm{T}}}$), and then these are sorted from lowest to highest and plotted on the vertical axis. Along the horizontal (theoretical quantiles) axis, the percentile of each innovation is plotted where it would lie on an ideal Gaussian distribution, showing the real GMST "weather" measurements from HadCRUT5 are distributed around the EBM-KF GMST climate state in precisely the expected Gaussian distribution.

As we hoped, the EBM-KF GMST climate state estimate over 1850 to present is not substantively different from the 30-year running average except for the impact of major volcanoes (see Fig. 10a, r^2=0.923), thus $\hat{\mathrm{T}}_{\mathrm{t}} \approx \overline{{}_{30}Y_t}$ in non-volcano years. The LENS2 hindcasts depart from both in the interval from 1940 – 2000 (see Fig. 10a) causing a lower r^2=0.906 over all 174 years between EBM-KF and LENS2. The EBM-KF with unfiltered volcanic forcing can thus be interpreted as a middle ground between the 30-year running average and a LENS2 ensemble average (which are farther apart with r^2=0.820). The performance of the GMST and OHCA portions of EBM-KF model do vary; the most noticeable biases (see Fig. 3) are that the blind OHCA is significantly corrected toward the Zanna et al. (2019) reconstruction of OHCA from 1875-2005 (assimilation of this data reconstruction continues through 2018), but these correction periods are not evident as persistent biases in the EBM-KF (Fig. 5). Forward model biases may be ameliorated by automated, optimized tuning of parameters. This is addressed in Section 4c and is well-studied in Kalman filter applications (Zhang and Atia 2020); the potential adoption of these tools to climate science is a key advantage of the EBM-KF hybrid.

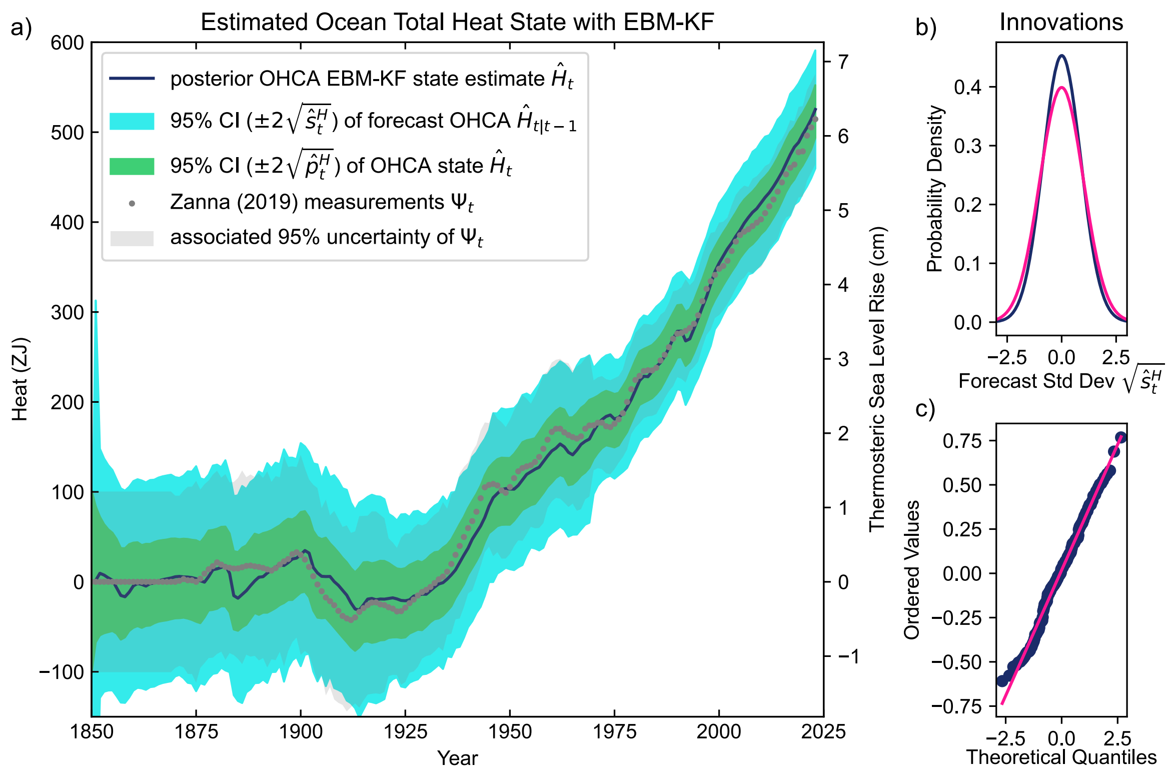

Fig. 5: EBM-KF state estimate (navy blue) for deep ocean OHCA in units of mean potential temperature from the same EBM-KF run as in Fig. 3. 95% CI of forecast estimate is drawn in light blue, and posterior 95% CI is drawn in green. Annual-mean Zanna et al. (2019) reconstructions are assimilated (gray dots and gray area almost entirely within the light blue). Other panels and colors as in Fig. 4. All panels demonstrate that the uncertainty window of the assimilated OHCA data (gray) closely corresponds to the 95% CI of the forecast uncertainty (light blue) around the EBM-KF state estimate (navy blue).

Fig. 5 shows the deep OHCA component of the EBM-KF and its associated uncertainties. While the OCHA measurements from the Zanna et al. (2019) hybrid product are more autocorrelated than the HadCRUT5 GMST (relatively less year-to-year variability), the innovations for OHCA are again approximately Gaussian (panels 5b, 5c). In the context of this empirical probability distribution, each member of the Gaussian mixture has a larger gray window given by the time-varying measurement uncertainties R_t^{var} from the OHCA measurements. In simpler language, the light blue forecast window is large because it must encapsulate the gray measurement uncertainty window, which moves around within it. To achieve the nearly Gaussian empirical probability distribution in panel 5b, it is unsurprising that most EBM-KF estimated states are pulled very close to the autocorrelated OHCA measurements in Figs. 5a & 3b. This is a situation dominated by measurement uncertainty R_t^{var}, which is different than observable dynamic “weather variability” (innovations z_t^T)

filling the full forecast distribution (light blue) in Fig. 4a. As a result, the OHCA component of the EBM-KF pays much more attention to these measurements ψ_t than relying mostly on the blind EBM (see Fig. 3b). This *updates* the OHCA state estimate ($\widehat{H}_t - \widehat{H}_{t|t\text{-}1}$) after 1855 by 0.05±3.72 ZJ/yr, range [-8.16 – 9.78]ZJ/yr; comparable with the OHCA change in $\widetilde{H}_t$ from the blind, forward EBM contribution 3.07±5.30ZJ/yr, up to [-25.31 – 14.72]ZJ/yr. Unsurprisingly, the EBM-KF takes a substantially different track than the blind EBM, yielding an *accumulated correction* of up to +91.6ZJ in 1998. Reflecting this improvement in measurement accuracy (as incorporated via R_t^{var}), the OHCA components of both state uncertainty $2\sqrt{\hat{p}_t^H}$ and forecast uncertainty $2\sqrt{\hat{s}_t^H}$ shrinks dramatically over the 174-year run. $2\sqrt{\hat{p}_t^H}$, the envelope for the OHCA climate state estimate, has a very slow initial convergence that reaches ±45.1ZJ by 1865 and then gradually falls to ±29.4ZJ by 2000, a 35% decrease. $2\sqrt{\hat{s}_t^H}$, the 95% forecast envelope for OHCA, drops from ±115.0ZJ by 1865 to ±66.2ZJ by 1970 (42% decrease) and then remains near this value through the present, range [±63.4 – ±71.2]. This reduction in forecast uncertainty directly reflects a 48% decrease in the uncertainty from the Zanna et al. (2019) hybrid product overt the equivalent time period.

b. Using the EBM-KF to determine Policy Threshold Crossing

A single GMST measurement is not an accurate measurement of anthropogenic climate change due to the large internal variability of the system, and so a single annual temperature above a particular policy threshold is not a guarantee of the climate state crossing that threshold. One interpretation of "crossing" is when the climate state underlying GMST (e.g. the "standard climate normal", or 30-year running mean of GMST) is determined with a given probability to have passed a policy threshold. This "climate state above" the threshold definition was used by Tebaldi and Knutti (2018) for regional thresholds and the IPCC AR6 (Lee, Marotzke et al.) who state "the time of GSAT exceedance is determined as the first year at which 21-year running averages of GSAT exceed the given policy threshold."[1] A second interpretation would be the chance that next year's annual-mean GMST will exceed the policy threshold, or "annual temperature forecast above" the threshold. The EBM-KF

[1] We use a 30-year averaging window nearly everywhere, but for consistency with these practices we use a 21-year averaging window for raw ESM simulations (only in Fig. 6b and Fig 12a-e). The EBM-KF climate state covariance is chosen to reflect the uncertainty in the 30-year average of real-world GMST (see Section 2c)—using R^{const} and Q matrices reflecting the 21-year means to match the IPCC definition would be a trivial modification.

generates probability distributions for both the "climate state above" and the "annual temperature forecast above" interpretations of whether a policy threshold has been crossed.

This climate state threshold, as in the IPCC definition, is given in the EBM-KF by a Gaussian distribution (green in Fig. 6a) about the state $\hat{T}_t$ with a variance $\hat{\mathrm{p}}_t^T$. The IPCC has an ensemble of models to draw upon over both the historical period and future projections (including those from LENS2 in Fig. 6b), so the fraction of the climate states (21-year means) of each (*j*) of the ensemble members $(\overline{{}_{21}Y_t})_j$ found above a given policy threshold determines the overall probability that the climate policy threshold was crossed (see Fig. 6d). Within our notation, we reuse Y_t to represent a GMST timeseries, but with the additional j subscript this timeseries is indicated to be the j$^{\text{th}}$ LENS2 hindcast, not a historical record. This empirical approach assumes the ensemble spread is a good representation of GMST uncertainty – yet recent IPCC reports discount the 90% ensemble spread to a 66% confidence range because coarse climate models under-represent internal variability and model uncertainty (Collins, Knutti et al. 2013; Lee, Marotzke et al.). The EBM-KF does not require a future projection to arrive at a present-day climate state, because it already provides an instantaneous and continual estimate of $\hat{T}_t$. The uncertainty $2\sqrt{\hat{\mathrm{p}}_t^T}$ around the *posterior* climate state $\hat{T}_t$ is used to calculate the probability of threshold crossing (see Fig. 6a) as follows: The probability of the climate state exceeding the policy threshold q is the integral of the probability density of the GMST climate state above q, equivalently 1 minus the integrated probability below q. The Gaussian cumulative distribution function centered at $\hat{T}_t$ with variance set to $\hat{\mathrm{p}}_t^T$, evaluated at q, is this cumulative probability below the threshold.

$$\mathbb{P}(\hat{T}_t \geq \mathrm{q}) = 1 - \mathrm{CDF}_{\mathcal{N}(\hat{T}_t, \hat{\mathrm{p}}_t^T)}(\mathrm{q}) = \frac{1}{2}\left(1 + \mathrm{erf}\left(\frac{\hat{T}_t - q}{\sqrt{2\ \hat{\mathrm{p}}_t^T}}\right)\right) \qquad (23)$$

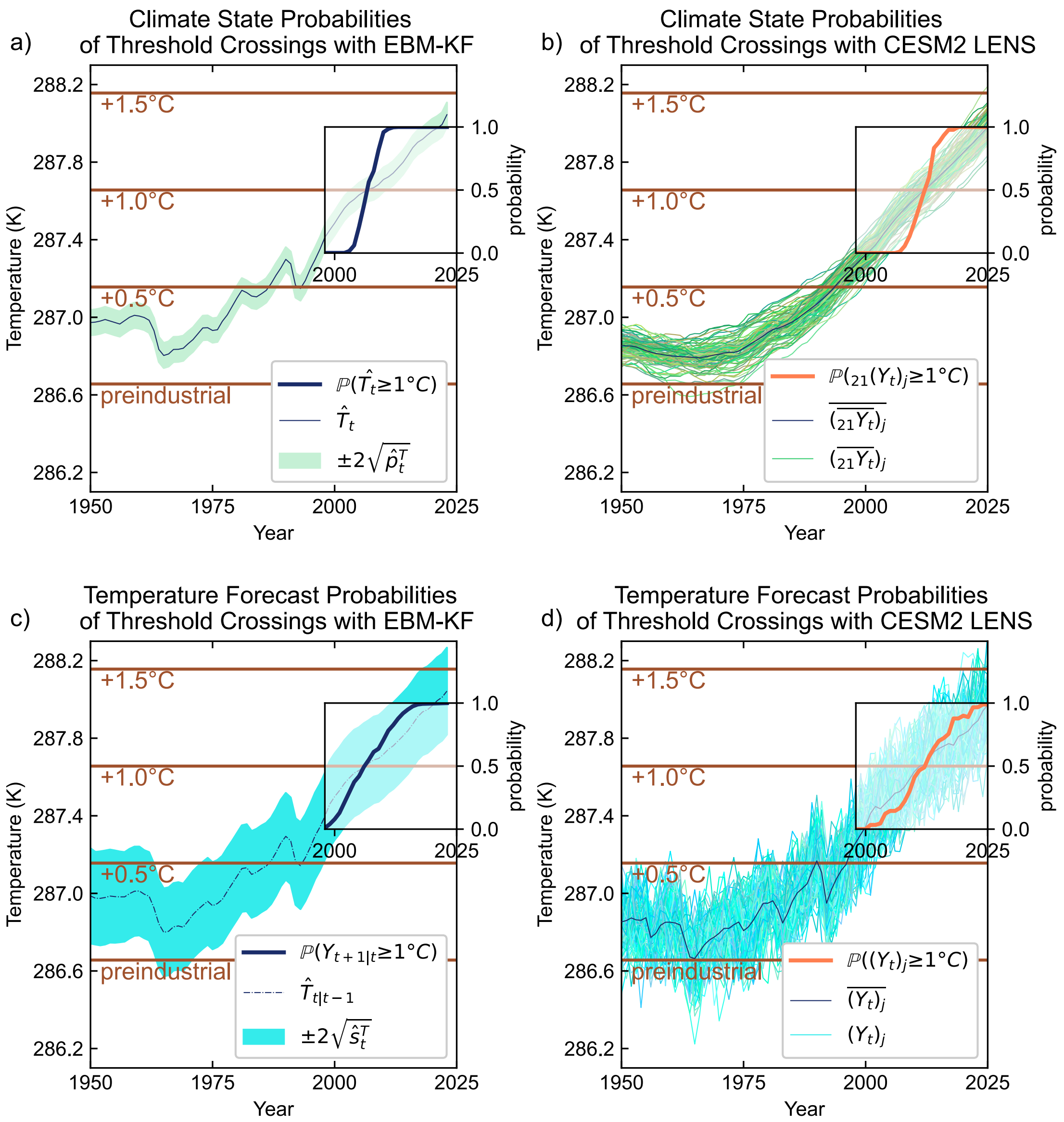

Fig. 6a: EBM-KF and climate state crossing policy thresholds: As in Fig. 4, there are the EBM-KF GMST state estimate (navy blue line) $\hat{T}_t$, 95% CI of the climate state (light green) $\pm 2\sqrt{\hat{p}^T_t}$, and GMST measurements (gray dots) Y_t. Additionally, policy thresholds (brown lines) are shown at 286.7K (+0°C), 287.2K (+0.5°C), and 287.7K (+1.0°C), compared to the preindustrial baseline. One inset axis indicates the +1°C), threshold crossing probability, with a y-axis of cumulative probability (thick navy blue; from 0 to 1) and the x-axis in time (years). Fig. 6b: each of the 21-year running means of the LENS2 ensemble is plotted in green, along with the ensemble-average in black. Individual LENS2 ensemble members are symbolized $(Y_t)_j$ with the subscript j indicating that this GMST timeseries is a hindcast simulation. The fraction of these running means above the +1°C policy threshold is plotted within the inset probability axis. Fig. 6c: The projected GMST "weather" 95% CI: $\pm 2\sqrt{\hat{s}^T_t}$ is shown in light blue around the forecast EBM-KF GMST state estimate (navy blue dashed-dotted line) $\hat{T}_{t|t-1}$. Actual annual GMST measurements (gray dots) Y_t are also shown. The inset axis indicates the likelihood that the actual GMST measurement will be above the +1°C), particular policy threshold based on this projection, a y-axis of cumulative probability (purple; from 0 to 1) and the x-axis in time (years). Fig. 6d: each of the LENS2 ensemble members is plotted as a blue or green line, along with the ensemble-average in dark blue. The

fraction of these members with annual GMST above the +1°C weather threshold is plotted within the inset probability axis.

For the second interpretation of temperature forecast above the policy threshold, the EBM-KF predicts a relevant window (blue in Fig. 6c) of possible next-year GMST measurements. This EBM-KF window is a Gaussian distribution centered at the projected state $\hat{T}_{t|t\text{-}1}$ (dashed dark blue line) with a variance $\hat{s}_t^T$: in other words, a simulated draw from the *forecast* state. This uncertainty range reflects and encapsulates actual annual GMST measurements, not the uncertainty in the climate. For an ensemble of ESMs, the analogous temperature forecast probability is the fraction of unfiltered individual ensemble members $(Y_t)_j$ at year *t* that are warmer than the policy threshold (blue lines in Fig. 6d).

There is additional ambiguity regarding whether "crossing a policy threshold" should specify an instant or a brief period. Here we define (based on the 1σ confidence interval, or the *likely* range in IPCC terminology) the "policy threshold crossing period" to span from the earliest year when ≥15.9% of climate states or temperature forecasts exceed the policy threshold to the latest year when ≤84.1% of climate states or temperature forecasts exceed that policy threshold. A "policy threshold crossing instant" is the year when the probability of exceeding the policy threshold is nearest to 50% while continuing to increase (or *as likely as not* to have crossed the policy threshold in IPCC terminology).

Regardless of whether an ESM ensemble (see Fig. 6b,d) or EBM-KF (see Fig. 6a,c) is used, the temperature forecast above threshold period (Fig. 6c,d) has a longer duration than the climate state above period (Fig. 6a,b) because the uncertainty/ensemble spread in the annual forecasts is wider than the uncertainty/ensemble spread of the time-averaged states. Both ESM ensemble and EBM-KF methods report similar policy threshold crossing instants (Fig. 11). Interestingly, the Mt. Pinatubo eruption in 1991 resets the +0.5K threshold crossing repeatedly in both the EBM-KF and ESM ensemble by its perturbation of elevated aerosols.

Fig. 6 quantifies the probability of crossing policy thresholds as a function of time (dark blue or orange), inset on top of the relevant GMST timeseries and spread. The EBM-KF climate state estimate in Fig. 6a and annual temperature forecast in Fig. 6c are aligned by year, although these two quantities are in entirely different probability domains. As the EBM-KF state estimate approaches any given policy threshold, the cumulative temperature policy threshold approaches 0.5, or 50% at a "policy threshold crossing instant". The +1.0K policy threshold's crossing instant was in 2010. For the annual temperature forecast in Fig. 6c, the crossing period was 2003-2015 for +1.0K. The crossing period for the climate state in Fig. 6a is briefer: 2008-2012 for +1.0K. For comparison using LENS2 the analogous climate state thresholds are plotted in Fig. 6b,d, although these do not precisely align temporally due to the cold bias of LENS2 during this decade. All threshold crossing periods and instants including future projections under SSP3-7.0 are compared directly in Fig. 12.

c. The spread from one member – using EBM-KF to generate an analog for an ESM large ensemble spread

There are many more past and future climate scenarios that researchers wish to investigate than there are computational resources to run a full large ensemble for each scenario. Fortunately, the EBM-KF allows for one or a handful of ESM simulations to approximate the distribution of an entire ensemble spread (similar to an approach taken for ensembles of ice sheet models in (Edwards, Nowicki et al. 2021; van Katwyk, Fox-Kemper et al. 2023). There are inter-annual differences which persist between runs of the ensemble and skew some climate states persistently cooler and others warmer (Supp. Fig. 6), an effect not captured by a Kalman filter framework.

Figure 7a shows the comparison between the EBM-KF GMST climate state uncertainty distribution (light green) and the LENS2 time-Filtered ensemble members. This time-Filtering was performed using the same EBM-KF, momentarily assuming that one of the ensemble members' hindcast was the actual measured temperature record. Each of the orange lines is a climate state central estimate that is comparable to the blue line of the real observed GMST climate state. Sometimes the observations' EBM-KF climate state uncertainty distribution contains the time-Filtered LENS2 ensemble members, such as in 1900 and 1935, but at other times it does not, such as in 1950. In corresponding panels within Supp. Fig. 14, we show the histogram Supp. Fig. 14a and quantile-quantile comparison Supp. Fig. 14b: both demonstrating a clear bias. This is what it means to say that the LENS2 climate state disagrees with the observed climate state within the EBM-KF framework.

We could interpret the time-Filtered ensemble spread versus the climate state uncertainty distribution of one ensemble member in a similar fashion. This has a different purpose, as now we are testing whether the EBM-KF can predict the spread of the time-Filtered LENS2 ensemble correctly, regardless of whether the LENS2 ensemble matches the observed temperatures. If so, that would indicate that from one ensemble member simulation we could effectively predict all the other ensemble members. As expected, there is a distribution of results, where some of the ensemble members are close to the center of the distribution and others are outliers.

We can statistically calculate the expected error in our predicted ensemble of time-Filtered LENS2 states from a single member versus the true ensemble of time-Filtered LENS2 states. Panel 7b shows the error in spread (standard deviation) and error in bias by repeatedly making this prediction of a distribution from single members of LENS2 and comparison to the whole time-Filtered LENS2 ensemble. Examining the centroid, this is an unbiased estimate of the distribution (as it should be). However, the ensemble of time-Filtered LENS2 is distributed with a standard deviation that is 1.22 times larger than the average prediction from one ensemble member. At worst, it is 1.54 times larger than any single ensemble member's estimate. Figure 7b labels two examples of where one ensemble member predicts the whole ensemble: a good fit (best quartile) is shown as a circle, and the worst fit is shown as a square. Supp. Fig. 14c,d show these two comparisons in more detail. This error in spread, as well as the distribution of biases are all better than the comparison between the LENS2 time-Filtered states and the observed record's EBM-KF state uncertainty (green star). From this we conclude that the error in predicted distribution from one ensemble member is negligible in comparison to the distance between the model and reality. Thus, this approach is effective in making such comparisons, with a typical bias error in single ensemble member estimate of order ±0.007K with range (-0.0265–- 0.0268)K.

Within panel 7c, the Kullback-Leibler divergence is utilized to evaluate the utility of using the EBM-KF state uncertainty as a prior estimate of the spread between time-Filtered LENS2 ensemble members. At each year, this GMST state variance $\hat{\mathrm{p}}_t^T$ is combined in a weighted mean with the sample variance of a small subset of LENS2 members (shown in yellow violin plots, with a number indicating the number of members taken: 1, 2, 3, or 8). Taking 2 LENS2 members does not improve the predicted distribution, as there is a significant chance that two members which are close together will be selected, incorrectly shrinking the predicted ensemble spread. With 3 LENS2 members, the predicted distribution slightly improves. Without using this prior estimate (and allowing the sample variance to change over time, red violin plots), at least 8 LENS2 members are required to generate a predictive ensemble distribution that is comparable to using a single LENS2 member and the Kalman Filter $\hat{\mathrm{p}}_t^T$ as the ensemble's variance. Panel 7c demonstrates this with 3 and 8 LENS2 members with a time-varying sample mean (red: 3 or 8). Thus, Fig. 7 shows the power of the parametric Gaussian statistics generated by the EBM-KF over a raw ensemble member sample estimate.

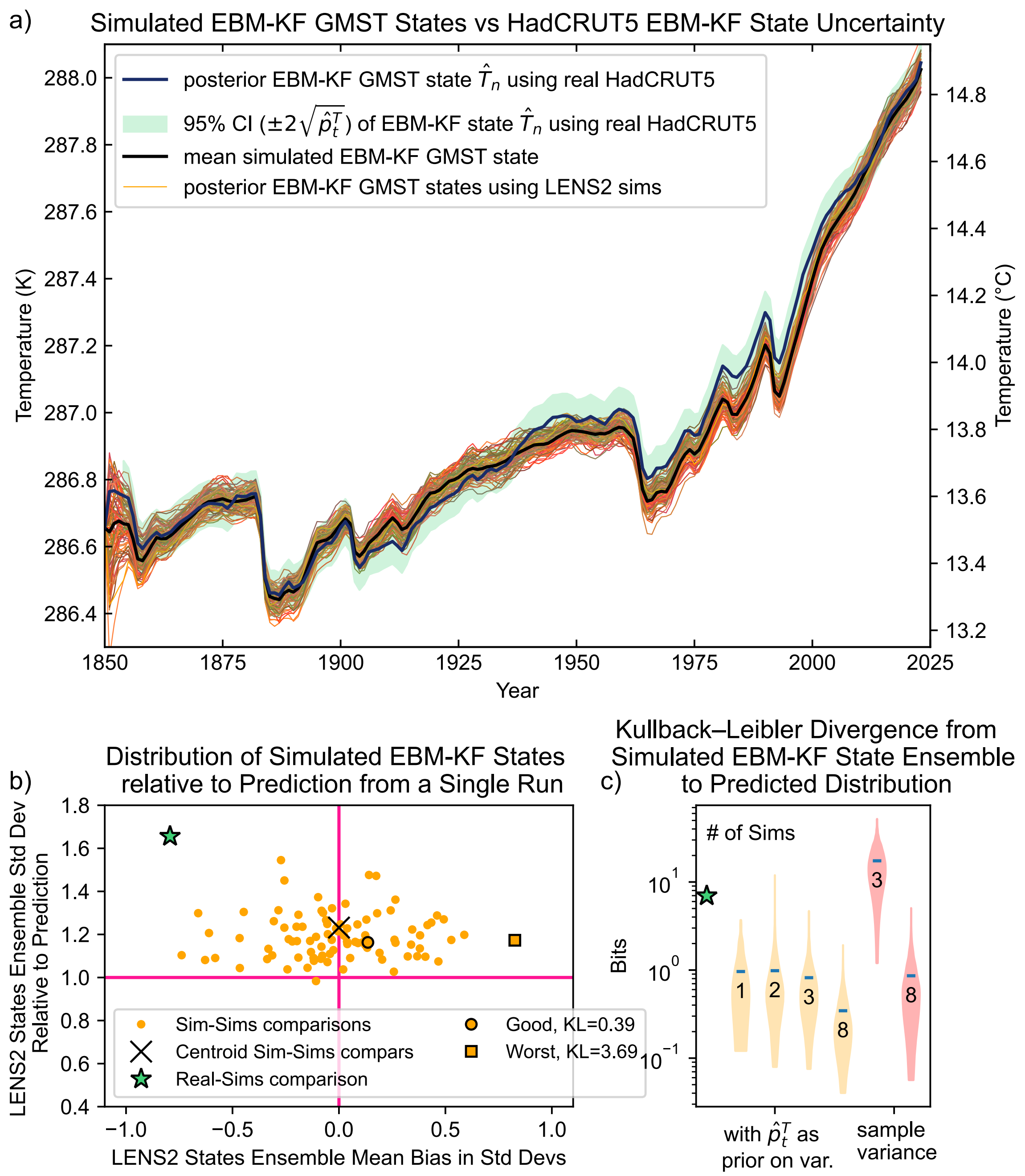

Fig. 7: Comparison of the GMST Kalman Filter states across the LENS2 ensemble. a) The EBM-KF posterior HadCRUT5 state estimate (thick blue) and its 95% confidence interval (light green), along with EBM-KF state estimates for each individual CESM2 ensemble member (orange lines) and their mean (thick black line). b) Climate states and associated uncertainties arising from each of 90 LENS2 simulations and HadCRUT5 are compared to all other LENS2 climate states, and the relative bias and standard deviation of the resulting empirical distributions with respect to a particular ensemble member's $\hat{p}_t^T$. c) Violin plots are shown comparing the Kullback-Leibler divergence (on a log scale, smaller indicates a better

match) for a variety of methods of generating a distribution predicting the LENS2-time-Filtered ensemble spread. In yellow, the mean $\hat{p}_t^T$ from 1, 2, 3, or 8 EBM-KF LENS2 runs is averaged, and used in combination with the time-varying sample variance. In red, 3 or 8 of these time-Filtered ensemble members are used to predict an ensemble distribution from time-varying sample variance alone. Taking a single EBM-KF LENS2 run with $\hat{p}_t^T$ approximates the time-Filtered LENS2 ensemble equivalently well to taking the time-varying sample variance of 8 time-Filtered ensemble members.

LENS2 runs are more similar to each other than to the real Earth, especially regarding outputs such as OHCA (see Supp. Fig. 13) and Arctic or Antarctic sea ice extent (Rosenblum and Eisenman 2017; Roach, Dörr et al. 2020; Horvat 2021). Also, the current generation of ESMs tend to underestimate the appropriate full spread of climate variability. For instance, some weather models use stochastic noise to push their distribution wider than dynamic variation alone (Buizza, Milleer et al. 1999), and other numerical climate models perturb parameters to achieve the same distribution-widening effect (Keil, Schmidt et al. 2021; Duffy, Medeiros et al. 2023).

Again, Fig. 7 shows that the EBM-KF climate state based on HadCRUT5 temperatures or based any one of the LENS2 ensemble members show the expected level of consistency and (potentially biased) Gaussian differences with the rest of the LENS2 ensemble. The GMST was estimated from the GSAT of each LENS2 ensemble member. Thus, the EBM-KF on any one of the ensemble members does a good job of estimating the GMST climate state (i.e., averaged over internal variability) and its uncertainty as simulated by the entire LENS2 ensemble.

d. Sampling Future Projections from a Non-Gaussian Volcanic Distribution

In standard climate assessments (e.g., IPCC 2021), future volcanism has long been singled out as an unknown aspect of projected climate change in any given future year, particularly regarding tropical eruptions' contribution to planetary albedo (Marshall et al. 2022). The forcing of historical-period climate models includes the effects of known past volcanoes, while the forcing of future climate models includes only "background forcing from volcanoes", i.e., an expected average forcing value in future years. Applying an average forcing misses the potential impact of individual volcanic events on the global climate state (compare blue line to black lines in Fig. 8), and because of the nonlinearities and feedbacks in the climate system, these volcanic events can gradually shift the distribution's center. Individual volcanoes can shift weather crossing thresholds by construction (as seen from

Pinatubo in Fig. 12f), and so they are important for understanding weather forecasts especially in the next decade (see Fig. 11a,b). However, running an ESM ensemble of sufficient size to explore the low probability of a large volcanic eruption in any potential year is computationally challenging using traditional ESMs, motivating specialized model intercomparison projects (Zanchettin, Khodri et al. 2016; Timmreck, Mann et al. 2018). By contrast, robust sampling of rare events is easily accomplished with the inexpensive EBM-KF. The added contribution of CO_2 and other greenhouse gases from random volcanic eruptions is not included in this analysis, both because all volcanoes at all latitudes make this contribution (so the distribution is more uniform), and because this annual contribution is miniscule compared to anthropogenic greenhouse gasses: 20x smaller in 1900, 130x smaller in 2010) (Gerlach 2011). Slightly different climate responses have been modeled to occur when volcanic events were simulated at different phases of climate oscillation patterns, such as the Pacific Decadal Oscillation (PDO) and North Atlantic Oscillation (NAO) (Illing, Kadow et al. 2018). Due to its low-dimensional state space, the EBM-KF neglects such complexities.

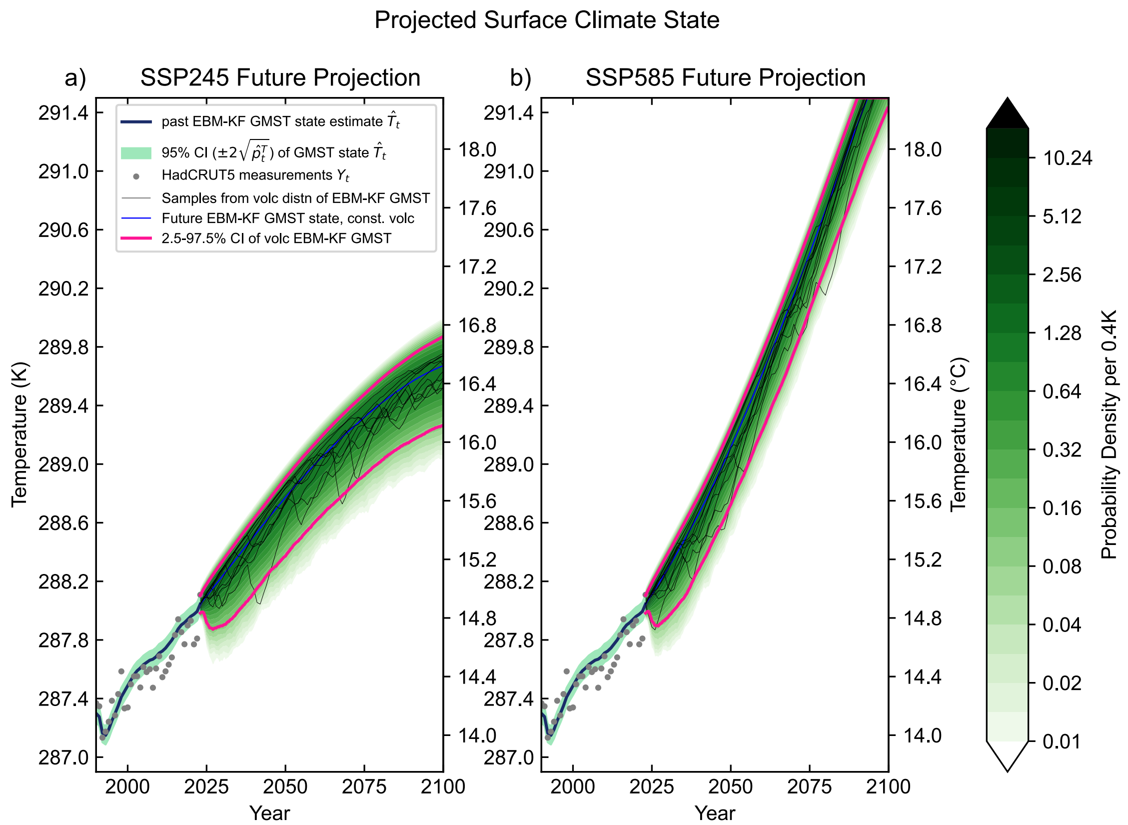

Fig. 8: Future GMST projections of SSP1-2.6 (a) and SSP3-7.0 (b) scenarios using sampled measures of volcanic activity and greenhouse gas concentrations calculated according to

MAGICC7.0 (Meinshausen, Nicholls et al. 2020). The historical Mt. Pinatubo eruption in 1991 is shown in the lower left corner of both graphs for scale. 10 of the sampled 6000 potential future climate states from the volcanic probability distribution are graphed (thin black), along with a future climate state projection that uses constant volcanism with average AOD (blue). The probability density function formed by taking the summation of all sampled Gaussian kernels at each time point is shaded in green on a logarithmic scale (note these probability densities are not probabilities so they can exceed 1). Pink lines show the 2.5-97.5% confidence interval of these probability density functions, which are very asymmetrical (negatively skewed) due to the sampled volcanic eruptions' impact on GMST.

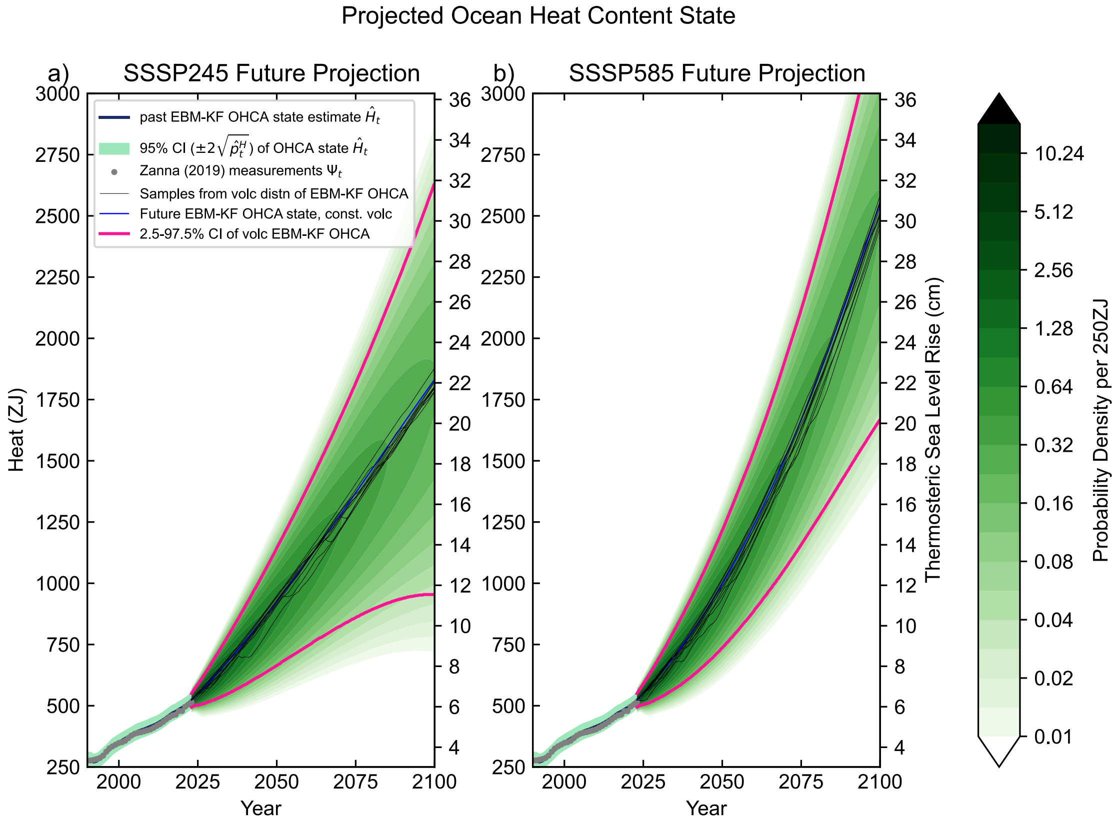

Fig. 9: Future OHCA projections of SSP1-2.6 (a) and SSP3-7.0 (b) scenarios using sampled measures of volcanic activity and greenhouse gas concentrations calculated according to MAGICC7.0 (Meinshausen, Nicholls et al. 2020). 10 of the sampled 6000 potential future climate states from the volcanic probability distribution are graphed (thin black), along with a future climate state projection that uses constant volcanism with average AOD (blue). The probability density function formed by taking the summation of all sampled Gaussian kernels at each time point is shaded in green on a logarithmic scale (note these probability densities are not probabilities so they can exceed 1). Pink lines show the 2.5-97.5% confidence interval of these probability density functions, which are only slightly asymmetrical because the sampled volcanic eruptions have a much smaller impact on OHCA.

Figs. 8 & 9 show the future projections of GMST and OHCA using EBM-KF, including sampling for future volcanoes for two scenarios. SSP1-2.6 shown in Figs. 8a & 9a has CO_2 emissions that sharply decline after 2020 to keep GMST rise below 2K (van Vuuren, den Elzen et al. 2007; van Vuuren, Stehfest et al. 2017). SSP3-7.0 shown in Figs. 8b & 9b is a higher emission scenario in which CO_2 emissions double by 2100 (Fujimori, Hasegawa et al. 2017). Note that the volcanic ensemble probability density is not symmetrical for GMST - there is a much longer tail on the cooler side because of intermittent cooling by volcanic aerosols. In Fig. 8 the cooler side of the distribution takes a few years to fully expand out because large eruptions generally did not produce their maximal effect on AOD (and thus the GMST climate state) until 1-2 years after the eruption began, and there are no major eruptions ongoing at present. Indeed, the volcanic eruptions dominate the future uncertainty over the slowly growing GMST climate state uncertainty and rival or exceed the scenario uncertainty up until about 2050 (assuming known model parameters, Fig. 11a). By contrast, the LENS2 using "constant background" future volcanism has a symmetrical distribution for future projections of the same SSPs (Supp Fig. 6, right of dashed line). The effects of volcanism on OHCA (Fig. 9) are much smaller than on GMST (Fig. 8), but there is still a longer tail toward the low OHCA side. Regarding future GMST policy threshold crossings, the uncertainty regarding volcanic eruptions lessens the difference between the climate state threshold crossing interval and the temperature prediction threshold crossing interval.

Across many future simulations the dynamic model Jacobian matrix Φ_t happens to remain nearly constant at values of: $\Phi_t \approx \begin{bmatrix} 0.893 & 0.000253\, K/\frac{W\,yr}{m^2} \\ 11.1\ \frac{W\,yr}{m^2}/K & 0.999 \end{bmatrix}$, nearly unit triangular. Due to this Jacobian matrix shape and the 0.893 factor, $\hat{p}_t^T$ grows sub-linearly, with yearly growth less than the GMST-exclusive component of Q = $\sqrt{0.01107\ K^2}$= 0.00037 K^2. (see Eq. 22) Over a 78-year future projection (2023-2100) the GMST state 95% confidence interval $2\sqrt{\hat{p}_t^T}$ only grows from 0.0625K to between 0.1757K and 0.1792K. This 2.8-fold increase is small over the 21st century compared to the GMST dips that occur under volcanic eruptions (see Figs. 8 & 10). The effect of volcanoes on historical state (Figs. 3 & 4) and future projections (Fig. 8) is therefore worthy of specialized treatment in addition to measurement uncertainty and internal chaotic variability (see Section 3d). In contrast, the OHCA component of the state uncertainty 95% confidence interval $2\sqrt{\hat{p}_t^H}$ grows exponentially due to the 11.1 value in the lower-left entry of Φ_t, and volcanoes have a negligible effect on of projected OHCA trajectories (see Fig. 9). The ocean state uncertainty 95% CI = $2\sqrt{\hat{p}_t^H}$, initially at $2.57\frac{W\,yr}{m^2}$ (29.4 ZJ) in 2023, balloons to 76.1-$77.1\frac{W\,yr}{m^2}$ (870-880 ZJ) by 2100.

4. Discussion

The EBM-KF climate state estimate resembles other standard estimates of climate state, but it has advantages they do not share. The EBM-KF algorithm, because of its relationship to a forward or "blind" EBM, can be projected forward in time without temperature observations and thus can be used in many situations. Unlike an ESM, the EBM-KF benefits from data assimilation due to its Kalman filter nature and thus remains close to observations or synthetic data (e.g., the examples in Section 4 of reproducing the LENS2 from few ensemble members or the ensemble of potential volcanic activity futures). The OHCA component is particularly sensitive to assimilated observations (see Fig. 3b), largely because of reduced understanding of the ocean dynamics that drive deep ocean heat uptake compared to atmospheric radiative feedbacks. The EBM has a correspondingly simpler model of this process. Unlike an Ensemble Kalman filter approach that can reweight a full-physics ESM ensemble toward observations, the EBM-KF has negligible computational cost and can thus examine rare, long-tailed events such as volcanoes. Additionally, tuning of the EBM parameters and uncertainty quantification of these results can benefit from the literature and algorithms to optimize our Kalman Filter parameters.

a. Comparison to Previous Estimation Methods of the Climate State

Although they are different types of average, a direct comparison (Fig. 10) of the state estimated from the EBM-KF (Fig. 4) and that estimated by the 30-year running mean (Fig. 1) and the LENS2 ensemble mean (Supp. Fig. 6), the EBM-KF has slightly more year-to-year variation than the 30-year mean and less than the LENS2 ensemble mean. Departures from the main Gaussian cloud in all methods represent volcanoes. The 5 largest eruptions which caused the largest dip in EBM-KF state are labeled in Fig. 10, corresponding to the 5 peaks in AOD ≥ 0.06 plotted in Fig. B1a in the appendix. The climate effects of these major tropical volcanic eruptions have been studied extensively *(McCormick, Thomason et al. 1995; Jones and Kelly 1996)*. Note for the eruptions listed, plus many others, the dips in the EBM-KF mean state correspond with dips in the sample mean of the LENS2 simulations. However, the earliest AOD values provided by Sato (1993) also demonstrate a major spike at 1856, which is not reflected in the LENS2 simulations. This may correspond to either the 1856 eruptions of Komaga-take, Japan or Mt. Awu, Indonesia, and we labeled this with the latter eruption because tropical volcanic eruptions typically have a much larger climate impact (Marshall et al. 2022).

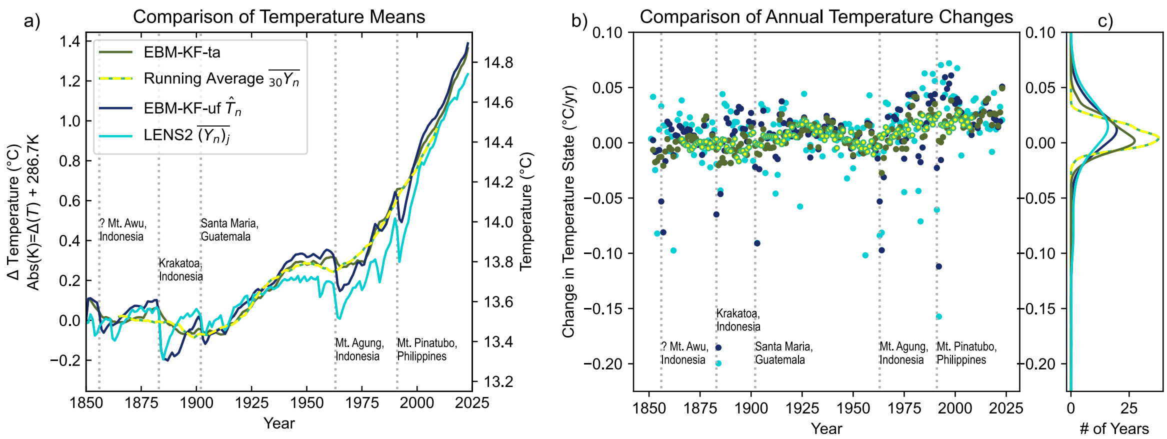

Fig. 10 a) Direct GMST "climate state" comparison of the 30-year averaged GMST (yellow-green dashed), the EBM-KF-ta state (dark green), the EBM-KF-uf state (navy blue), and the ensemble mean of GSAT in the LENS2 simulations (sky blue). b) For "climate states" with the same colors as in panel a, the distribution of innovations (derivative) is plotted against time. c) A smoothed empirical density with respect to yearly change in temperature is linked to panel b. (This empirical density is simply an approximation of a histogram, and the kernel densities are not provided by elements of the Kalman Filter as in Fig 4b and 5b). Major volcanic eruptions are labeled in both panels a and b with dotted vertical light gray lines. In all panels, the 30-year averaged GMST (yellow-green dashed) is close to the EBM-KF-ta state (dark green), whereas the EBM-KF-uf state (navy blue) resembles the LENS2 simulations (sky blue) in responses to volcanic eruptions.

Based on this interpretation of Fig. 10, we now see that the LENS2 ensemble average (light blue) is closer to the (EBM-KF-uf, navy blue, with uf abbreviating "un-filtered" AOD forcing) regarding sensitivity to volcanoes than the 30-year running mean (yellow-green). In response to this, we will distinguish three preparations of AOD forcing: one that takes a centered 30-year running average of AOD (EBM-KF-ca, not shown, with ca abbreviating "centered average"), one that directly uses the annual measured values of AOD (EBM-KF-uf, navy blue, as above), and one that takes a 15-year trailing average combined in equal weight with the overall timeseries AOD mean (EBM-KF-ta in dark green, the best point estimate for the 15-years of future AOD and displayed as a green line in Fig. B1a, with ta abbreviating "trailing average"). As demonstrated in Fig. 10, this EBM-KF-ta preparation of the AOD (dark green) brings the EBM-KF-tf close to the 30-year running mean (yellow-green) regarding sensitivity to volcanoes (their maximum separation was in 1962 with the 30-year running average -0.073°C cooler, otherwise their average absolute separation ±0.025°C, standard deviation ±0.030°C, r^2=0.986). We prefer EBM-KF-ta (dark green) to EBM-KF-ca (not shown) as the latter involves future information, and the EBM-KF-ca does not get closer to the 30-year running mean (maximum separation was in 1976 with the 30-year running average -0.080°C cooler than the EBM-KF-ca, otherwise their average absolute separation was ±0.025°C, standard deviation ±0.031°C, r^2=0.987). If we are trying to directly match the behavior of ensembles such as LENS2 (light blue), the EBM-KF-uf (navy blue) is the correct choice. As noted in Section 3a, LENS2 is farther from EBM-KF-uf because LENS2 has a cold bias over 1940-2000 (max separation in 1983 of 0.262°C, average absolute separation ±0.088°C, standard deviation ±0.085°C, r^2=0.928), but the responses to volcanic events are very similar (highlighted in Fig. 10b). If we are trying to estimate both the weather and climate state without bias, for next-year predictions and beyond (so AOD will be unavailable), then the optimal method is to run many predictions using EBM-KF-uf and a volcanic probability distribution, as in section 4d. Filtering of other forcing, e.g., [eCO_2], yielded negligible changes as these fields are more slowly evolving than volcanic AOD.

It is beyond the scope of this paper to detail the characteristics of the large and growing variety of "mean state" definitions, but a summary is useful. For all methods we have examined regarding the GMST (30-year mean – Fig. 1, EBM-KF – Fig. 4, LENS2 model ensemble mean – Supp. Fig. 6, purely statistical methods – Supp. Fig. 4c, 4d, 5), the differences in the estimated climate state are relatively small in available years (on the order of 0.1K – see Supp. Fig. 7, column 1). The largest differences seen between these methods lie in the spread of the changes from year to year (see Supp. Fig. 7, column 2) which can be addressed by preparations filtering the forcing and persistent mean anomalies relative to observations, particularly so in the forward, blind LENS2 ensemble (see Supp. Fig. 7, column 4).

The primary distinction of our EBM-KF method and all existing alternative definitions is the integrated quantification of uncertainty. While many methods exhibit a relationship between the "mean state" and "sample" that varies in time, the EBM-KF (and the related RTS) quickly converge to a stable state uncertainty of 0.034K (and 0.023K for the RTS, see Supp. Fig. 2). Our choice of method was motivated by the mathematical compatibility between the governing equation for a Kalman filter and that of an EBM, which is not true of many alternatives, e.g., a Butterworth filter or Bayesian changepoint analysis and an EBM. We also emphasize again that our EBM-KF infers the climate state directly via yearly signal processing, which is a different approach than extrapolating the future weather to the next 15 years using reforecasts with a weather emulator and calculating many 30-year means. In the next section we compare the EBM-KF uncertainties to those of ESM ensembles.

b. Comparison to Earth System Models (CESM2 Large Ensemble and CMIP5)

The chief advantages of EBM-KF over an ensemble of ESMs is that it replicates many statistical features while being trivial to compute. Fig. 7 suggested that any of the ensemble members or the observed temperature record could be used together with EBM-KF to recreate the climate state, but now we examine if we can anticipate or improve on the ensemble statistics *without a single ensemble member*.

First, we examine the basic statistical character of LENS2. The distribution of annual differences of all ESM trajectories from the ensemble mean are remarkably close to Gaussian (see Supp. Fig. 10a). Therefore, again due to the central limit theorem, this fundamental assumption of the EBM-KF is also met by GSAT simulated by the CESM2. The standard deviation rises insignificantly with time in LENS2 over the entire simulation duration ($p=0.168$). Before 2065 this rise is significant ($p=1.2*10^{-6}$, see Supp. Fig. 10b) while relatively small (linear trend $r^2=0.105$ and only 8.9% rise in σ from 1850-2065). The time-averaged standard deviation 0.127K was close to both chosen total GMST-exclusive (top-left) measurement noise from R_t (range 0.107 – 0.136K, see section 2c, Eq. 21) and half the converged values in the EBM-KF of $\sqrt{\hat{s}_t^T}$: 0.13K in 1865, later 0.112K in 2000. Examining skewness and kurtosis, the distribution of simulations about the LENS2 GSAT ensemble mean is not meaningfully altered as the climate warms (see Supp. Fig. 10c,d).

Next, we evaluated how well the LENS2 captures the overall shape of the observed HadCRUT5 temperatures, given that it is not constrained directly by these observations. The absolute temperature of the LENS2 runs had to be revised down by a full 1.75K to match its ensemble 1850-1950 100-year average GMST to HadCRUT5. Other authors have also noted this high absolute temperature as well as the high climate sensitivity of CESM2, the model used in LENS2 (Gettelman, Hannay et al. 2019; Feng, Otto-Bliesner et al. 2020; Zhu, Otto-Bliesner et al. 2022). Recall HadCRUT5 was recalibrated to a 1960-1990 30-year climate normal (Jones and Harpham 2013) of 13.85°C (287.00K), and the LENS2 average has a slightly lower temperature during this 30-year climate normal of 13.71°C (286.86K).

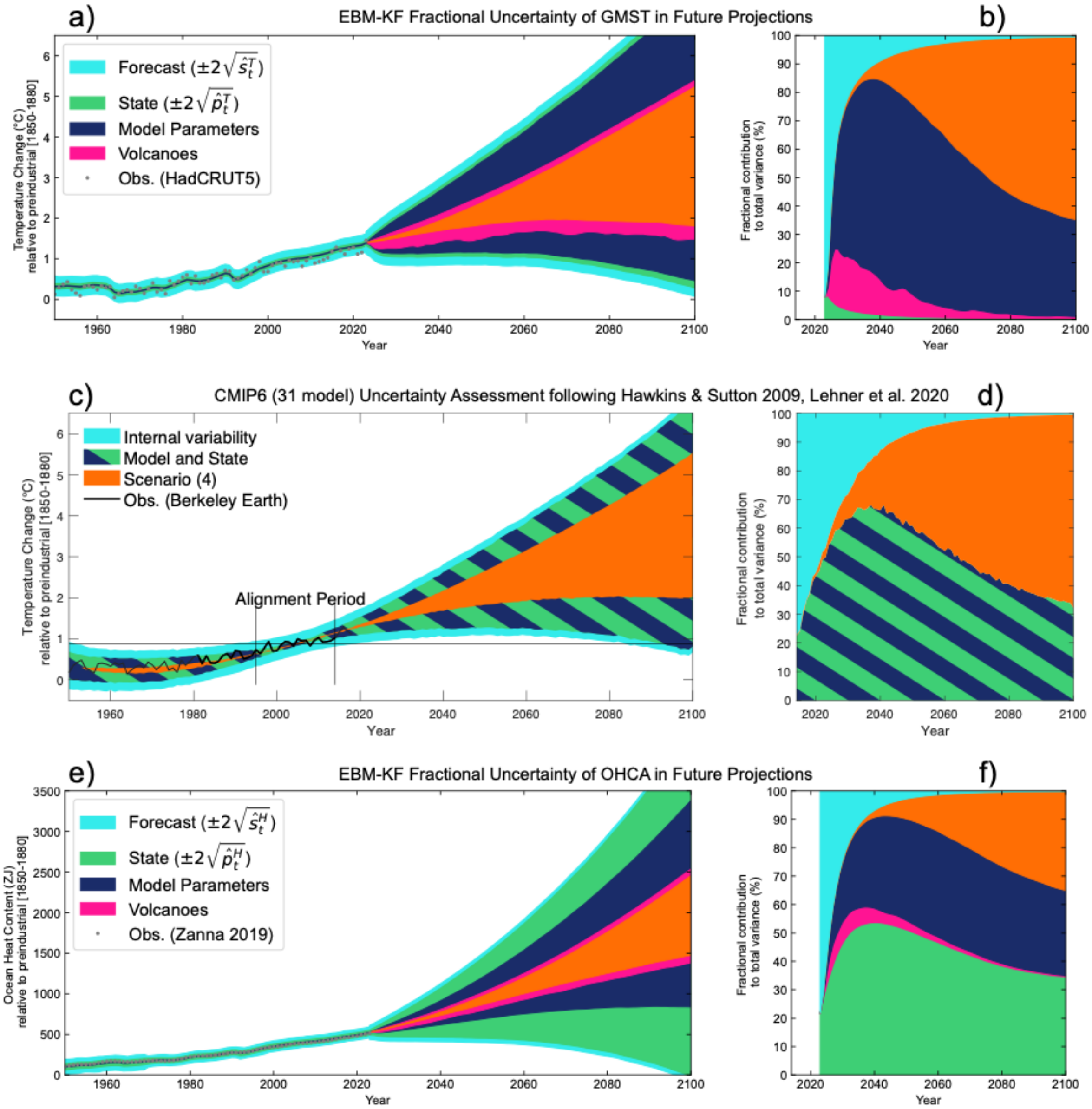

Fig. 11: "Hawkins Plots" (Hawkins and Sutton 2009) of the sources of uncertainty (95% CI on left, fractional variance on the right) in future projections, with the top row (a, b) showing the GMST state projections from the EBM-KF, the middle row (c,d) showing global mean surface air temperature GSAT from CMIP6, and the bottom figure (e,f) showing the OHCA projections from the EBM-KF. Refer to (Lehner, Deser et al. 2020) for details on how panels c and d were generated (relying on 4th-degree polynomials for smoothing), but here showing 95% CI and annual (rather than decadal) variability. Note that noise from internal dynamical variability is marked in light blue in all figures, and while initially dominant (≥80%), it quickly falls off within the first decade, to eventually be replaced with emissions scenario uncertainty in orange. The extent to which different smoothed ESMs disagree within CMIP6 for the same emissions scenario ("model uncertainty") is plotted in green, as it is roughly analogous to the climate state covariance within the EBM-KF, Future uncertainty related to volcanoes (in magenta) is negatively skewed and very important in the first 3-25 years of the EBM-KF's projections of GMST.

We also compared EBM-KF projections (Fig. 8) with LENS2 projections (Supp. Fig. 6). Both Fig. 8b and the right side of Supp. Fig. 6 trace out roughly the same shapes, as both are forced by the SSP3-7.0 projections. The largely symmetric variation in the LENS2 is driven by dynamical instability. This is fundamentally different from the EBM-KF, which in addition to scaled, weather-driven uncertainty samples a noisy distribution of volcanic eruptions, yielding an asymmetrical distribution. LENS2 projections based on SSP3-7.0 achieve a slightly higher mean temperature in 2100 (291.3K, +4.6K warming) than the equivalent EBM-KF projection (290.9K, +4.2K warming, see Fig. 8b), despite the LENS2 simulations being cooler throughout most of the 20th century and early 21st century (see Fig. 10a). Across all CMIP6 models (Lee, Marotzke et al. 2021; Tebaldi, Debeire et al. 2021) the projected warming under this scenario is 3.9K with 5-95% range (+2.8K, +5.5K), closer to the EBM-KF projection.

Continuing beyond LENS2 to compare against the multi-model CMIP6 ensemble, a projected uncertainty decomposition is created following Hawkins and Sutton (2009) and Lehner, Deser et al. (2020) in Fig. 11. By the methods in Section 4c, the EBM-KF adds the volcanic uncertainty into this picture (pink). A second advantage is that the climate state uncertainty (due to the cumulative reliability of measurements with respect to a model) and the model uncertainty (due to the confidence in the model structure and parameters) can be distinguished. For simplicity, we estimated the model and parameter uncertainty of the EBM-KF by just varying the cloud feedback parameter (samples from $\mathcal{N}(0.42, 0.36^2)$, based on Figure 7.10 and Table 7.10 of AR6 (Forster, Storelvmo et al. 2021)) and the ocean heat conductivity (samples from $\mathcal{N}(0.42, 0.36^2)$, based on Geoffroy et al. Part II (2013)). Incomplete understanding of cloud feedback is a primary source of uncertainty within ESMs, leading to diverging predictions within CMIP6 (Zelinka, Randall et al. 2017; Ceppi and Nowack 2021), and as noted above the OHCA dynamics of EBM-KF are oversimplified (Cheng, von Schuckmann et al. 2022; Newsom, Zanna et al. 2023) and sparse long-term records yield disparate OHCA reconstructions before 2005 (see Figure 2.26 of AR6). (Gulev, Thorne et al. 2021).

Regarding the various types of climate policy thresholds, the LENS2 can be used to generate very similar results to the EBM-KF (Figs. 6 & 12). Differences in absolute probability and policy threshold crossing instants reflect differences in the modeled climate states: particularly that the LENS2 ensemble was slightly cooler than the EBM-KF model after correcting to the same preindustrial temperature, so policy thresholds were crossed 3-5 years later (Fig. 12). The eruption of Mt. Pinatubo caused the policy threshold of +0.5K to be crossed in three instants within the EBM-KF model, because this eruption temporarily cooled the climate state back below the threshold temperature. The first of these EBM-KF crossings coincides very closely with the (single) policy threshold crossing instant of the 30-year running mean (indicated by orange asterisks). The 21-year running averages of the LENS2 simulations only crossed the 0.5K threshold once, illustrating how the EBM-KF state estimate fundamentally differs from a running mean. Future threshold crossings (1.5K, 2.0K, 2.5K) under the SSP3-7.0 projection scenario show close temporal alignment in the threshold instants between LENS2 and the EBM-KF estimates that sample for volcanic uncertainty. Although shifted, the overall shapes of these cumulative distribution functions and spans of the threshold crossing windows are more similar between LENS2 and a single EBM-KF future estimate that like LENS2 keeps AOD constant (see Fig. 12). In contrast, there is a long tail on the >90% portion of the future sample volcanism (pink lines, lower row) regarding temperature forecast thresholds, extending the later bound of the crossing period by about 5 years, because there remains a modest chance that a large volcano will erupt and tip the temperature forecast below that threshold.

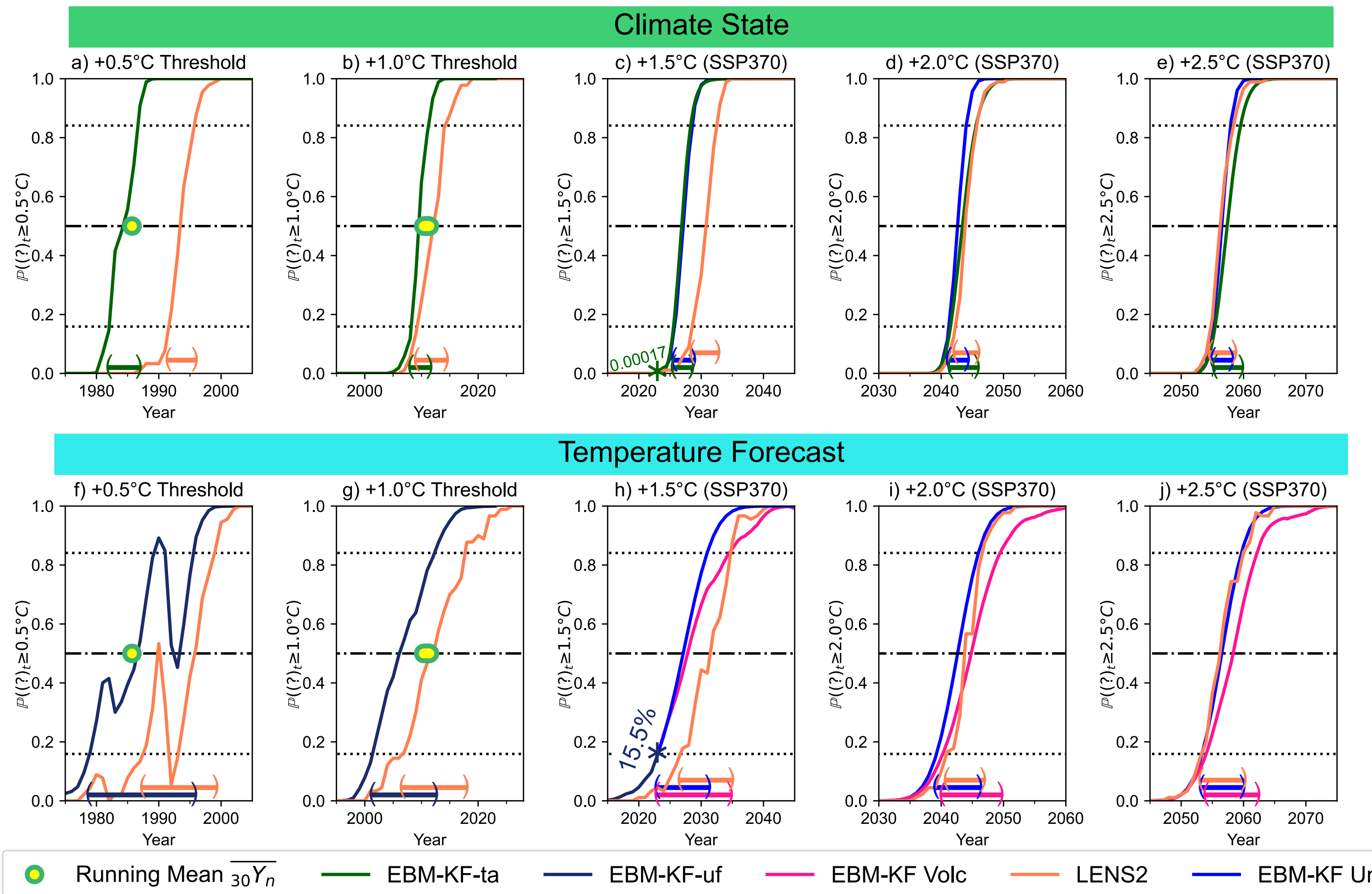

Fig. 12: Comparison of 0.5-2.5K GMST policy threshold crossing probabilities for various relevant preparations of the EBM-KF and CESM2 LENS simulations (orange). The top row

of panels a-e compare climate states in the EBM-KF with 21-year averages of the LENS2 simulations. The bottom row f-j compares next-year temperature forecasts from the EBM-KF directly with the LENS2 simulations. Recall from Section 3b that these are the integrals of all probability densities of the GMST climate states or temperature forecasts below that policy threshold. Historical EBM-KF-uf estimates of temperature forecasts are in dark blue in panels f,g, with the latter reproduced from Fig. 6c. Regarding climate state thresholds, the EBM-KF-ta states are shown in green in panels a,b. These EBM-KF-ta state estimates come the closest to matching the instants (yellow-green dots) when the 30-year running average crossed the 0.5°C threshold in 1985 (or very likely from a linear trend will have crossed the 1.0°C threshold in 2010 or 2011). Two versions of future EBM-KF state estimates are shown: an amalgamation of samples in pink (in h,i,j) from the volcanic distribution shown in Fig. 8, and a single run in bright blue (in c,d,e,h,i,j) with uniform AOD mirroring how LENS2 treats volcanism. In future climate state projections (green in c,d,e), samples of future volcanism are first pre-processed according to EMB-KF-ta. Policy threshold crossing instants (intersecting horizontal and vertical lines) and crossing windows (arrows at bottom) are also shown.

c. Potential Issues with the EBM-KF and Future Extensions

This first climate Kalman filter does not generate regional temperatures nor other essential climate variables, such as precipitation. These variables are often highly non-Gaussian and may require an understanding of regional "dynamical tipping points" or other important nonlinear process aspects of climate change. Additionally, this 2-component EBM-KF lacks a "memory ENSO state" to allow for prediction of 2-7 year quasi-periodic El Nino events (Hu and Fedorov 2017), and without such a state our EBM-KF wrongly assumes that weather innovations z_t^T have no autocorrelation. Therefore, this first EBM-KF is far from generating the information required to replace many aspects of large ensembles. An expanded global climate state vector, including precipitation, seasonal temperature, or eigenvalues of spatially decomposed principal components (e.g., El Nino / Southern Oscillation) might be appended into this statistical framework with appropriate physical forward modeling (Yang, Li et al. 2018).

Astute readers may note the estimated climate state and covariance within the EBM-KF are influenced by the choice of reconstructed HadCRUT5 GMST and Zanna et al. (2019) OHCA. With only minor modifications, the EBM-KF method could be used with multiple annual reconstructions at the same time, e.g., GISTEMP GMST (Lenssen, Schmidt et al. 2019) or other OHCA reconstructions (Cheng, Trenberth et al. 2017; Ishii, Fukuda et al. 2017), considering each as only an estimate of the true GMST or OHCA (Willner, Chang et al. 1977). Reconstructions of sea level rise could be used from different sources as further constraints on OHCA (Palmer, Howard et al. 2018; Fox-Kemper, Hewitt et al. 2021; Palmer, Domingues et al. 2021).

Here we use pre-selected, constant parameters at their published values in the EBM-KF. However, methods for tuning parameters, including time-dependent parameters, within Kalman filters are much more extensively studied mathematically (Chen, Heckman et al. 2018; Zhang and Atia 2020; Chen, Heckman et al. 2021) than the methods thus far applied in climate sciences to diagnose parameter variations within energy balance models (e.g., the regional effects diagnosed in Armour, Bitz et al. (2013) and the global effects found by Gregory and Andrews (2016)). Our EBM-KF hybrid presents an opportunity to adopt KF parameter optimization methods for the GMST, OHCA projection optimization problem. In a preliminary experiment with Bayesian parameter search to give better estimates of the coefficients in the blind EBM, the prior distributions of these coefficients (rather than point estimates) were extracted from climate science literature, followed by a Metropolis-Hastings search. Several parameters required further care or tuning to achieve desired constraints (e.g., balanced energy transfer in the preindustrial climate), such as the main longwave radiation coefficient and the temperature exponent. However, identifiability and overfitting are challenges of this approach and deserve more attention than the scope of this paper allows. In this first illustration of the system, opportune imperfections in the point estimates given by literature sources allow demonstration of the course-corrective properties of the EBM-KF (Fig. 4).

d. Policy Utility

Has the climate already passed the 1.5°C policy threshold? Real-time, accurate knowledge of policy threshold crossing will allow for more prudent planning and more comprehensible communication of climate science to the public. For instance, while the "Climate Clock" (https://climateclock.world) intends to communicate the urgency of the climate crisis with a countdown to the estimated expenditure of our remaining carbon budget, only a static date informs it. In contrast, an EBM-KF threshold reflects the most recently measured state of the Earth system and up-to-date emissions and present limits on future emissions. As can be seen in Fig. 12h, there was a substantial (15.5%) chance that 2023's GMST measurement could have exceeded the 1.5°C threshold. Indeed, the HadCRUT5 2023 number came close at 1.45°C, and others with slightly different methodologies reported 1.52°C above preindustrial (Burgess 2024). Rather than relying on sponge proxy data to posit that the climate state has exceeded this threshold (McCulloch, Winter et al. 2024), or using an overlap window between ESM projections and smoothed observations that is challenging to translate into probability (Hausfather 2024), the EBM-KF-ta can simply give a p-value (subject to the key, but quantifiable uncertainties of the EBM-KF mapped in Fig. 11). By this method, assuming our original point-estimates of key feedback parameters, we have not yet crossed the 1.5°C climate policy threshold: the EBM-KF-ta states (Fig. 12c) that there is p=0.00017 that the climate state exceeded 1.5°C in 2021.

Climate modeling with ESMs is slow, computationally expensive, and typically performed with blind models that do not respond to the latest observations. The relatively simple question, "How did the COVID-19 lockdowns and the 8% reduction in CO_2 emissions impact the near-term climate?" required hundreds of ESM simulations to yield a statistically insignificant answer (Jones, Hickman et al. 2021). That sort of modeling effort, arriving months or years after the question was posed, is an unsatisfactory prize for many aspects of communication and decision making for the annual profit or election term. The EBM-KF can produce the result that an 8% emissions reduction over 2 years cools the climate state by ~0.0017K and pushes back subsequent threshold crossing time by 1.2 months – an insufficient reduction in climate change, but at least precisely and rapidly quantified. The EBM-KF is sufficiently fast that, once fully calibrated, it could be easily embedded as an interactive web tool for such exploration. This demonstrates that, like "attributable anthropogenic warming" the EBM-KF is an "anti-fragile index" and therefore of greater use to planning climate mitigation strategies (Otto, Frame et al. 2015).

Additionally, Kalman filters are often used for process control (Myers and Luecke 1991; Lee and Ricker 1994), and in this case an EBM-KF could be used to optimize climate change mitigation or intervention strategies (Filar, Gaertner et al. 1996; MacMartin, Kravitz et al. 2014; Kravitz, MacMartin et al. 2016). For instance, within carbon offset / sequestration and geoengineering accreditation markets, credits could be assigned based on the projected delay in crossing policy thresholds. Once a space of potential climate solutions has been defined, the EBM-KF can work seamlessly with a variety of optimizers to find the maximum climate benefit at the lowest societal cost.

5. Conclusion

The EBM-KF presented in this paper takes the best features from a 30-year running mean of GMST (the historical definition of climate) and state-of-the-art ESM large ensembles such as CESM2 LENS. The EBM-KF GMST climate state, which also tracks the ocean heat content anomaly (OHCA), is constructed to be very close to that of a running 30-year mean but generates this climate state 15 years sooner: it has no lag in reporting after annual observations are collected. This filtered climate state captures the overall shape of the

30-year means of measured GMST ($r^2 = 0.922$) and OHCA ($r^2 = 0.989$). In comparison to the ensemble spread of a hindcast ensemble of an ESM (LENS2), which is the state-of-the-art method for quantifying internal variability and probabilistic futures, the EBM-KF provides a similar Gaussian distribution. Using this distribution, EBM-KF can annually assess the likelihood of whether a policy threshold, e.g., 1.5 or 2°C over preindustrial, has been crossed. The EBM-KF is also accurate at inferring the behavior of an entire climate model large ensemble using only one or a few ensemble members. In future projections of climate under SSP trajectories, the efficiency of the EBM-KF allows for sampling non-Gaussian probabilistic futures, e.g., the impact of rare but significant events such as future volcanic eruptions. An exponential mixture model of future volcanic eruptions causes the EBM-KF GMST climate states to be negatively skewed, unlike LENS2 which remains Gaussian by constant forcing design.

The EBM-KF approach has transparent, clean physical parameters of the EBM that can be directly measured or taken from estimates in modeling literature, leading to trivial uncertainty quantification by the Kalman filter machinery under fixed parameters. This uncertainty quantification revealed important aspects of GMST and OHCA uncertainty, both in hindcast and future projections contexts, with and without volcanoes. We considered that the EBM-KF may be improved with time-varying EBM parameters or other extensions, although a thorough treatment is left for future work. While the EBM-KF does not predict all climate variables of interest, it is a powerful, transparent, and inexpensive tool that may be readily combined with other approaches.

Acknowledgments.

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Data Availability Statement.

This study performed re-analysis of existing datasets openly available at locations provided in Appendix A regarding historical CO_2 and AOD, for SSP projections at https://greenhousegases.science.unimelb.edu.au/, and for LENS2 at https://www.earthsystemgrid.org/dataset/ucar.cgd.cesm2le.atm.proc.monthly_ave.TS.html.

For critical measurements of the climate state, GMST via HadCRUT5 is at https://www.metoffice.gov.uk/hadobs/HadCRUT5/data/current/download.html and OHCA from Zanna et. al. (2019) is at https://zenodo.org/record/4603700#.ZDuFNxXMI88. Further documentation about data processing, copies of the utilized datasets, and EBM-KF Python code is available through Harvard Dataverse at http://doi.org/10.7910/DVN/XLY8C2.

APPENDICES

Appendix A: *Derivation of the Blind Energy-Balance Model*

A1: Overall Structure of the Model

In the schematic diagram Fig. 2, one stream of incoming solar shortwave energy $\frac{G_{SC}}{4}$ is successively fractionated by three reflective layers until a portion warms the ground and surface ocean. Then this surface layer radiates longwave infrared energy back to space $j^{\star}$, again with greenhouse "reflection" in two layers. The surface ocean warms the deep ocean with fixed thermal insulation between them.

Temperature-dependent feedbacks are shown as cyclical arrows, with positive and negative feedback indicated relative to the overall energy balance. Positive feedbacks increase the energy flowing to the surface at higher surface temperatures T either by decreasing the fraction of shortwave reflection or increasing the greenhouse "reflection". Prescribed forcings are indicated by gear symbols. Unknown coefficients $\beta_0\,\beta_1\,\beta_2\,\beta_3$ exist respectively within the terms: $\tilde{g}(t), f_{H20}(T)$, $f_{\alpha A}(T,t)$, $f_{\alpha S}(T)$ in addition to the unknown exponent η. All these symbols are defined below.

Reiterating the overall structure in the model with discrete difference equations, T_t is the temperature of the surface in calendar year t (e.g. 2000), θ_t is the potential (or conservative) temperature of the deep ocean in that same year, and H_t is the total ocean heat content combining the heat in the surface ocean and deep ocean. The time step (abbreviated k in Kalman filter literature) is 1 year. Units are omitted in this section for brevity.

$$\Delta\text{energy\_total} = \mathcal{F}_{\mathrm{SW}} - \phi_{\mathrm{LW}} \tag{A1}$$

$$\Delta\text{energy\_surf} = \mathcal{F}_{\mathrm{SW}} - \phi_{\mathrm{LW}} - Q_{\mathrm{surf\_deep}} \tag{A2}$$

$$\frac{T_{t+1}-T_t}{1}\mathrm{C_{surf}} = \left(\frac{G_{SC}}{4}\right)_t * \tilde{\mathrm{d}}_{\mathrm{t}} * f_{\alpha A}(T_t) * f_{\alpha S}(T_t) - \mathrm{j}^{\star}(T_t) * \tilde{\mathrm{g}}_{\mathrm{t}} * f_{H2O}(T_t) - \gamma * (T_t - \theta_t - \zeta_0) \tag{A3}$$

$$\frac{\theta_{t+1}-\theta_t}{1}\mathrm{C_{deep0}} = \gamma * (T_t - \theta_t - \zeta_0) \tag{A4}$$

$$H_t = (T_t - T_{1850}) * \mathrm{C_{upper0}} + (\theta_t - \theta_{1850}) * \mathrm{C_{deep0}} \tag{A5}$$

$$\theta_t = \big(H_t - (T_t - T_{1850}) * \mathrm{C_{upper0}}\big)/\mathrm{C_{deep0}} + \theta_{1850} \tag{A6}$$

$$H_{t+1} = (T_{t+1} - T_{1850}) * \mathrm{C_{upper0}} + \gamma * (T_t - \theta_t - \zeta_0) + (\theta_t - \theta_{1850}) * \mathrm{C_{deep0}} \tag{A7}$$

$$H_{t+1} - H_t = (T_{t+1} - T_t) * \mathrm{C_{upperO}} + \gamma * (T_t - \theta_t - \zeta_0) \qquad \text{(A8)}$$

Derivatives of θ_n: $\frac{\partial \theta_t}{\partial H_t} = 1/\mathrm{C_{deepO}}$ (A9a) $\frac{\partial \theta_t}{\partial T_t} = \mathrm{C_{upperO}}/\mathrm{C_{deepO}}$ (A9b)

On the right side of equation A3, both the incoming shortwave radiative flux $\mathcal{F}_{\mathrm{SW}}$ and outgoing longwave radiative flux ϕ_{LW} take the same form: (source $\{\frac{G_{SC}}{4}, \mathrm{j}^{\star}(T_t) = \sigma_{sf} {T_t}^4\}$) * (prescribed attenuation $\{\tilde{\mathrm{d}}(t), \tilde{\mathrm{g}}(t)\}$) * (attenuations with feedback $\{f_{\alpha A}(T_t, t) * f_{\alpha S}(T_t), f_{H2O}(T_t)\}$). C_{surf}, the heat capacity of the surface (including the atmosphere, thermally active soil, and an 86m upper layer of the ocean), was known least precisely of all coefficients: 17 ± 7 W (year) m^{-2} K^{-1}, (Schwartz 2007). The deep ocean layer (technically the zone where most of the ocean warming occurs) was chosen for the purpose of heat capacity estimation to be an additional 1141m within the 71% of area covered by ocean based on previous work of this heat transfer process. (Geoffroy, Saint-Martin et al. 2013; Hall and Fox-Kemper 2023). This gives C_{deepO} = 1141m *0.71 * 1030kg/m^3 * 4180Ws/kg/K * 1 yr/ (3.154*10^7s) = 155.7 W (year) m^{-2} K^{-1}. Constants γ, ζ_0 form a linear heat flux $Q_{\mathrm{surf\_deep}}$ into the deep ocean, as discussed below. Radiative fluxes are signified in this text by the symbol ϕ, followed by specific details of that flux.

A2: Functional Forms of Components

For brevity, derivations and detailed explanations of each of these components has been moved to the Supplement (A1 & A2). Here the functional form of each component is provided.

$$Q_{surf-deep} = \gamma * (\Delta T_t - \Delta \theta_t) = \gamma * (\mathrm{T_t} - \theta_\mathrm{t} - \mathrm{T_{1850}} + \theta_{1850}) \qquad \text{(A9)}$$

The heat flowing from the surface layer into the deep ocean.

$$\tilde{\mathrm{d}}(\mathrm{t}) \approx \frac{9.07}{\mathrm{AOD_t} + 9.73} \qquad \text{(A10)}$$

The fraction of shortwave (incoming) light reflected by stratospheric (volcanic) aerosols.

$$\tilde{\mathrm{g}}(t) = 1 - \beta_0 \log_{10}([\mathrm{eCO_2}]_\mathrm{t}) < 1 \qquad \text{(A11)}$$

The fraction of longwave radiation absorbed by greenhouse gases.

$$j^{\star}(T_t) = \sigma_{sf} T_t^4 \tag{A12}$$

Blackbody radiation, source of longwave outgoing radiation.

$$\phi_{LW}(\mathrm{out}) = j^{\star}(T_t) - \frac{\phi_{LW}(\mathrm{absorbed})}{2} = j^{\star}(T_t) * \tilde{g}(t) * f_{H2O}(T_t) \tag{A13}$$

Two simplified expressions of how longwave radiation is blocked, in this paper we use the one at right which relates CO_2 to a faction absorbed (similarly to albedo). Other authors favor the expression in the center, as it relates the absorption of a greenhouse gas to a power (in W/m^2) rather than an expression.

$$\phi_{LW}^{CO2} = 12.74 \log_{10}([eCO_2]_t) - 31.55 \tag{A14}$$

The expression reported by Forster (2023) for the blocked outgoing longwave radiation – for our EBM this expression must be converted into a fraction to solve for β_0.

$$f_{H2O}(T_t) \doteq \beta_1 (1/T_t)^{\eta} \approx 1-\left(1+\beta_1 (T_{2002})^{-\eta} - \beta_1 \eta (T_{2002})^{-\eta-1}(T_t - T_{2002})\right) \tag{A15}$$

$$f_{\alpha A}(T_t, t) \doteq 0.834 \left(1+\beta_2(T_t - T_{2002})\right) + \frac{AC_n - AC_{2002}}{\frac{G_{SC}}{4}\widetilde{d_{2002}}} \tag{A16}$$

$$f_{\alpha S}(T_t) \doteq 0.909 \left(1+\beta_3(T_t - T_{2002})\right) \tag{A17}$$

Functions proposed by the authors for the water feedback (on absorbing fraction of longwave radiation), atmospheric albedo feedback, and surface albedo feedback. Note that the values of 0.834 and 0.909 came from the CERES satellite in the early 2000s. Solving for all the coefficients, we find from feedbacks assessed in ESM (CMIP6 & AR6):

$$\eta = 1.615,\ \beta_2 = 0.00136\ K^{-1},\ \beta_3 = 0.00163\ K^{-1} \tag{A18}$$

And by assuming the climate was at equilibrium before 1850 and assimilating the longwave anthropogenic greenhouse gas and anthropogenic aerosol energy fluxes published by Forster (2023) at https://github.com/ClimateIndicator/forcing-timeseries/tree/main/output:

$$6593.57 \approx \beta_1 \quad \text{and} \quad 0.04660 \approx \beta_0 \tag{A19}$$

This yields the following energy fluxes in 2002 (compare to Wild, Folini et al. (2015), Wild, Hakuba et. al. Wild, Hakuba et al. (2019))

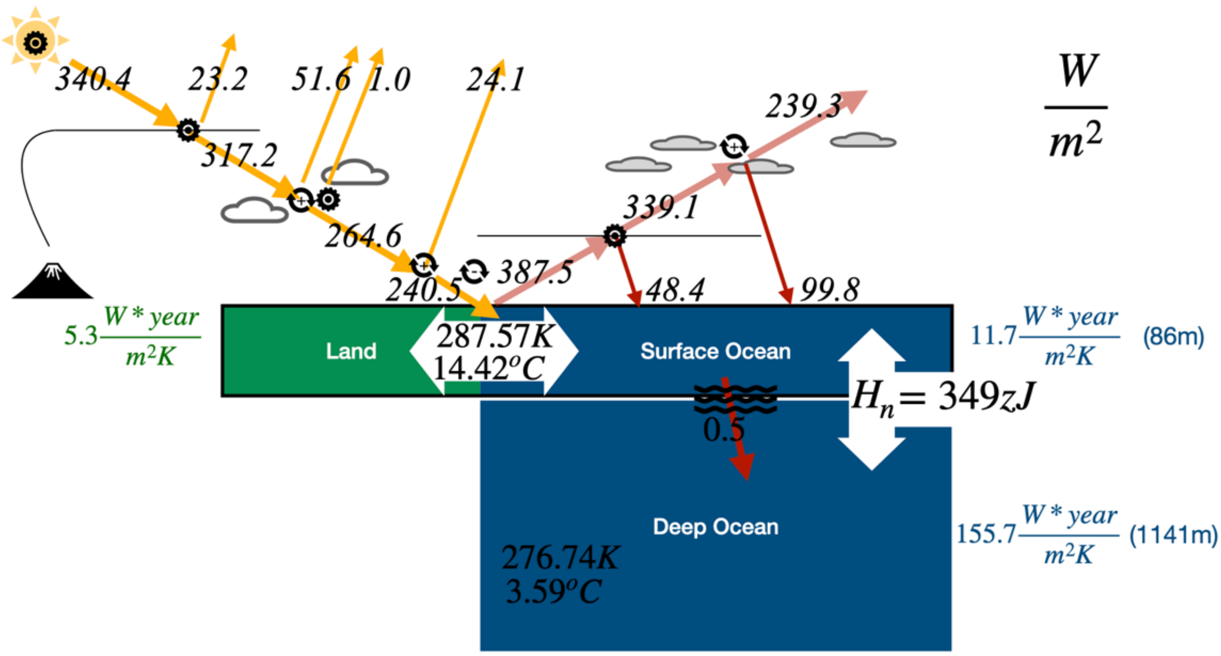

Fig. A1: Diagram with energy fluxes, temperatures, and total ocean heat content for the blind run of energy balance model in 2002 (when many of the reflectivity values were first measured by the CERES satellite). All numbers without units are in W/m^2.

A3: Differentiating to Find the Jacobian Matrix

This yielded a blind energy-balance model with good skill at predicting the GMST (orange dashed line in Fig. 2), $r^2 = 0.902$. Rewriting the overall model with β coefficients,

$$T_{t+1}=T_t+\frac{\left(\frac{G_{SC}}{4}\right)_t 0.758 * 9.068}{C_{surf}\,(AOD_t+9.73)}\left(1+\beta_2(T_t\text{-}287.5)+\frac{AC_t-AC_{2002}}{\frac{G_{SC}}{4}\,\widetilde{d_{2002}}\,0.834}\right)\left(1+\beta_3(T_t\text{-}287.5)\right)$$

$$-\frac{\sigma_{sf}\beta_1}{C_{surf}}(T_t)^{2.39}\left(1\text{-}\beta_0\log_{10}([eCO_2]_t)\right)\quad -\frac{\gamma}{C_{surf}}(T_t-\theta_t-10) \qquad \text{(A20)}$$

Derivatives of θ_n: $\frac{\partial\theta_t}{\partial H_t}=1/C_{deep}$ (A9a) $\qquad \frac{\partial\theta_t}{\partial T_t}=C_{upper0}/C_{deep}$ (A9b)

$$\frac{\partial T_{t+1}}{\partial T_t}=1+\frac{137.6}{AOD_t+9.73}\left(\beta_2+\beta_3+2\beta_2\beta_3(T_t\text{-}287.5)+\beta_3\frac{AC_t-AC_{2002}}{G_0^{\star}\,\widetilde{d_{2002}}\,0.834}\right)$$

$$-\frac{2.39\,\sigma_{sf}\beta_1}{C_{surf}}(T_t)^{1.39}\left(1\text{-}\beta_0\log_{10}([eCO_2]_t)\right)-\frac{\gamma}{C_{surf}}\left(1\text{-}C_{upper0}/C_{deep}\right) \qquad \text{(A21)}$$

$$\frac{\partial T_{t+1}}{\partial H_t}=\frac{\gamma}{C_{surf}}*\frac{\partial\theta_t}{\partial H_t}=\frac{\gamma}{C_{surf}\,C_{deep}} \qquad \text{(A22)}$$

The ocean heat content update equation ($r^2 = 0.907$ blind) and related partial derivates are:

$$H_{t+1}=(T_{t+1}-T_{1850})*C_{upper0}+\gamma*(T_t-\theta_t-\zeta)+(\theta_t-\theta_{1850})*C_{deep} \qquad \text{(A23)}$$

$$\frac{\partial H_{t+1}}{\partial H_t} = \mathrm{C_{upperO}}\,\frac{\partial \mathrm{T_{t+1}}}{\partial \mathrm{H_t}} + \gamma * \left(0 - \frac{\partial \theta_{\mathrm{t}}}{\partial H_t}\right) + \mathrm{C_{deep}}\,\frac{\partial \theta_{\mathrm{t}}}{\partial H_t} = \frac{\gamma}{\mathrm{C_{deep}}} * \left(\frac{\mathrm{C_{upperO}}}{\mathrm{C_{surf}}} - 1\right) + 1 \qquad \text{(A24)}$$

$$\frac{\partial H_{t+1}}{\partial T_t} = \mathrm{C_{upperO}} * \frac{\partial \mathrm{T_{t+1}}}{\partial \mathrm{T_t}} + \gamma * \left(1 - \frac{\mathrm{C_{upperO}}}{\mathrm{C_{deep}}}\right) + \mathrm{C_{upperO}} \qquad \text{(A25)}$$

Appendix B: *Generation of Volcanic Eruption Samplings*

As can be appreciated in Fig. B1a, long periods of no major volcanic eruptions (for instance 1935-1960) alternated with periods of many eruptions occurring in rapid succession (1883-1914, 1960-1994). Perhaps this observed pattern has some relation to magma or tectonic dynamics, but it prevented one Poisson distribution from describing the data well.

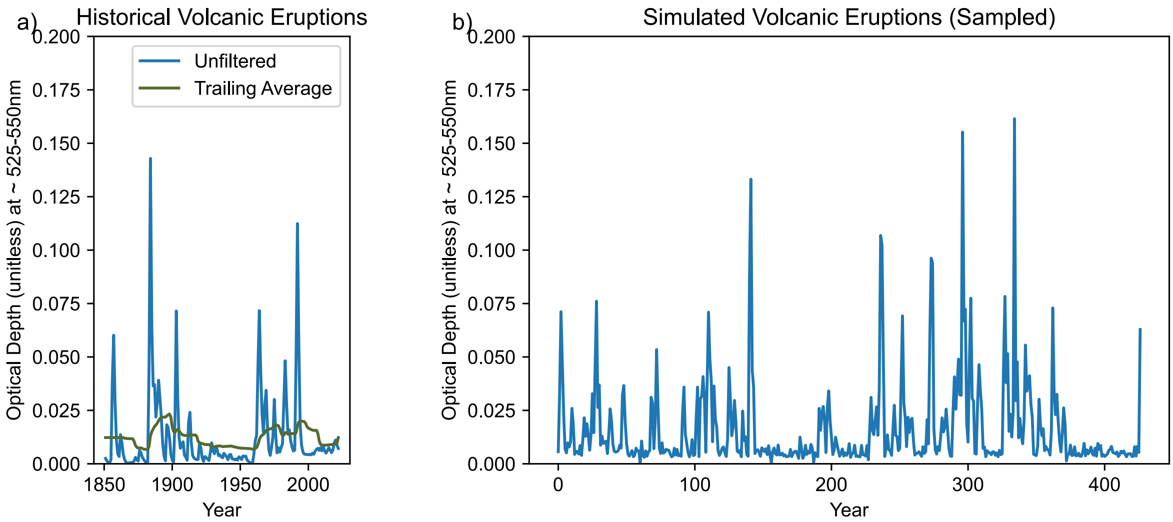

Fig. B1: Comparison of Historical Volcanic Eruptions (B1a) with Simulated Volcanic Eruptions (B1b), generated from a combination of several probability distributions. Observe in panel a that the unfiltered aerosol optical depths (AOD_t) are plotted in blue, whereas the trailing average filter is plotted in green (it combines 15 years of a trailing average and 15 years of future projections at the mean AOD).

Eruptions that occurred within 3 years were indistinguishable in the historical dataset, so the minimum time interval between simulated volcanic eruptions was 2.6 years plus a sample (Table B1) from the exponential mixture model i_t (Okada, Yamanishi et al. 2020). These intervals were rounded to integers. Similarly, the size of each volcanic eruption h_t was approximated using another shifted exponential distribution. The preceding year and two years following the eruption peak were positive fractions of the maximum aerosol optical depth, with Gaussian blur. Similarly, non-volcanic years were positive Gaussian noise (Table B2). Fig. B1b shows a sample from this combined generating function.

Exponential Distribution	**Rand. Var.**	**Scale (units)**	**P(if mixture)**
Interval Between: round($i_{t,0}$ + 2.6)	$i_{t,0}$ ~ Exp	2.263 (years)	88.9%
Interval Between: round($i_{t,1}$ + 2.6)	$i_{t,1}$ ~ Exp	24.2 (years)	11.1%
Peak Size: $AOD_t = h_t + 0.0082$	h_t ~ Exp	0.0339 (m)	**only "eruption" years**

Table B1. Exponential Parameters of Volcano Generating Function. This generating function starts with a list of zero values for all AOD_t, and first samples several of these *t* years to be major volcanic eruptions. "Interval Between" refers to the interval in years between two successive major volcanic eruptions.

Gaussian Distribution	**Rand. Var.**	**Mean μ (units)**	**Std Dev σ**
Pre-Peak: $AOD_{t-1} = a_{(-1)} * AOD_t$	$a_{(-1)}$ ~ Norm>0	0.51	0.25
Post-Peak 1: $AOD_{t+1} = a_1 * AOD_t$	a_1 ~ Norm >0	0.61	0.16
Post-Peak 2: $AOD_{t+2} = a_2 * AOD_t$	a_2 ~ Norm >0	0.32	0.16
Other Years: $AOD_t = a_0$	a_0 ~ Norm >0	0.00371 (m)	0.00286 (m)

Table B2. Gaussian Parameters of Volcano Generating Function. These distributions are sampled after the major erruptions have already been filled in by the exponential distributions in Table B1.

1420 **Appendix C:** ***Glossary of Mathematical Symbols and Notation***

Symbol	**Meaning**	**Context**	**Units**
p , $\mathbb{P}(\text{ event })$	Probability of the observed result for a particular hypothesis test (e.g. that the slope is positive)	Statistics	[0-1]
r^2	Coefficient of determination: fraction of variance explained by a model	Statistics	[0-1]
σ ±2σ = 95% CI	Standard deviation (√Variance) 95% confidence interval (extremely likely) under Gaussian distribution	Statistics	any
Cov()	Covariance of a random vector (here length 2)	Statistics	sq. matrix
t, k	Time index, time step	KF, EBM	year
T_t	GMST surface temperature climate state, idealized	EBM-KF	K (°C)
θ_t	Deep ocean potential temperature state, idealized	EBM-KF	K
H_t	Ocean heat content anomaly, idealized	EBM-KF	$\frac{W\,yr}{m^2}$ (ZJ)
u_{t}=$[eCO_2]_t, AOD_t, AC_t, (\frac{G_{SC}}{4})_t$	Time-varying concentrations in the atmosphere	EBM	ppm, ∅, W/m^2
$[\tilde{T}_{t+1}, \tilde{H}_{t+1}] = \boldsymbol{F}(\tilde{T}_t, \tilde{H}_t, u_{\mathrm{t}})$	Blind energy balance model, which is entirely deterministic based on prior climate state	EBM	[K, $\frac{W\,yr}{m^2}$]
$\Phi_t = \frac{\partial \mathbf{F}(\boldsymbol{x};u_t)}{\partial \boldsymbol{x}}\Big\vert_{x=\hat{x}_{t-1}}$	Linearized tensor derivative of the (blind) EBM model	EBM-KF	$\begin{bmatrix} \varnothing & K / \frac{W\,yr}{m^2} \\ \frac{W\,yr}{m^2}/K & \varnothing \end{bmatrix}$
$\boldsymbol{x_t}$=$[T_t, H_t]$	Idealized true climate state, with dynamic model noise	EBM -KF	[K, $\frac{W\,yr}{m^2}$]
$\hat{\boldsymbol{x}}_{\mathrm{t}} = [\hat{T}_{\mathrm{t}}, \hat{H}_t]$	Estimate of the underlying climate state	EBM -KF	[K, $\frac{W\,yr}{m^2}$]

$\boldsymbol{y}_t = [Y_t, \psi_t]$	Measurements with noise of the climate state, from HadCRUT5 (Morice, Kennedy et al. 2021) and Zanna et al. (2019).	EBM -KF	[K, $\frac{W\,yr}{m^2}$]
Q=COV[w_t]	Assumed dynamic model error and model covariance matrix	KF	$\begin{bmatrix} K^2 & K\frac{W\,yr}{m^2} \\ K\frac{W\,yr}{m^2} & \left(\frac{W\,yr}{m^2}\right)^2 \end{bmatrix}$
R=COV[v_t]	Assumed measurement error and measurement covariance matrix	KF	$\begin{bmatrix} K^2 & K\frac{W\,yr}{m^2} \\ K\frac{W\,yr}{m^2} & \left(\frac{W\,yr}{m^2}\right)^2 \end{bmatrix}$
$\overline{{}_{30}\boldsymbol{y}_t}$ $\overline{{}_{30}Y_t}$	30-year running mean of measurements, undefined before 1865 or after 2008	Prior climate methods	[K, $\frac{W\,yr}{m^2}$] K
R$_t$= $R_t^{var} + R^{const}$ Q= $R^{const}/30$	Actual covariance matrices used in the EBM-KF, defined to mimic the statistics of the 30-year running mean	EBM-KF	$\begin{bmatrix} K^2 & K\frac{W\,yr}{m^2} \\ K\frac{W\,yr}{m^2} & \left(\frac{W\,yr}{m^2}\right)^2 \end{bmatrix}$
$\hat{\boldsymbol{x}}_{t\|t-1}$ P$_{t\|t-1}$	KF forecast state projection and forecast covariance projection (before new measurement)	KF	[K, $\frac{W\,yr}{m^2}$] $\begin{bmatrix} K^2 & K\frac{W\,yr}{m^2} \\ K\frac{W\,yr}{m^2} & \left(\frac{W\,yr}{m^2}\right)^2 \end{bmatrix}$
$\boldsymbol{z}_t$ S$_t$	Innovation residual, Innovation covariance	KF	[K, $\frac{W\,yr}{m^2}$] $\begin{bmatrix} K^2 & K\frac{W\,yr}{m^2} \\ K\frac{W\,yr}{m^2} & \left(\frac{W\,yr}{m^2}\right)^2 \end{bmatrix}$
K$_t$	Kalman gain: weight on innovation to correct state	KF	$\begin{bmatrix} \varnothing & \varnothing \\ \varnothing & \varnothing \end{bmatrix}$

$\hat{\boldsymbol{x}}_t$ P_t	KF posterior estimated state projection and state variance (after measurement)	KF	$[\mathrm{K}, \frac{W\,yr}{m^2}]$ $\begin{bmatrix} K^2 & K\frac{W\,yr}{m^2} \\ K\frac{W\,yr}{m^2} & \left(\frac{W\,yr}{m^2}\right)^2 \end{bmatrix}$
$\hat{\hat{K}}_t$, $\hat{\hat{x}}_t$, $\hat{\hat{P}}_t$	RTS re-estimated Kalman gain, state estimate, and state covariance, following backward sweep	RTS	as above
$(Y_t)_j$, $(\psi_t)_j$	The j[th] ensemble member's annual mean at time t of air temperature or ocean heat content	LENS2	K, $\frac{W\,yr}{m^2}$
$\overline{(Y_t)_J}$	Ensemble average (across all members) at year t	LENS2	K
$(\overline{_{21}Y_t}\,)_j$	The 21-year running mean of ensemble member j	LENS2	K
$\overline{(\overline{_{21}Y_t}\,)_J}$	The cross-ensemble average of all 21-year running means	LENS2	K
γ	Thermal conductivity between layers of the ocean	EBM	$\frac{W\,yr}{m^2\,K}$
$\mathcal{F}_{\mathrm{SW}}$, ϕ_{LW}	Net radiative fluxes (shortwave and longwave) at the top of the atmosphere	EBM	W
ΔEnergy_surf $Q_{\mathrm{surf\_deep}}$	Net heat flow into the surface and deep ocean layers respectively	EBM	W

C_{surf} ; C_{upper0} ; C_{deep0}	Heat capacities of the surface, surface ocean, and deep ocean	EBM	$\frac{W\ yr}{m^2\ K}$
$(\frac{G_{sf}}{4})_t$ $j^{\star}=\sigma_{sf}T^4$	Sources of shortwave (total solar irradiance) and longwave (blackbody or Planck feedback)	EBM	$\frac{W}{m^2}$
$\tilde{d}_t, \tilde{g}_t$	Prescribed, time-varying attenuations from atmospheric dust and longwave radiation respectively	EBM	∅
$f_{\alpha A}(T,t) * f_{\alpha S}(T)$ $f_{H2O}(T)$	Attenuations due to albedo of the atmosphere, albedo of the surface, and longwave absorbing water vapor (all with feedback from T_t)	EBM	∅
ζ_0	Equilibrium temperature difference between the surface and deep ocean.	EBM	K
ζ_1	Baseline temperature for HadCRUT5 to achieve the appropriate 1960-1989 climate normal (Jones and Harpham 2013)	EBM	K
σ_{sf}	Stefan-Boltzman constant = 5.670 10^{-8}	EBM	$\frac{W\ m^2}{K^4}$
β_0	Solved coefficient on $\log_{10}([eCO_2]_t)$ within a	EBM	∅

	sequential filter atmosphere approximation		
β_1, η	Solved coefficient and exponent for the $f_{H2O}(T)$ water vapor feedback on longwave	EBM	∅
β_2, β_3	Solved coefficients for $f_{\alpha A}(T,t) * f_{\alpha S}(T)$, atmosphere and surface albedo feedbacks.	EBM	∅
$c_1, c_2,$ c_3, c_4	Simplifications of constants within the EBM, see Eq. 4-6 and Table 1.	EBM	$K^{-3+\eta}$, $\frac{K\,m^2}{W}$, $\frac{W}{m^2}$, ∅ (AOD)
q	Location of a climate policy threshold	EBM-KF thresholds	K
$i_{t,0}$ $i_{t,1}$	Exponential mixture random variables to determine the interval between major eruptions	Volcanoes	years
h_t	Exponential random variable to determine size of a particular major eruption	Volcanoes	∅ (AOD)
$a_{(-1)}, a_1, a_2, a_0$	Truncated gaussian distributions to determine the atmospheric optical depth in eruption-adjacent and non-eruption years.	Volcanoes	∅ (AOD)

Table C1: Glossary of Mathematical Symbols

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SUPPLEMENT TO

Efficient Estimation of Climate State and Its Uncertainty Using Kalman Filtering with Application to Policy Thresholds and Volcanism

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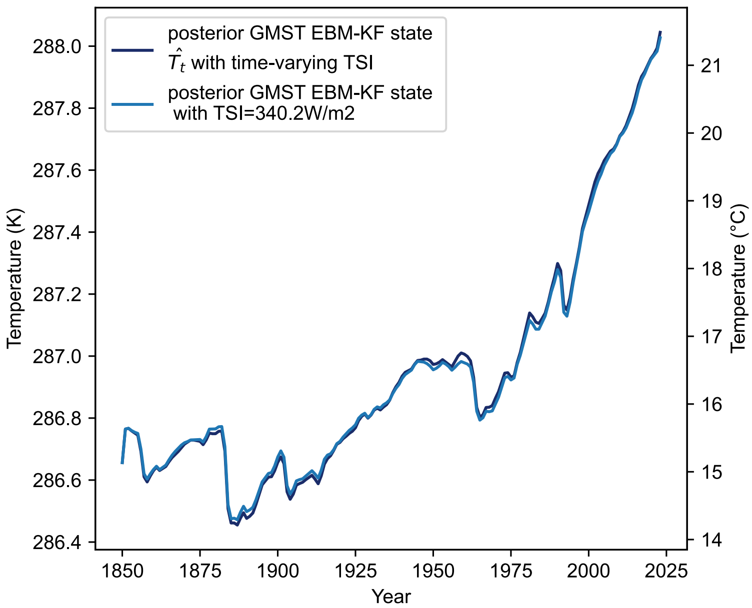
Section A: Derivation of EBM-KF

A1: Individual Functional Parts and Derivation

$(\frac{G_{SC}}{4})_t$ is the total solar irradiance (TSI) normalized to the Earth's surface area at ~1360 W/m^2 / 4 = 340.2 W/m^2 . Estimates of this normalized total solar irradiance indicate that it has varied since 1850 between 340.06 W/m^2 and 340.49 W/m^2 according to the Naval Research Laboratory 2 solar irradiance model (NRLTSI2_v02r01 (Coddington, Lean et al. 2017)). Within the hindcast EBM-KF model these NRL2 estimates were used, but this had a negligible effect on the model results compared to a constant $\overline{\frac{G_{SC}}{4}}$=340.2 W/m^2 value.



Supp. Fig. 1: Comparisons of the used EBM-Kalman Filtered climate state with time-varying total solar irradiance (navy blue) with an EBM-Kalman Filtered climate state with constant solar irradiance (light blue) set at 340.2 W/m^2. These differed by at most 0.028°C in 1960.

$\tilde{d}(\boldsymbol{t})$ is the prescribed shortwave radiation attenuation due to volcanic dust, the direct radiative effect of anthropogenic aerosols, and non-cloud atmospheric effects. This stochastically varying quantity can be calculated from the (unitless) stratospheric optical depth AOD_n (Sato, Hansen et al. 1993; Vernier, Thomason et al. 2011), according to the formula given by Harshvardan and King (1993; Schwartz, Harshvardhan et al. 2002). (g=0.853 is the middle of the given range). The AOD_t values used are forcings for the GISS climate model from 1850 – 1978 (https://data.giss.nasa.gov/modelforce/strataer/tau.line_2012.12.txt, AOD_n at 550nm) and globally averaged measurements from the GloSSAC_V2.21 satellite measurement product (Nasa/Larc/Sd/Asdc 2018) from 1979 – 2022 (https://asdc.larc.nasa.gov/project/GloSSAC/GloSSAC_2.21, AOD_t at 525nm). These wavelengths are at the shorter end of the 0.25-4 µm range of incoming solar shortwave energy $\boldsymbol{\mathcal{F}}_{\mathbf{SW}}$, allowing satellites to detect dust reflectance. As the CALIPSO satellite mission ended in 2023, the year 2023 was extrapolated from a linear trend of the AOD_t values from 2025-2022.

$$\tilde{d}(t)=\frac{4/3}{AOD_t*(1\text{-}g)+2q'},\quad g\in[0.834\text{ - }0.872],\ q' = 0.715 \qquad \text{(SA1)}$$

$$\tilde{d}(t)\approx\frac{9.07}{AOD_t+9.73} \qquad \text{(SA2)}$$

Utilizing the equation above to calculate the dry-atmosphere reflected energy during a relatively aerosol-free period (2000-2005), when the aerosol optical depth was about 0.002m:

$$\mathcal{F}_{SW}{}^{refl\ by\ dryatm}_{clearsky} = \frac{\overline{G_{SC}}}{4} * (1 - \tilde{d}(2002)) = 340.2\,\frac{W}{m^2}\,(1-\frac{9.07}{0.002+9.73}) = 23.1\,\frac{W}{m^2} \qquad \text{(SA3)}$$

This value agrees with the clear-sky reflected energy (53 [52-55] W/m^2) minus reflected surface energy (33 [31-34] W/m^2), of 20 [18-24] W/m^2 reported by Wild, Hakuba et. al. (2019). Furthermore, the measured and inferred aerosol optical depth measurements already include those contributions from the anthropogenic sources.

$f_{\alpha A}(T_n,t)$ is the additional atmospheric shortwave attenuation due to cloud albedo, while $f_{\alpha S}(T_n)$ is the surface shortwave attenuation due to ground albedo. A portion of this varying cloud albedo is direct thermal feedback, whereas another portion is due to cloud seeding by anthropogenic aerosols AC_t. To contain the EBM model's complexity, the changing ground albedo is assumed to be only thermal feedback: the shortwave aspect of land use changes are neglected. Taken together, these two terms and $\widetilde{d_n}$ yield an overall absorption of 0.707 as measured from March 2000 to February 2005 by the CERES satellite (Wielicki, Barkstrom et al. 1996; Loeb, Wielicki et al. 2009), or equivalently a top-of-atmosphere, all-sky albedo of 0.293. Decomposition of this overall albedo into its clear-sky component (0.153) yields a ground * dry atmosphere absorption fraction of 0.847.

$$0.847 = \widetilde{\mathrm{d}_{2002}} * f_{\alpha S}(\mathrm{T}_{2002}) = 0.932 \ * f_{\alpha S}(\mathrm{T}_{2002}), \quad \text{thus} \qquad f_{\alpha S}(\mathrm{T}_{2002}) = 0.909 \quad \text{(SA4)}$$

$$0.707 = \widetilde{\mathrm{d}_{2002}} * f_{\alpha A}(\mathrm{T}_{2002}, 2002) * f_{\alpha S}(\mathrm{T}_{2002}) = 0.847 * f_{\alpha A}(\mathrm{T}_{2002}, 2002),$$

$$\text{thus}, \quad f_{\alpha A}(\mathrm{T}_{2002}, 2002) = 0.834 \qquad \text{(SA5)}$$

Verifying the reflected energies:

$$\mathcal{F}_{\mathrm{SW}\,clearsky}^{\,refl\ by\ gnd} = \frac{\overline{G_{SC}}}{4} * \tilde{\mathrm{d}}(2002) * \left(1\text{-}f_{\alpha S}(\mathrm{T}_{2002})\right)$$

$$= 340.2\,\frac{W}{m^2} * 0.932 * 0.091 = 28.8\,\frac{W}{m^2} \qquad \text{(SA6)}$$

$$\mathcal{F}_{\mathrm{SW}\,allsky}^{\,refl\ by\ gnd} = \frac{\overline{G_{SC}}}{4} * \tilde{\mathrm{d}}(2002) * f_{\alpha A}(\mathrm{T}_{2002}, 2002) * \left(1 - f_{\alpha S}(\mathrm{T}_{2002})\right) = 24.1\,\frac{W}{m^2} \ \text{(SA7)}$$

$$\mathcal{F}_{\mathrm{SW}\,allsky}^{\,refl\ by\ clouds} = \frac{\overline{G_{SC}}}{4} * \tilde{\mathrm{d}}(2002) * \left(1 - f_{\alpha A}(\mathrm{T}_{2002}, 2002)\right) = 52.6\,\frac{W}{m^2} \ \text{(SA8)}$$

There is a slight discrepancy in the clear-sky ground-reflected energy relative to the literature value (33 [31-34] W/m^2), but the all-sky reflected energies are much more closely aligned: the ground reported value is 25 [23-26] W/m^2, and the dry atmosphere + cloud reported value is 75 [71-77] W/m^2, compared to this inferred value of 52.6 + 24.1 = 76.7 W/m^2 (Wild, Folini et al. 2015). Note that this shortwave flux equation does not consider shortwave energy absorbed into the atmosphere, a substantial simplification.

$\mathrm{j}^{\star}(T_t) = \sigma_{\mathrm{sf}} T_t^4$ is the ideal black body radiation or Planck feedback, which derives from quantum mechanics, particularly the Stefan-Boltzmann law (Boltzmann 1884), which gives the Stefan-Boltzman constant $\sigma_{\mathrm{sf}} = 5.670\ 10^{-8}\mathrm{Wm^{-2}K^{-4}}$ as a coefficient. For the Earth, because the temperature is in the neighborhood of 287K, this black body radiation is primarily in the infrared spectrum, between 200 and 1200 $\mathrm{cm^{-1}}$ (Zhong and Haigh 2013).

$\tilde{\mathrm{g}}(\boldsymbol{t})$ is the prescribed longwave attenuation due to CO_2 and other anthropogenic greenhouse gases (CH_4, NO_2, O_3, halogens), which is half of the fraction of radiative energy absorbed by those gases (because half is re-emitted upwards and half downwards). This absorbed, downwards-emitted fraction increases linearly by a factor of β_0 with respect to the logarithm of the CO_2 concentration measured in ppm (see Figure 6b of (Zhong and Haigh 2013)). CO_2 concentrations were taken as the historical concentrations used in the NASA GISS climate model 1850-1979 (https://data.giss.nasa.gov/modelforce/ghgases/Fig1A.ext.txt) and the NOAA global averages from 1980-2021 (https://gml.noaa.gov/webdata/ccgg/trends/co2/co2_annmean_gl.txt).

$$\phi_{\mathrm{LW}}(\mathrm{out}) = \mathrm{j}^{\star}(T_t) - \frac{\phi_{\mathrm{LW}}(\mathrm{absorbed})}{2} = \mathrm{j}^{\star}(T_t) * \tilde{g}(t) \; * f_{H2O}(T_t) \tag{SA9}$$

$$\tilde{g}(t) * f_{H2O}(T_t) = \left(1- \frac{\phi_{\mathrm{LW}}(\mathrm{CO2\ absorb})}{2\,\mathrm{j}^{\star}(\mathrm{T_t})}\right) * \left(1- \frac{\phi_{\mathrm{LW}}(\mathrm{H2O\ absorb})}{2\,\mathrm{j}^{\star}(\mathrm{T_n})}\right) \approx \left(1- \frac{\phi_{\mathrm{LW}}(\mathrm{CO2\ absorb}) + \phi_{\mathrm{LW}}(\mathrm{H2O\ absorb})}{2\,\mathrm{j}^{\star}(\mathrm{T_t})}\right) \tag{SA10}$$

$$\tilde{g}(t) = \; 1\text{-}\beta_0 \log_{10}([\mathrm{eCO_2}]_{\mathrm{t}}) < 1 \tag{SA11}$$

Equation SA9 refers to a single-layer atmosphere assumed by prior researchers such as Kravitz, Rasch, et. al. (2018). While the technically correct separation of SA9 is shown on the right hand side of SA10, the form for the product of $\tilde{g}(t) * f_{H2O}(T_t)$ was chosen specifically to resemble the previous shortwave energy expressions, essentially representing CO_2 in an atmospheric layer above H_2O (sequential filtering in the middle expression of SA10). Relating these two representations demands the simplification that both the longwave radiative fluxes absorbed by CO_2 and H_2O are each smaller than twice the total ground-emitted longwave radiative flux, so their product is yet smaller and can be neglected. Indeed, for CO_2 this ratio $\frac{\phi_{\mathrm{LW}}(\mathrm{CO2\ absorb})}{2\,\mathrm{j}^{\star}(\mathrm{T_t})} = \beta_0 \log_{10}([\mathrm{CO_2}]_{\mathrm{t}})$ is in the range [0.165 - 0.176] and for H_2O the analogous ratio is in the range [0.250 - 0.259] so their product (the difference between the RHS and LHS of A12) is at most 0.045. This difference in energy flux would be large enough to cause significant inaccuracies in the energy balance model (larger than the anthropogenic global warming signal), should parameters from a single-layer atmosphere be used in a sequential filter model. Thus, the critical parameters β_0 and β_1 must be calculated within the framework of the chosen model (here a sequential filter – see below), after which this distinction only matters to the higher-order terms of the deviations from the preindustrial energy flux (0.176-0.165) * (0.259-0.250) ≈ 0.0001, a negligible fraction.

More complex functions for $\tilde{g}(t)$ exist involving functions for each individual greenhouse gas (Meinshausen, Nicholls et al. 2020) but for the purposes of simplifying this energy balance model, only one “effective greenhouse” concentration is used. Our “effective greenhouse gas concentration" includes CH_4, N_2O, O_3, contrails, stratospheric water vapor, land use, and black carbon on snow but excluding anthropogenic atmospheric aerosols (Forster, Smith et al. 2023). Formally, land use and black carbon on snow should be included as a prescribed change to the $f_{\alpha S}$ function on the shortwave side but in combination these two amount to within -0.15 W/m^2, less in absolute value than all the other aforementioned "combined greenhouse forcing" components aside from contrails and stratospheric water vapor. Similarly, the prescribed contribution of stratospheric water vapor should formally be within the $f_{H2O}(T_t)$ function not lumped with the other greenhouse gases, but as this represents only 0.05 W/m^2 at most, this is inconsequential (variations in incoming solar insolation are of a similar magnitude). We determined the "effective CO2 concentration" by first fitting a function relating CO2 concentrations reported above to the CO2 radiative forcings reported by Forster (2023) at https://github.com/ClimateIndicator/forcing-timeseries/tree/main/output.

$$\phi_{LW}^{CO2} = 12.74\ \log_{10}([eCO_2]_t) - 31.55 \qquad \text{(SA12)}$$

Then by summing all "effective greenhouse gas" reported energy fluxes, the above function was inverted to determine the "effective CO2 concentration." These ranged from 278 ppm (or $\log_{10}([eCO_2]) = 2.444$ when there was no "effective greenhouse gas" energy flux to 558.7ppm or $\log_{10}([eCO_2]) = 2.747$ in 2022, the last date of this timeseries. Within this timeseries, the datapoint corresponding to the year 2023 was not yet published at the time of this study's publication, but was inferred from a linear projection of the ratio between Mona Loa CO_2 concentrations since 2000 (https://gml.noaa.gov/webdata/ccgg/trends/co2/co2_annmean_mlo.txt) and recent eCO2 concentrations (563.4 ppm = $[eCO2]_{2023} \approx 1.34 * [MLo\ CO_2]_{2023}$).

$f_{H2O}(T_t)$ is the additional atmospheric longwave attenuation due to water vapor and other gasses, including both lapse rate and relative humidity. The precise functional form of this feedback function is unknown, as is the functional form of the two shortwave feedbacks, partially due to disagreements between paleoclimate inferences and ESMs. We thus introduced the following 3 functions, which incorporate an additional 3 positive β coefficients and 1 exponent η. (Note $f_{H2O}(T_t)$ can be either linearized into a form like these other feedbacks or rewritten in the $(1\text{-}\ \frac{\phi_{LW}(\text{H2O absorb})}{2\, j^{\star}(T_t)})$ form.)

$$f_{H2O}(T_t) \doteq \beta_1 (1/T_t)^{\eta} \approx 1-\left(1+\beta_1 (T_{2002})^{-\eta} - \beta_1 \eta (T_{2002})^{-\eta-1}(T_t - T_{2002})\right) \quad \text{(SA13)}$$

$$f_{\alpha A}(T_t) \doteq 0.834 \left(1+\beta_2 (T_t - T_{2002})\right) + \frac{AC_n - AC_{2002}}{\frac{G_{SC}}{4}\widetilde{d_{2002}}} \quad \text{(SA14)}$$

$$f_{\alpha S}(T_t) \doteq 0.909 \left(1+\beta_3 (T_t - T_{2002})\right) \quad \text{(SA15)}$$

Finally returning to the heat flux between the surface and the deeper layer of the ocean, other researchers have modeled this $Q_{surf\text{-}deep}$ as a simple thermal conductivity γ multiplied by the difference in deviation temperatures between the surface ($\Delta T_t - \Delta \theta_t$), with these deviations measured relative to the pre-industrial equilibrium.

$$Q_{surf-deep} = \gamma * (\Delta T_t - \Delta \theta_t) = \gamma * (\mathrm{T_t} - \theta_t - \mathrm{T}_{1850} + \theta_{1850}) \quad \text{(SA16)}$$

If we take T_{1850} = 286.66K = 13.51°C and θ_{1850} = 276.66K = 3.51°C, then $\zeta_0 = 10\text{K}$. This consistent equilibrium temperature difference exists because the ocean is temperature stratified. We used γ from the CMIP5 reported by Geoffroy et al. Part II (2013) to be 0.67±0.15 W/m^2/K. Estimates of γ from the CMIP6 coupled model comparison project were almost unchanged, 0.64±0.14 W/m^2/K (Hall and Fox-Kemper 2023). The deep ocean heat content record was extended back from 1850-1869 by prepending zero values. Since this is an equilibrium value, the deviation from the equilibrium deep ocean temperature θ_{1850} = 276.66K is given by the deviation from this baseline heat content.

The ocean heat content anomaly is obtained from Zanna (Zanna, Khatiwala et al. 2019) from 1870-2018. Before 1870, the OHCA was set to 0, with a standard deviation taken to be the 1870-1889 average: 50.2 ZJ. After 2018, the standard deviation was continued as the 2009-2018 average of 25.2ZJ. The additional increase in OHCA after 2018 was provided from a separate NCEI dataset (Levitus, Antonov et al. 2017). This NCEI dataset disagrees with the Zanna, Khatiwala et al. (2019) dataset regarding the change in OHCA from 2005-2018 by a factor of 1.71. NCEI reports 134.2ZJ compared to Zanna (Zanna, Khatiwala et al. 2019) reporting 78.5ZJ. However, the NCEI dataset is more directly derived from observations, especially the Argo array of autonomous floats, and thus is preferred when that array has been fully available.

A2: Solving for unknown β coefficients:

Following the definition of climate feedback of w as ∂*N*/∂w * dw/dT, where *N* is the TOA radiative flux (the entire EBM model), we equated the climate feedbacks of each of the three $f_?$ feedback functions and the Planck response $j^\star$, with the values (in $W/m^2/K$) reported in Table 7.10 and Figure 7.10 of AR6 (Forster, Storelvmo et al. 2021).

$$\frac{\partial N}{\partial j^\star} * \frac{d j^\star}{dT_t} = -\tilde{g}(t) * f_{H2O}(T_t) * 4\sigma_{sf}(T_t)^3 = -3.22 \quad \text{(SA17)}$$

$$\frac{\partial N}{\partial f_{H2O}(T_t)} * \frac{d f_{H2O}(t)}{dT_t} = -j^\star(T_t) * \tilde{g}(t) * -\beta_1 p_1(T_t)^{-\eta-1} = 1.30 \quad \text{(SA18)}$$

$$\frac{\partial N}{\partial f_{\alpha A}(T_t)} * \frac{d f_{\alpha A}(T_t)}{dT_t} = 340.2 * \tilde{d}(t) * f_{\alpha S}(T_t) * 0.834\,\beta_2 = 0.35 \quad \text{(SA19)}$$

$$\frac{\partial N}{\partial f_{\alpha S}(T_t)} * \frac{d f_{\alpha S}(T_t)}{dT_t} = 340.2 * \tilde{d}(t) * f_{\alpha A}(T_t) * 0.909\,\beta_3 = 0.42 \quad \text{(SA20)}$$

Solving for the exponent by taking the ratio of the first two equations yielded η =1.615. Furthermore, based on the CERES measurements from 2000-2005, everything to the left of both β_2 (SA19) and β_3 (SA20) is the overall absorbed SW irradiance of 340.2*0.707=240.5 W/m^2, so $\beta_2 = 0.00136\ K^{-1}$ and $\beta_3 = 0.00163\ K^{-1}$.

Figure 3.3 from Zhong and Haigh (2013) shows that per log10 order of magnitude of [CO2] increase, an additional 15.45 W/m^2 is absorbed. However, in Forster (2023), the "greenhouse gas" absorption increases by 12.74 W/m^2 per log10 order of magnitude of effective [CO2] increase (eq. SA12). This measurement approximating a partial derivative was presumably made recently, so we used the more recent 2002 temperature of ~287.5K (14.4°C), but this choice is relatively inconsequential: $\beta_0\beta_1$ would be only 0.66% larger if the pre-industrial temperature was used instead. In the pre-industrial climate, we assumed a steady-state equilibrium with a constant black body temperature of 286.66K (13.6°C) and a log10([effective CO2]) ≈ 2.444. This allows us to solve for β_0 and β_1 as follows:

$$12.74 = \frac{\partial N}{\partial \tilde{g}_n} * \frac{d\,\tilde{g}_n}{d\ \log_{10}([eCO_2]_n)} = -\sigma_{sf}(T_n)^4 \beta_1 (T_n)^{-1.61}\left(-\beta_0\right) \quad \text{(SA21)}$$

$$307.24 = \beta_1\beta_0 \quad \text{using} \quad T_{2002} = 287.5 \quad \text{(SA22)}$$

$$0 = 340.2 * \tilde{d}_n * f_{\alpha A}(T_{1850}) * f_{\alpha S}(T_{1850}) - \sigma_{sf}(T_{1850})^4 \beta_1 (T_{1850})^{-1.61}\left(1-\beta_0(2.444)\right) \quad \text{(SA23)}$$

$$240.56 = \sigma_{sf}(286.7)^{2.39}\left(\beta_1\right)\left(1-\beta_0(2.444)\right) \quad \text{(SA24)}$$

$$5842.68 = \left(\beta_1\right)\left(1-\beta_0(2.4)\right) \quad \text{(SA25)}$$

$$6593.57 \approx \beta_1 \quad \text{and} \quad 0.04660 \approx \beta_0 \quad \text{(SA26)}$$

Checking that Planck partial derivative is accurate, we obtained a value for climate sensitivity of $j^\star$ to be -3.34 W/m^2/K at current conditions and the sensitivity of f_{H2O} to be 1.35 W/m^2/K, within the likely range of AR6. With an instantaneous doubling or quadrupling of CO_2 the sensitivity of $j^\star$ becomes -3.30 W/m^2/K or -3.22 W/m^2/K respectively, matching the reported value. Because they were defined to have proportional climate sensitivities, f_{H2O} exactly matches AR6 in a 4xCO_2 scenario, with 1.30 W/m^2/K.

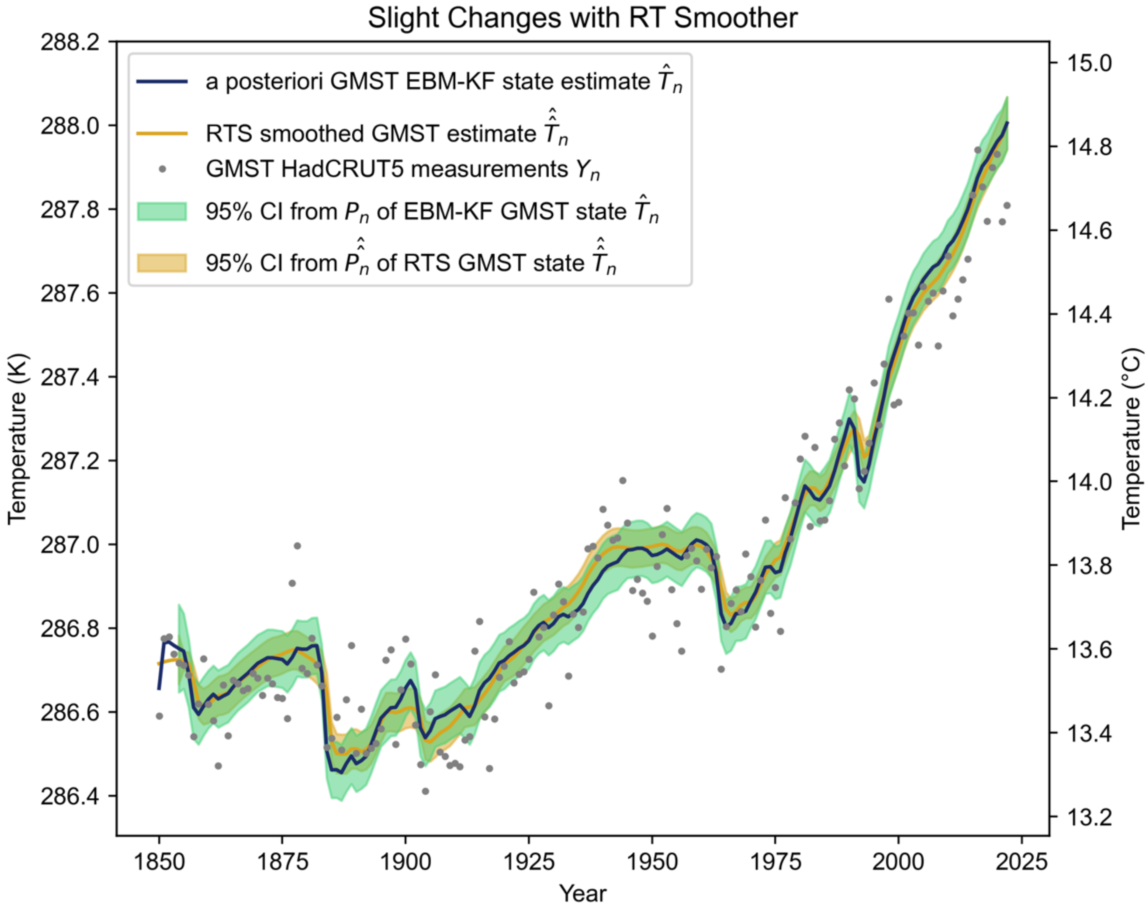
Section A3: RTS Smoother

$$\hat{\hat{K}}_t = \mathrm{P}_t \Phi_t (\mathrm{P}_{t|t\text{-}1})^{-1} \quad \text{back-updated Kalman gain} \quad \text{(SA26)}$$

$$\hat{\hat{x}}_t = \hat{x}_t + \hat{\hat{K}}_t \left(\hat{\hat{\mathrm{x}}}_{\mathrm{t}} \text{-} \mathbf{F}(\hat{x}_t; u_{t+1}) \right) \quad \text{back-updated state estimate} \quad \text{(SA27)}$$

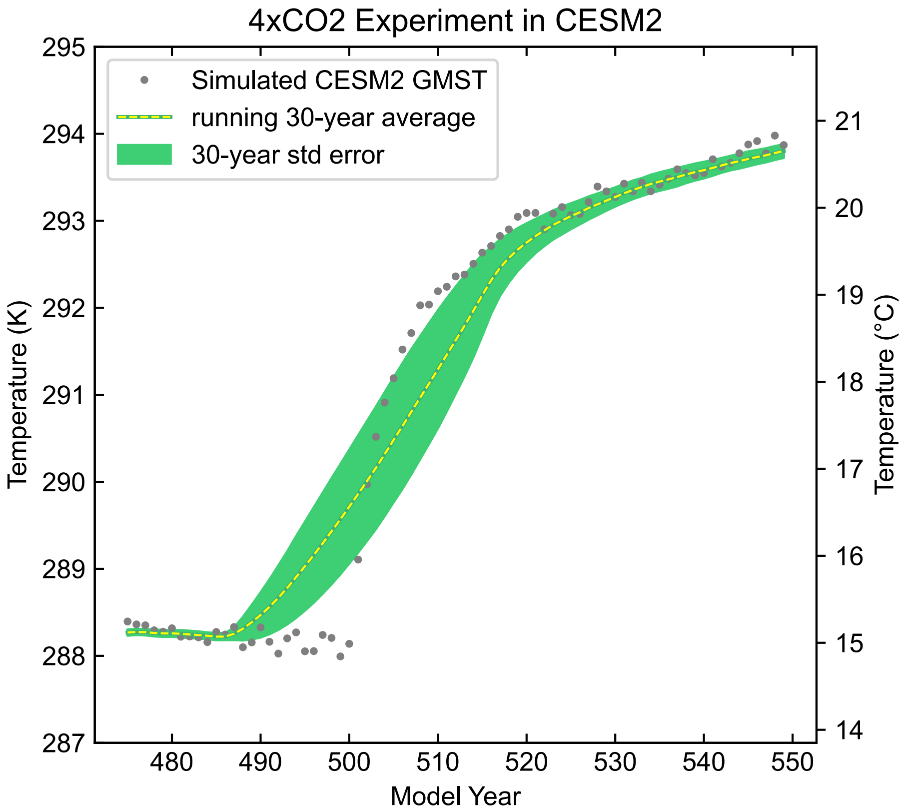
$$\hat{\hat{P}}_t = \mathrm{P}_t + \hat{\hat{K}}_t (\hat{\hat{P}}_{t+1} \text{-} \mathrm{P}_{t|t\text{-}1}) \hat{\hat{K}}_t^{\mathrm{T}} \quad \text{back-updated state covariance} \quad \text{(SA28)}$$

This RTS has a theoretical advantage of blending abrupt changes in the model state over greater time periods, while also slightly reducing the state covariance. For instance, if the measurements suddenly and persistently diverged from the blind, forward EBM (unrelated to a known volcanic eruption), an EBM-Kalman Filter model state would only react as these measurements diverge, whereas an EBM-RTS would slightly foreshadow this jump because it can see future as well as past measurements. This occurred in 1900: even though the EBM-KF estimated state is trending up, the EBM-RTS state moves cooler to reflect the colder GMST measurements from 1902-1907, colder than the EBM predicted from the Santa Marina volcanic eruption alone (see Supp. Fig. 2). Generally, the EBM-RTS just provides a second "nudge" toward measurements. However, for the purposes of this paper, these distinctions make little difference between $\hat{\hat{\boldsymbol{x}}}_{\boldsymbol{t}}$ and $\hat{\boldsymbol{x}}_{\mathrm{t}}$, as is demonstrated in Supp. Fig. 1 below.

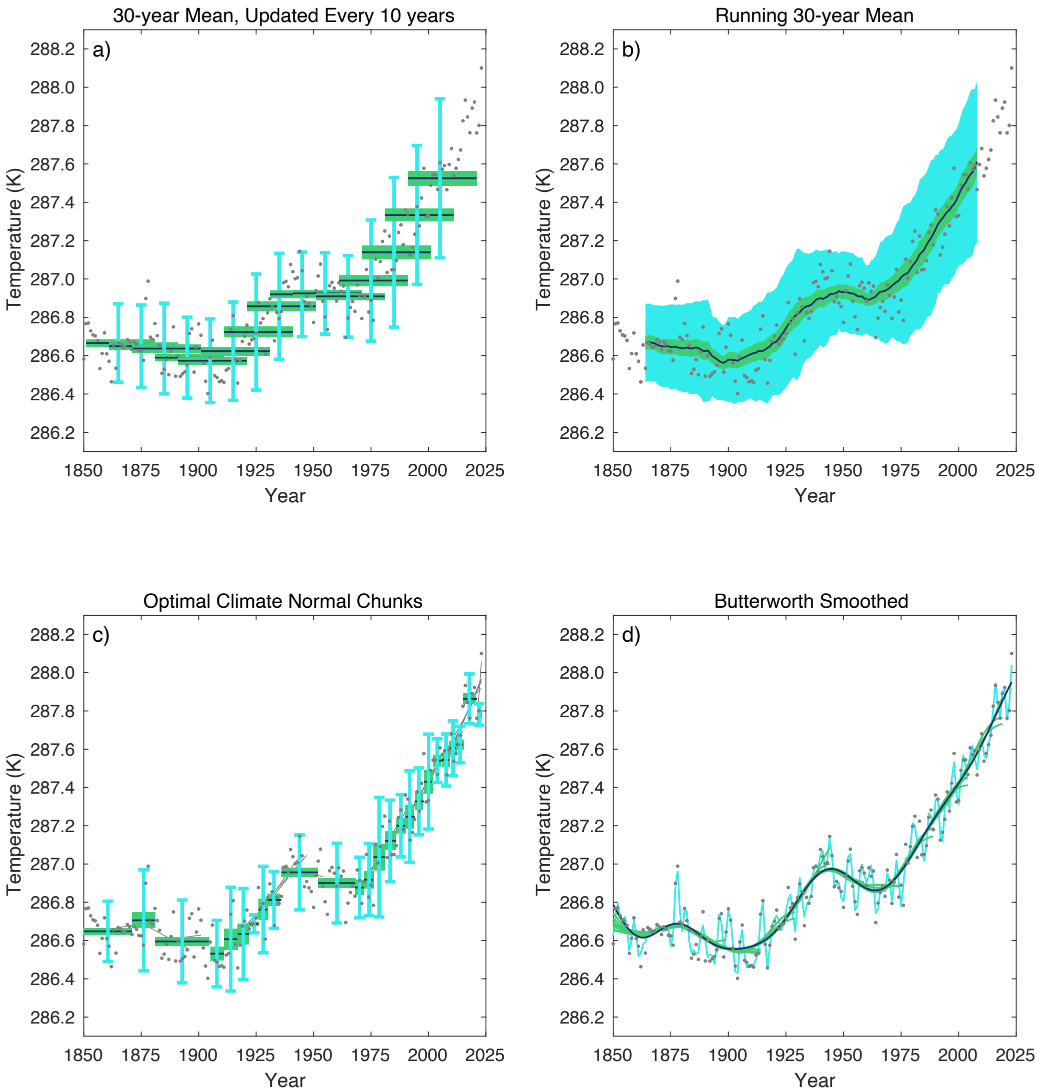


Supp. Fig. 2: Comparisons of the original EBM-Kalman Filtered climate state (navy blue line with green 95% uncertainty window) with an EBM-RTS climate state (orange line with orange 95% uncertainty window). Note that the temperatures on y-axis are zoomed in relative to all other figures to demonstrate these minute differences. From 1905-1930 and 2000-2020 when there are repeated cooler GMST temperature measurements than the EMB-KF state prediction, the EBM-RTS climate state doubly takes these annual temperature measurements into account, so it has a greater cooling deflection in these periods. Other years are warmer in the EBM-RTS than the EBM-KF climate state, although even these differences are slight - at most 0.1K during years of volcanic activity. However, there is greater certainty in the state estimate with the EBM-RTS: $\hat{\hat{P}}_t$ shrinks relative to P_t (see Supp. Fig. 10) by factors of 2.25 and 2.84 for the GMST ($\hat{p}_t^T$) and OHCA ($\hat{p}_t^H$) components respectively (everywhere except at the start and tail end of the timeseries). The off-diagonal heat-transfer uncertainty component of $\hat{\hat{P}}_t$ is negative and 29 times smaller than those of P_t.

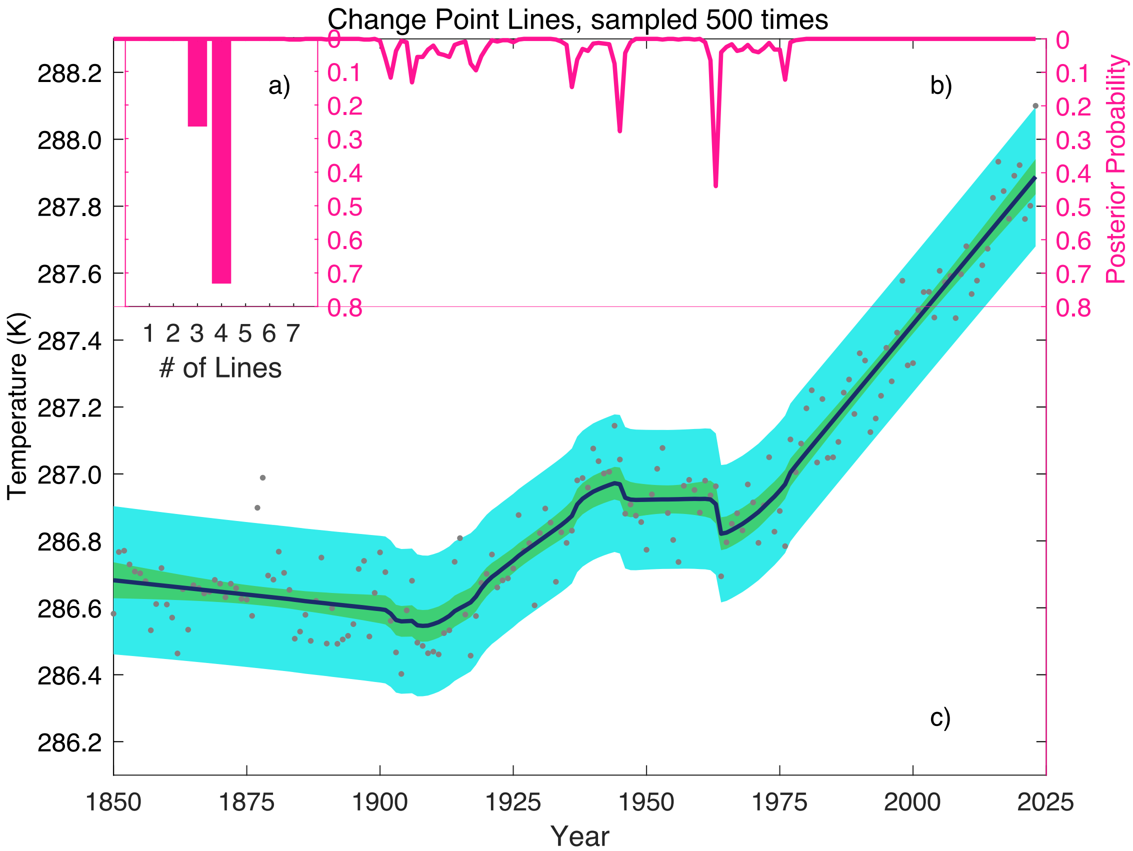
236 ***Section B: Alternative Definitions of the Climate State***



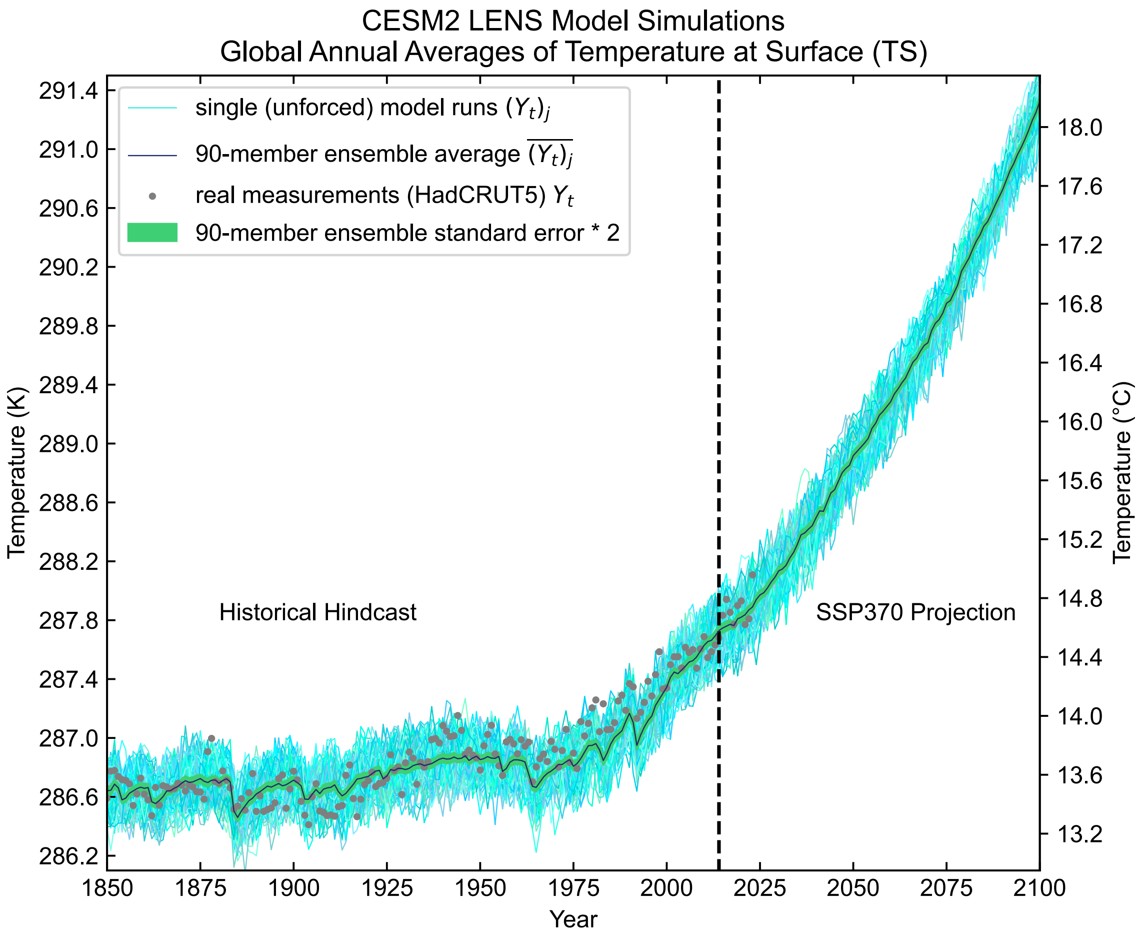
237
238 Supp. Fig. 3: In this modeling experiment conducted within CESM2, the CO2 concentration
239 was instantaneously quadrupled at year 500. The resulting modeled GMST values are plotted
240 in grey, along with their 30-year running mean (yellow dashed), and the standard error of this
241 mean (green window). The 30-year running average anticipates the jump for 15 years before
242 CO2 even began to increase, so that the 30-year average "climate" is several °C away from
243 the simulation year 500 temperatures. Then, it fails to increase at the appropriate rate, such
244 that a period of 6 years (505-511) is hotter than the 30-year running average's 95%
245 confidence interval. Only by simulation year 520 does the 30-year running average appear
246 visually to catch up and visually correspond with the simulated temperatures.



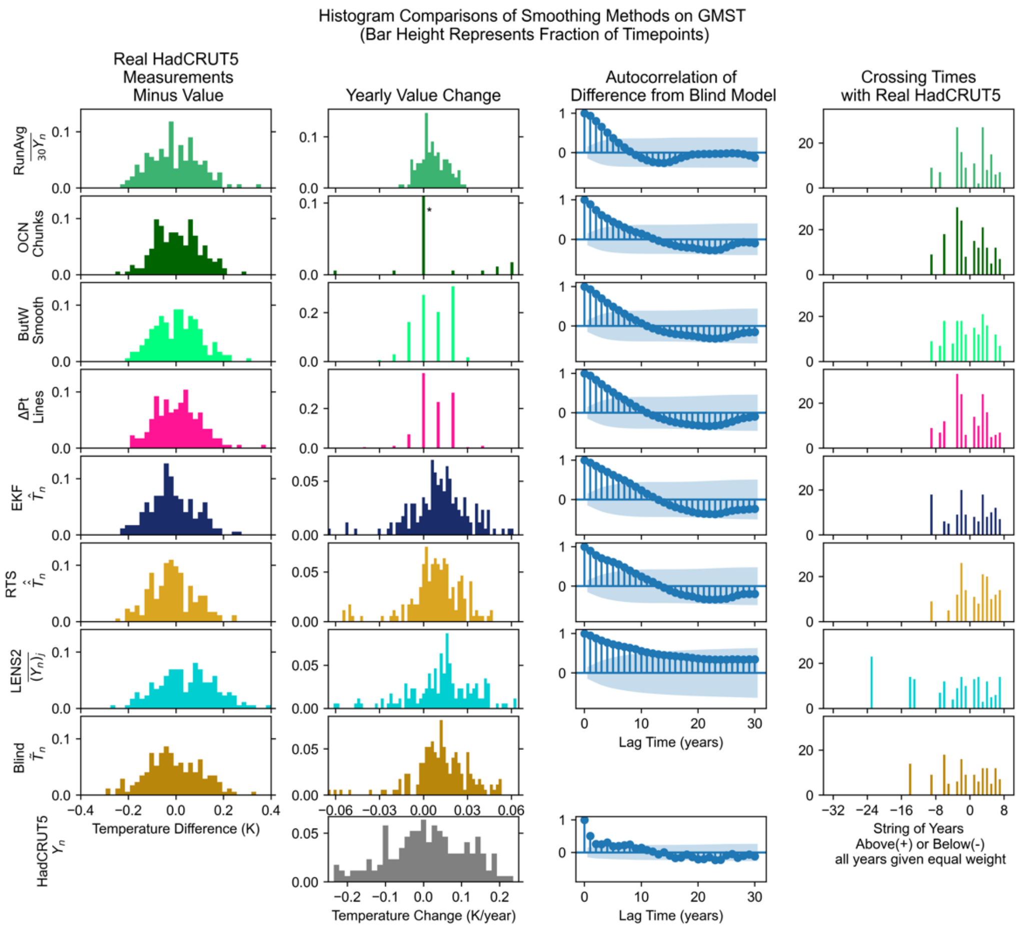
Supp. Fig. 4: Comparison of Prior Methods for Filtering or Smoothing the Climate as applied to the HadCRUT5 temperature dataset. (Morice, Kennedy et al. 2021) All metrics analogous to standard deviation are plotted at the 2σ level in light blue, and all metrics analogous to the standard error are plotted at the 1σ level in light green. a) The 30-year climate normals, updated every 10 years as per the World Meteorological Association in 1935. b) A running 30-year average. c) Adaptive periods of multiyear averages, known as the optimal climate normal (OCN). (Livezey, Vinnikov et al. 2007). Chunks became smaller as the rate of climate change increased in recent decades. d) The Butterworth Smoother applied to this temperature dataset. (Mann 2008) For the “standard error” highly smoothed lines, the lowpass adaptive, lowpass mean padded, and lowpass methods were applied to chunks of the timeseries data ranging from 50 to 170 years in increments of 15 years with a cutoff frequency of 1/30years. The black “best” line a lowpass adaptive curve extended to 2021. The blue “standard deviation” line is a lowpass mean padded filter with a cutoff frequency of 1/5years.



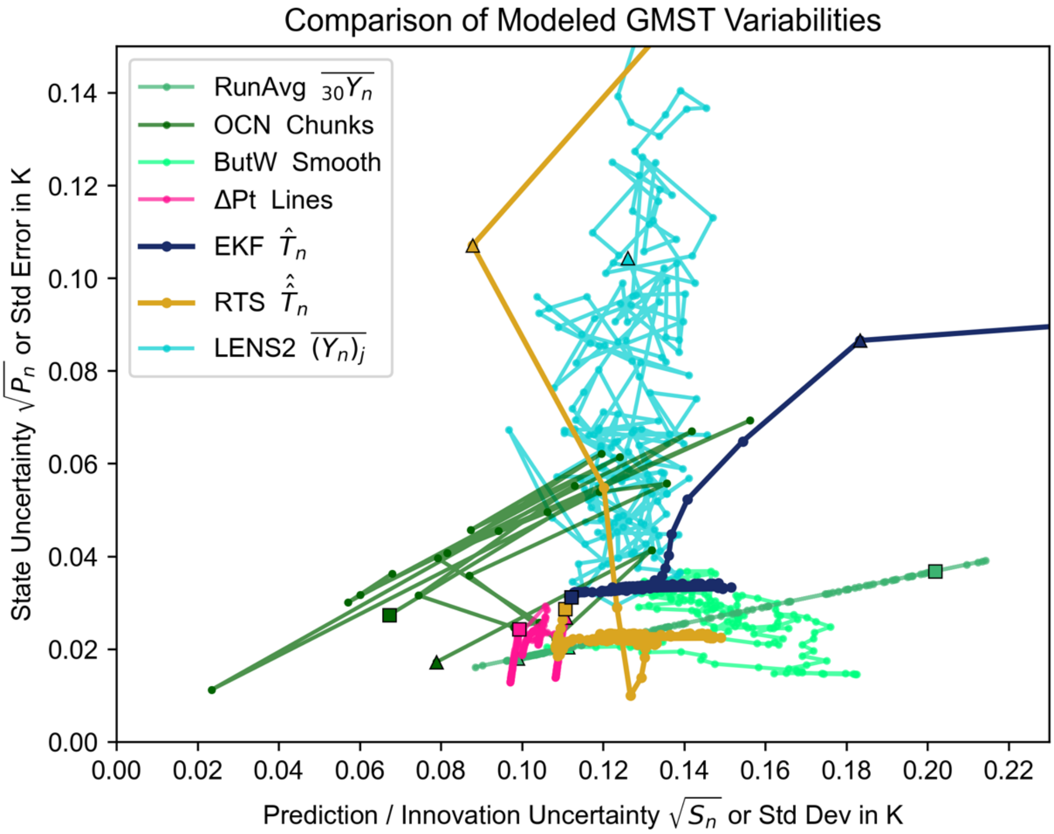
Supp. Fig. 5: Utilization of Bayesian Change Point on the HadCRUT5 data. (Ruggieri and Antonellis 2016) a) There are likely 4 trendlines with 72% of the posterior probability, and the remaining posterior probability on 3 trendlines. b) The posterior probability plot of where trendlines are most likely to occur: 51.2% of all samplings have a change point occur in 1963, and 26.4% of samplings have a change point occur in 1945. c) The posterior distribution of the trendlines in GMST, again with blue shading to indicate 2σ confidence interval of the data and green shading to indicate 2σ confidence interval of the mean trendline. These trend lines do not have to be continuous (note the dip at 1963), but over many samplings the average trend is smoothed.



273
274 Supp. Fig. 6: Comparison of the CESM2 Large Ensemble (LENS2) GSAT (Rodgers, Lee et
275 al. 2021) with HadCRUT5 GMST measurements. The various shades of thin light blue and
276 turquoise lines represent each individual simulation $(Y_n)_j$ of the 90-member ensemble. The
277 ensemble mean is plotted in a navy-blue line, and the ensemble mean standard error is plotted
278 around this line in green. This standard error is twice the standard deviation divided by the
279 square root of the number of ensemble members at that moment and shows the 2σ uncertainty
280 in the yearly simulated climate is roughly 0.026K. The ensemble mean has $r^2 = 0.83$ relative
281 to the HadCRUT5 measurements, lower than for the blind EBM (r^2=0.88). The dashed
282 vertical line represents when LENS transitions from historical to future forcing (SSP3-7.0).
283



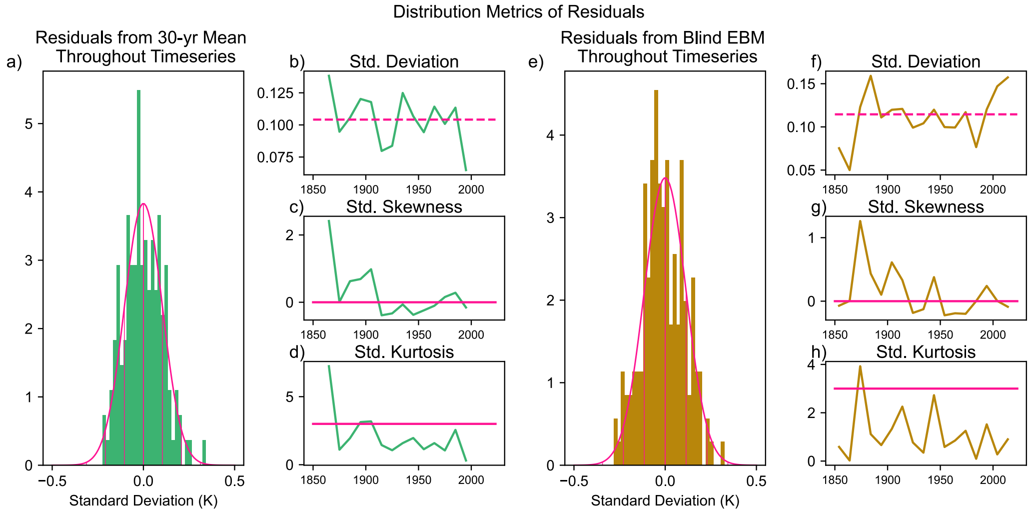
Supp. Fig. 7: Histogram comparisons of several aspects of many of the smoothing methods for generating a climate timeseries. The far-left column represents the absolute differences between the HadCRUT5 measurements and all the other models. All look similar in this respect. The center-left column shows the annual changes in the temperatures reported by each model. In this respect, the real HadCRUT5 measurements are the most spread out, because the stochastic change each year is large, whereas in most years the OCN Chunks do not change. The center-right column shows an autocorrelation plot, which demonstrates that every other model aside from HadCRUT5 (and to a lesser extent the running average) are autocorrelated with the blind energy-balance model to similar degrees. The far-right column shows how many continuous years are spent above or below HadCRUT5: both the LENS2 ensemble average and the blind energy-balance model had >20 year spans for which they were colder than the “real” HadCRUT5 data, illustrating the benefit of data assimilation.



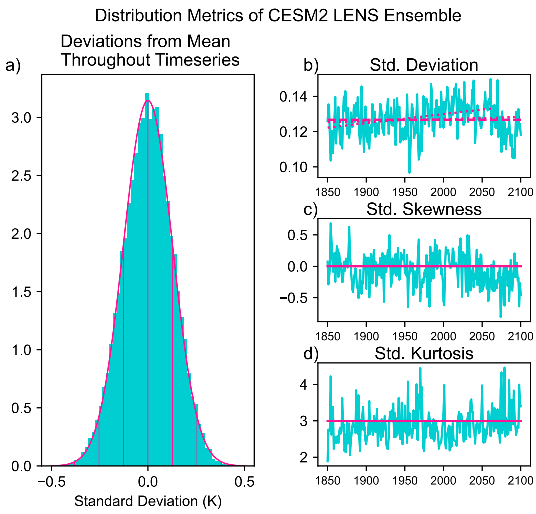
Supp. Fig. 8: Comparisons of the state and prediction (or equivalent) uncertainties of the smoothing methods for generating a climate timeseries. The x-axis represents the state uncertainty (colored light green in all other figures), and the y-axis represents the prediction uncertainty (colored light blue and doubled in all other figures). As these quantities change over time, all points in these smoothing timeseries are connected with colored lines, with the triangle △ representing the value of these quantities in 1850 or the first point that they entered the frame limits of this graph, and the square □ representing the value of these quantities in 2021 or the last point that they were within the frame limits. For instance, the running average draws a straight line because standard deviation and standard error are linearly correlated by a favor of 1/√30, and latter points have larger quantities for each variability due to the changing climate. The Butterworth Smoother traces a curve roughly in this region, with both the standard deviations and standard errors being twice the 15-year running average of the maximum of the absolute value of differences between colored and black curves. The EKF and RTS methods rapidly converge to an innovation uncertainty of 0.11-0.15K and state uncertainties of 0.034K and 0.023K respectively. The Change Point Regression variance also fluctuate the same region as the RTS, although change point method's standard error twice drops to 0.014K, and the prediction uncertainty is slightly smaller, 0.10-0.11K. Both the OCN and the LENS2 climates have standard errors that are above the other methods at most times. For LENS2, the standard deviation within the CESM2 ensemble generally remains between 0.11K and 0.14K, whereas the state uncertainty is taken to be the standard deviation of the 20 ensembles comprising CMIP6 in October 2021. (Meehl, Moss et al. 2014) These metrics are unrelated to Figure 10 in the main text. Within CMIP6, the 20 ensembles are closest to agreement in 1939, when the state uncertainty dipped down to only 0.029K between ensemble means, but this uncertainty was much greater at earlier and later time points, reaching 0.183K by 2014.

This manuscript has been submitted for publication to JOURNAL OF CLIMATE (AMS). Please note that this manuscript has undergone two rounds of peer review but has yet to be formally accepted for publication. Subsequent versions of this manuscript may differ slightly in content.

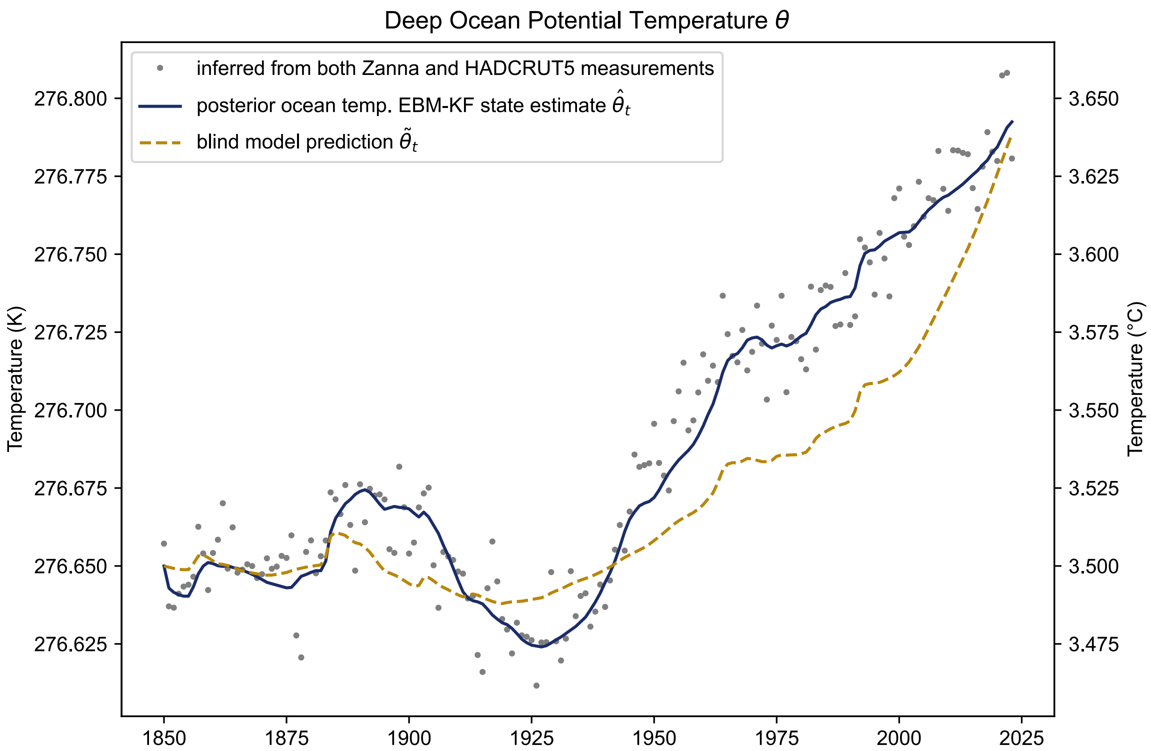
Section C: Miscellaneous Additional Figures



Supp. Fig. 9: Left panels show statistical features of the residuals between the HadCRUT5 measurements with respect to their 30-year running mean, which have a bias of -0.00339K. Pink lines in the histogram in (a) depict an ideal Gaussian distribution with standard deviation of 0.105K, and vertical lines drawn for each of these standard deviations. The dashed pink line (b) indicates the overall standard deviation. Solid pink lines for the skewness = 0.147 (c) and kurtosis = 1.904 (d) indicate the ideal values for a Gaussian distribution. Right panels show statistical features of the differences between the HadCRUT5 measurements with respect to the blind EBM, which have a bias of -0.00104K. Pink lines in the histogram in (e) depict an ideal Gaussian distribution with standard deviation of 0.115K, and vertical lines drawn for each of these standard deviations. The dashed pink line (f) indicates the overall standard deviation. The skewness = 0.123 (g) and kurtosis = 1.208 (h) differ from the ideal values for a Gaussian distribution indicated by solid pink lines.

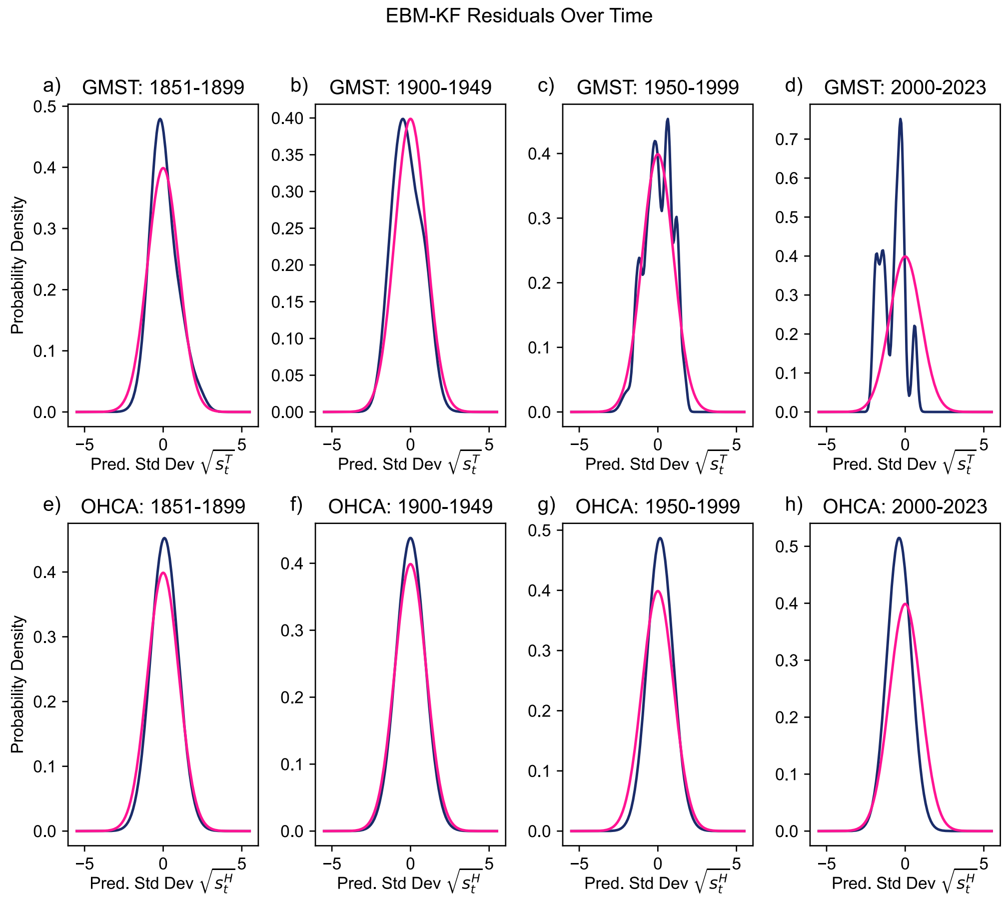


Supp. Fig. 10: Statistical Features of the CESM2 Large Ensemble. (Rodgers, Lee et al. 2021). Pink lines in the histogram in (a) depict an ideal Gaussian distribution with standard deviation of 0.127K, and vertical lines drawn for each of these standard deviations. The observed trend (b) up until 2065 ($p<0.001$) and overall ($p=0.168$) in the standard deviation over time is plotted in a dotted pink, while the dashed line indicates the overall standard deviation of 0.127K. The skewness = -0.069 (c) and kurtosis = 2.87 (d) differ from the ideal values for a Gaussian distribution indicated by solid pink lines.



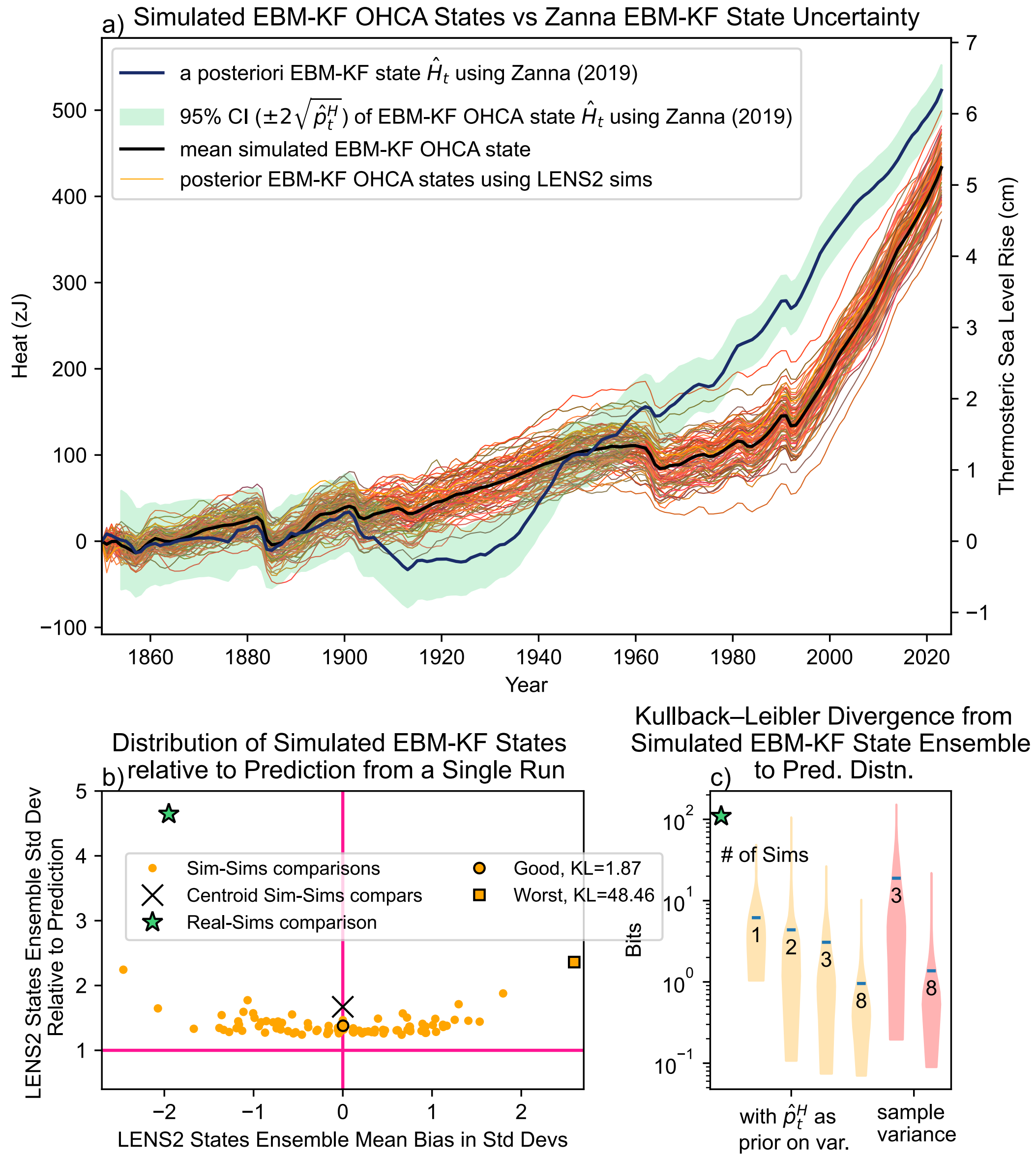
357

358 Supp. Fig. 11: As in Fig. 2, but regarding the deep ocean potential temperature. A comparison
359 of the blind model EBM, the a posteriori EKF state estimate, and the inferred deep ocean
360 potential by combining the Zanna (2019) and HadCRUT5 measurements with the surface and
361 deep ocean heat capacities specified in Section 2a and Appendix A.



Supp. Fig. 12: Deviation between the projected climate state (pink) and empirical PDFs of the Gaussian mixture of measurements with associated uncertainty (purple), plotted relative to the ideal distribution given by the innovation covariance. Each column indicates a different time window of the EMB-KF model's run length. The top row displays the empirical PDFs of the GMST HadCRUT5 measurements relative to the model's estimate of GMST state, whereas the bottom row displays empirical PDFs of the OHCA Zanna 2019 measurements relative to the model's estimate of OHCA state. Note the initial period begins at 1851 (and the 1850 measurement is excluded from main text Fig. 3 and 4) because this has comparison involves P_0, which was intentionally over-estimated (resulting in relatively too-narrow measurement kernel). Also note that the last period is less than half the time of the others, so the GMST empirical distribution is much choppier. The observations from this most recent period 2000-2023 are also shifted slightly colder than the EMB-KF predictions, possibly indicating that some of the parameters could be better tuned than the original literature values.

This manuscript has been submitted for publication to JOURNAL OF CLIMATE (AMS). Please note that this manuscript has undergone two rounds of peer review but has yet to be formally accepted for publication. Subsequent versions of this manuscript may differ slightly in content.



Supp. Fig. 13: As in Fig. 7, but focusing on the OHCA component rather than GMST. a) The EBM-KF *a posteriori* from Zanna (2019) state estimate (thick blue) and its 95% confidence interval (light green), along with EBM-KF state estimates for each individual CESM2 ensemble member (orange lines) and their mean (thick black line). b) The differences between the “real” measurement based Zanna (2019) climate state and all LENS2 climate states, scaled by the state standard deviation and plotted against the ideal normal distribution. This is a particularly ill-fitting distribution because the LENS timeseries of OHCA differ substantially from the Zanna (2019) observation. c) In the quantile-quantile plot, this disagreement is apparent between the “real” measurement based Zanna (2019) climate state and all LENS2 climate states of OHCA. d) Climate states and associated uncertainties arising from each of 90 LENS2 simulations and Zanna (2019) are compared to all other LENS2 climate states, and the bias and standard deviation respect to a particular member, of the resulting empirical distributions are plotted. e) An example of these empirical distributions is graphed, indicated by the point circled in black within the scatterplot. The expected difference across an entire simulation run between $(\widehat{H}_t)_i$ and $\overline{(\widehat{H}_t)_j}$ is $\pm 0.721(\sqrt{p_t^T})_i$ with range (-2.439 - 2.574), or 12.72ZJ with range (-40.47 - 42.85)ZJ.

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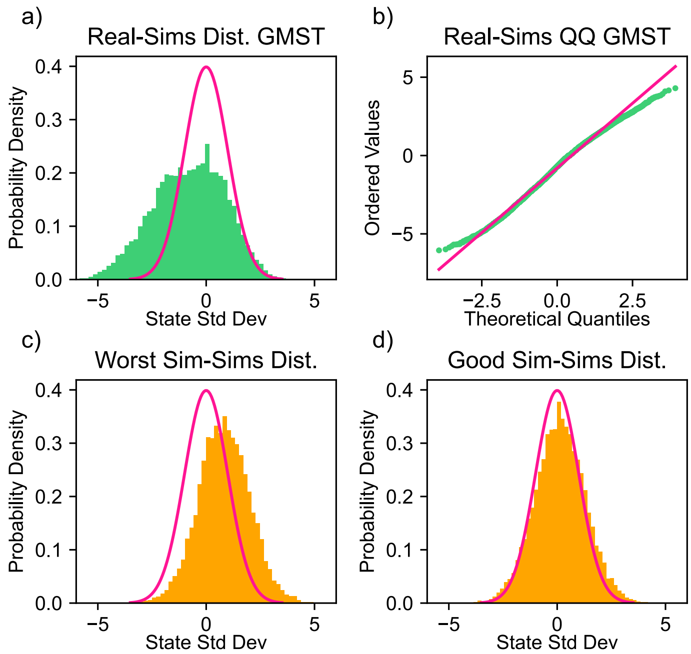
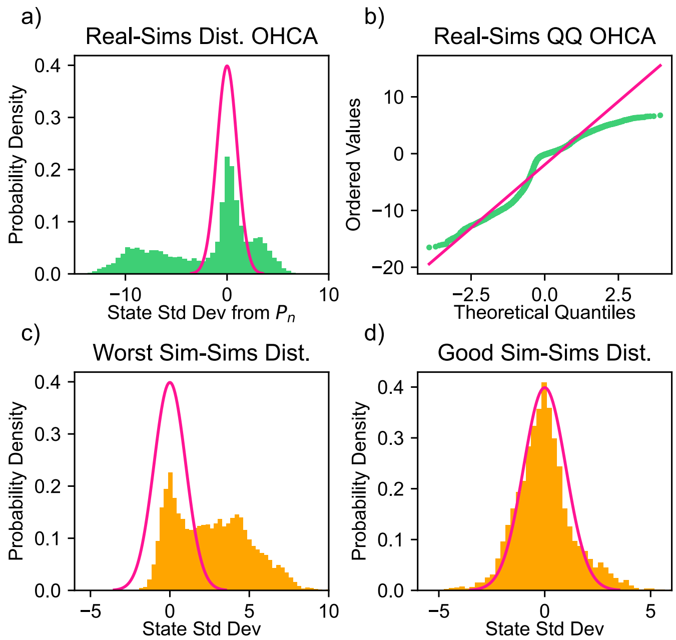


Supp. Fig. 14 (a) The differences between the “real” measurement based HadCRUT5 climate state and all LENS2 climate states, scaled by the state standard deviation and plotted against the ideal normal distribution. b) In the quantile-quantile plot, these differences between the “real” measurement based HadCRUT5 climate state and all LENS2 climate states distributions agree. c) The “worst” (by Kullback-Leibler divergence) correspondence between the predicted GMST ensemble distribution (pink) and the actual LENS2 ensemble (orange), indicated by the point outlined with a square within the Fig. 7b scatterplot. d) An example a “good” (25th percentile by Kullback-Leibler divergence) correspondence between the predicted ensemble distribution (pink) and the actual LENS2 ensemble (orange), indicated by the point circled in black within the Fig. 7b scatterplot.



Supp. Fig. 15: As for Supp. Fig. 14, but regarding OHCA instead of GMST.

Section D: Justification that the EKF is sufficient for nonlinearity, will not diverge

The issue of nonlinearity arises not in the computation of $\hat{\boldsymbol{x}}_{t|t-1}=\mathbf{F}(\hat{\boldsymbol{x}}_{n-1})$ but rather the covariance distribution P_t of points (infinitesimal probability masses) neighboring $\hat{\boldsymbol{x}}_{t-1}$, which are assumed to scale linearly around this transformation to maintain a normal distribution. The OHCA part of the model is linear, producing 2nd-order derivatives which are 0 (SC3, SC10). Nonlinear distortion may pile more probability density onto a state other than the transformed original projection $\mathbf{F}(\hat{\boldsymbol{x}}_{t-1})$, necessitating a new computation of $\hat{\boldsymbol{x}}_{t|t-1}$ as the mean of this distorted PDF. Thus, for an arbitrary point that is z standard deviations away from $\hat{\boldsymbol{x}}_{t-1}$, tracing out an ellipse that is symbolized as $\mathbf{z}\sqrt{\mathbf{P}_t}$ the remainder error R_1 (Lagrange mean-value form) induced in a single cycle is:

$$\mathbf{F}\left(\hat{\boldsymbol{x}}_{t-1}+\mathbf{z}\sqrt{\mathbf{P}_t};u_t\right) - \mathbf{F}\left(\hat{\boldsymbol{x}}_{t-1}\right) - \frac{\partial \mathbf{F}(x;u_t)}{\partial T}\left(\mathbf{z}\sqrt{\mathbf{P}_t}\right)|_T - \frac{\partial \mathbf{F}(x;u_t)}{\partial H}\left(\mathbf{z}\sqrt{\mathbf{P}_t}\right)|_H = R_1\left(\hat{\boldsymbol{x}}_{t-1}+\mathbf{z}\sqrt{\mathbf{P}_t}\right) \tag{SC1}$$

This is a vector equation with two components, T_{t+1} and H_{t+1} Splitting this remainder term into its two components, starting with T_{t+1}:

$$R_{1,T_{t+1}}\left(\hat{\boldsymbol{x}}_{t-1}+\mathbf{z}\sqrt{\mathbf{P}_t};u_t\right)=\frac{\partial^2 F_T(\xi_{T1},\xi_{H1};u_n)}{\partial T\,\partial T}\frac{\left(\xi_{T1}-T_{t-1}\right)^2}{2}$$
$$+\frac{\partial^2 F_T(\xi_{T1},\xi_{H1};u_n)}{\partial T\,\partial H}\left(\xi_{T1}-T_{t-1}\right)\left(\xi_{H1}-H_{t-1}\right)+\frac{\partial^2 F_T(\xi_{T1},\xi_{H1};u_n)}{\partial H\,\partial H}\frac{\left(\xi_{H1}-H_{t-1}\right)^2}{2}$$
$$\text{for}\quad [\xi_{T1},\xi_{H1}] = \hat{\boldsymbol{x}}_{t-1}+\mathbf{z}_{\xi1}\sqrt{\mathbf{P}_t}\qquad \text{, where } 0 \le |z_{\xi1}| \le z \tag{SC2}$$

$$\frac{\partial T_{t+1}}{\partial \boldsymbol{H}_t} = \frac{\gamma}{C_{surf}\,C_{deep}}, so\ \frac{\partial^2 F_T}{\partial T\,\partial H} = \frac{\partial^2 F_T}{\partial H\,\partial H} = 0 \tag{SC3}$$

$$\frac{\partial^2 F_T}{\partial T\,\partial T} = \frac{137.6*2\beta_2\beta_3}{AOD_t+9.73} - \frac{1.39*2.39\,\sigma_{sf}\beta_1}{C_{surf}}(T_t)^{0.39}\left(1-\beta_0\log_{10}([eCO_2]_t)\right) \tag{SC4}$$

$$R_{1,T_{t+1}}\left(\hat{\boldsymbol{x}}_{t-1}+\mathbf{z}\sqrt{\mathbf{P}_t};u_t\right) = \frac{\left(z\sqrt{\hat{p}^T_{t-1}}\right)^2}{2} * \left(\frac{0.00061}{AOD_t+9.73} - 7.26\,E-5\,(T_t)^{0.39}\left(1-\beta_0\log_{10}([eCO_2]_t)\right)\right) \tag{SC5}$$

$$|R_{1,T_{t+1}}\left(\hat{\boldsymbol{x}}_{t-1}+\mathbf{z}\sqrt{\mathbf{P}_t};u_t\right)| \le \frac{z^2\hat{p}^T_{t-1}}{2}\,|6.15\,E-5 - 5.685\,E-4| \le \frac{z^2\hat{p}^T_{t-1}}{2} * 0.0005 \tag{SC6}$$

$$\hat{p}^T_{t-1} \le 0.003 \text{ after t = 1855} \tag{SC7}$$

$$|R_{1,T_{t+1}}\left(\hat{\boldsymbol{x}}_{t-1}+\mathbf{z}\sqrt{\mathbf{P}_t};u_t\right)| \le 10^{-7}\,z^2 * 7.5 \tag{SC8}$$

This means that all probability masses that are within $|z|< 4$ standard deviations regarding the T component will have a one-step error of <0.000012K. Even if the error accumulates in the same direction in each cycle of the EKF, over the 174 year timeseries, the error will be within 0.002K compared to a particle method such as the Unscented Kalman Filter. (Julier and Uhlmann 1997; Wan and Van Der Merwe 2000)

Splitting this remainder term into its second component, H_{t+1}:

$$\mathbf{R}_{\mathbf{1},H_{t+1}}\left(\hat{\boldsymbol{x}}_{\boldsymbol{t-1}}+\mathbf{z}\sqrt{\mathbf{P}_{t}};\boldsymbol{u}_{t}\right)=\frac{\partial^2 F_H(\xi_{T2},\xi_{H1};u_n)}{\partial T\,\partial T}\frac{(\xi_{T1}-T_{t-1})^2}{2}+$$

$$\frac{\partial^2 F_H(\xi_{T1},\xi_{H1};u_n)}{\partial T\,\partial H}\left(\xi_{T1}-T_{t-1}\right)\left(\xi_{H1}-H_{t-1}\right)+\frac{\partial^2 F_H(\xi_{T1},\xi_{H1};u_n)}{\partial H\,\partial H}\frac{(\xi_{H1}-H_{t-1})^2}{2}$$

$$\text{for}\quad [\xi_{T2},\xi_{H2}]=\hat{\boldsymbol{x}}_{\boldsymbol{t-1}}+\mathbf{z}_{\xi 2}\sqrt{\mathbf{P}_{t}}\qquad,\text{ where } 0\le|z_{\xi 2}|\le z \tag{SC9}$$

$$\frac{\partial H_{t+1}}{\partial \boldsymbol{H}_t}=\frac{\boldsymbol{\gamma}}{C_{deep}}*\left(\frac{C_{upper0}}{C_{surf}}-1\right)+1,\ so\ \frac{\partial^2 F_H}{\partial T\,\partial H}=\frac{\partial^2 F_H}{\partial H\,\partial H}=0 \tag{SC10}$$

$$\frac{\partial^2 F_H}{\partial T\,\partial T}=C_{upper0}*\frac{\partial T_{t+1}}{\partial T\partial T} \tag{SC11}$$

$$R_{1,H_{t+1}}\left(\hat{\boldsymbol{x}}_{\boldsymbol{t-1}}+\mathbf{z}\sqrt{\mathbf{P}_{t}};u_t\right)=\frac{\left(z\sqrt{\hat{p}_{t-1}^{T}}\right)^2}{2}*C_{upper0}*R_{1,T_{t+1}}\left(\hat{\boldsymbol{x}}_{\boldsymbol{t-1}}+\mathbf{z}\sqrt{\mathbf{P}_{t}};u_t\right) \tag{SC12}$$

Repeating the logic above, this means that all probability masses that are within $|z|< 4$ standard deviations will have a one-step error of <0.0016 ZJ. Even if the error accumulates in the same direction in each cycle of the EKF, over the 174 year timeseries, the error will be within 0.28ZJ compared to a particle method such as the Unscented Kalman Filter. (Julier and Uhlmann 1997; Wan and Van Der Merwe 2000)

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