An analytical solution to the Navier–Stokes equation for incompressible flow around a solid sphere

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This paper is concerned with obtaining a formulation for the flow past a sphere in a viscous and incompressible fluid, building upon previously obtained well-known solutions that were limited to small Reynolds numbers. Using a method based on a summation of separation of variables, we develop a general analytical solution to the Navier–Stokes equation for the special case of axially symmetric two-dimensional flow around a sphere. For a particular set of mathematical conditions, the solution can be expressed generally as a hypergeometric function. It reproduces streamlines and flow velocities close to a moving sphere, and provides the angular location immediately behind the sphere where there is a separation between laminar flow and a stagnant region. The solution, however, is unable to reproduce eddies around a fast-moving sphere due to the simplification employed in the method of summation of separation of variables. To reproduce these phenomena, through a variable substitution, we present a solution that does not require the separation of variables and is a function of Bessel functions of the first and second kind. For particular boundary conditions, it exhibits eddies behind a fast-moving sphere.

Keywords: Navier–Stokes equation, hypergeometric function, angle of separation, Bessel functions of the first and the second kind

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I. INTRODUCTION

A basic unsolved problem of viscous incompressible flow theory is finding an analytical solution to the Navier–Stokes equation for the steady flow of a viscous fluid past a solid sphere. The problem was first approached by Stokes[1] (1850). Taking the reference frame as the center of the sphere, and assuming a steady solution of form $\psi = (1-\mu^2)f(r)$, where $\mu = \cos(\theta)$, r is the radius, and θ is the angle in axially symmetric spherical coordinates (`figure` 1), Stokes obtained the steady stream function about a sphere moving with constant velocity V_0

$$\psi(r,\mu) = \frac{1}{4}V_0 a^2\Big(\frac{2r^2}{a^2} - \frac{3r}{a} + \frac{a}{r}\Big)(1-\mu^2) \tag{1}$$

Stokes' solution ignores the role of inertial forces in the surrounding flow so the solution applies only to the Stokes flow regime where viscosity dominates the inertial force and the Reynolds number $\mathrm{Re} = \frac{V_0 d_p}{\nu}$ is less than unity, where d_p is the sphere diameter and ν is the kinematic viscosity of the fluid.

Boussinesq[2,3] (1885) and Basset[4] (1888) independently found a solution that allows for the sphere to move with time-varying velocity $V(t)$ starting from rest, although still omitting second-order inertial terms, including those proportional to the squares and products of velocities in the Navier–Stokes equation for flow surrounding the particle, as so was also restricted to small Reynolds numbers, i.e, $\mathrm{Re} < 1$. The Boussinesq-Basset solution for the unsteady stream function around a moving sphere is

$$\begin{aligned}\psi(t,r,\mu) =& \frac{1}{2}V_0 a^2\Big\{\frac{3\nu t}{ra} + \frac{6\sqrt{\nu t/\pi}}{r} + \frac{a}{r}\Big\}(1-\mu^2)\\ &- \frac{3}{\sqrt{\pi}}V_0 a^2(1-\mu^2)\int_{\frac{r-a}{2\sqrt{\nu t}}}^{\infty}\Big\{\frac{2\xi^2\nu t}{ra} + \frac{2\xi\sqrt{\nu t}}{r} + \frac{1}{2}\big(\frac{a}{r} - \frac{r}{a}\big)\Big\}e^{-\xi^2}d\xi\end{aligned} \tag{2}$$

The stream function around the sphere obtained by the Boussinesq-Basset is laminar and its form is identical to that obtained by Stokes, as shown in `figure` 1. The difference is that the stream function is unsteady due to acceleration of the fluid around the sphere. The value of unsteady stream function reduces to the Stokes stream function at the particle surface $r = a$, and in the limit $t \to \infty$ where motion becomes steady.

The Boussinesq-Basset solution requires omission of second-order inertial forces , or the advection terms, because the particle motion is considered to be "slow". Numerous theoretical studies

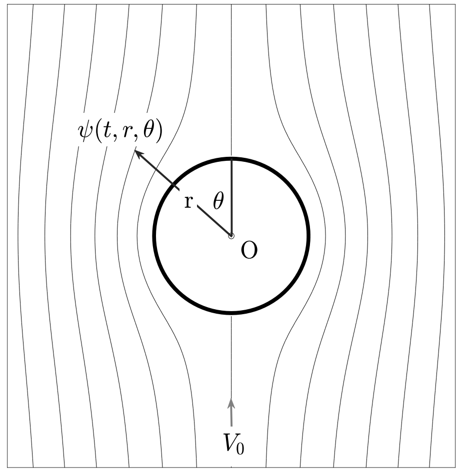

FIG. 1. The laminar stream function relative to the center of a quiescent sphere in spherical coordinates. To reduce the determination of the motion of the fluid about a sphere to a problem of steady motion, Stokes switched reference frames and treated the fluid as moving with velocity V_0 relative to a stationary sphere. Fluid moves uniformly with a small velocity so that the corresponding Reynolds number is less than unity.

have since sought approximate expressions for the Navier–Stokes equation retaining the advection terms at finite Reynolds numbers[5–13]. Proudman and Pearson[8] (1957) and later Michaelides[14] (1997) have provided a comprehensive review. Notably, the perturbation theory given by Oseen[6] (1910) emphasizes the importance of consideration of advection terms far from the sphere. By considering the advection terms where they are comparable with viscous forces, Oseen obtained an approximate solution for the stream function in spherical coordinates as

$$\psi(r,\mu) = \frac{1}{4}V_0 a^2\Big(\frac{2r^2}{a^2}+\frac{a}{r}\Big)(1-\mu^2) - \frac{3}{\mathrm{Re}}V_0 a^2(1+\mu)\Big(1-e^{-\mathrm{Re}\,r(1-\mu)/4a}\Big) \tag{3}$$

Oseen's solution simplifies to the Stokes solution if $\mathrm{Re} \ll a/r$.

Proudman and Pearson argued that Oseen's solution (3) is only a first approximation of a more general form of the stream function $\psi = \sum F_n(\mathrm{Re})\cdot\Psi_n(\mathrm{Re}\,r,\mu)$, so that (3) can be written as $\psi = F_0\Psi_0 + F_1\Psi_1$ where $F_0(\mathrm{Re}) = 1$, and $F_1(\mathrm{Re}) = 1/\mathrm{Re}$. Stokes' solution, on the other hand, is the leading term of a general expansion of the form $\psi = \sum f_n(\mathrm{Re})\cdot\psi_n(r,\mu)$, where ψ is independent of Re, and $f_0(\mathrm{Re}) = 1$. Since Oseen and Stokes expansions are both expansions of the same stream function for small values of Re, therefore, Oseen expansion of $\sum F_n(\mathrm{Re})\cdot\Psi_n(\mathrm{Re}\,r,\mu)$ if expanded about $R \to 0$, must becomes closed to the Stokes expansion $\sum f_n(\mathrm{Re}) \cdot \psi_n(r,\mu)$. Proudman and Pearson determined succeeding terms by substituting these expansions in the Navier–Stokes equation, applying the no-slip condition for the Stokes expansion, and the uniform-stream function condition for the Oseen expansion, and then using a matching procedure so that the two expansions are derived for the same solution. The second term in the Stokes expansion is then

$$\psi_1 = \frac{3}{32} V_0 a^2 \Big(\frac{2r^2}{a^2} - \frac{3r}{a} + \frac{a}{r}\Big)(1-\mu^2) - \frac{3}{32} V_0 a^2 \Big(\frac{2r^2}{a^2} - \frac{3r}{a} + 1 - \frac{a}{r} + \frac{a^2}{r^2}\Big)\mu(1-\mu^2) \tag{4}$$

with $f_1(\mathrm{Re}) = \mathrm{Re}$.

Continuing, Bentwich and Miloh[10] (1978) used the matched asymptotic expansion method to solve the unsteady low Reynolds number flow past a sphere whose velocity undergoes a sudden change. Similarly, Sano[11] (1981) obtained the higher-order term where $f_2(\mathrm{Re}) = \mathrm{Re}^2(\ln \mathrm{Re})$

$$\psi_2 = \frac{9}{160} V_0 a^2 \Big(\frac{2r^2}{a^2} - \frac{3r}{a} + \frac{a}{r}\Big)(1-\mu^2) \tag{5}$$

Later, Mei and Adrian[13] (1992) applied a successive orders of matched asymptotic expansion to solve the Navier–Stokes equation to O(Re) for the case of oscillating flow over a sphere, by considering small fluctuations in velocity when the Reynolds number is not negligibly small.

A matching procedure obtains higher order approximations to the flow surrounding a sphere for the case of finite but small Reynolds numbers $\mathrm{Re} \leq 100$. However the approach is not obviously able to be extended to Reynolds numbers of arbitrarily high values. This article attempts to derive a more generalized analytical solution for the motion of flow about a moving sphere in a viscous and incompressible fluid using two separate analytical approaches. The strategy is first to derive a compact form of the Navier–Stokes equation in a 2D spherical coordinate system expressible in terms of the stream function. A series solution is obtained using a technique initially developed by Basset (1888), by assuming that a solution stream function is amenable to a summation of separation of variables. It is shown that certain arbitrarily chosen coefficients yield stream function solutions that depict streamlines' deformation consistent with the development of a wake and vortices behind a moving sphere. One specific result is the successful identification of the angular location on the sphere surface where the main flow detaches from the sphere. We also derive a solution to a compact form of the Navier–Stokes equation using a change of variable. The solution appears able to reproduce a train of closed vortices behind a fast moving sphere. We use the

Maple™ software[15] as an analytical solver for the partial differential equations described in this article.

II. THE STREAM FUNCTION AROUND A MOVING SPHERE

To start, suppose a sphere of radius a surrounded by a stationary viscous fluid that is moving with a constant velocity V_0 along a straight axis z. Placing the center at the origin, v_r and v_θ represent the components of the flow velocity around the sphere, where r is the radius and θ is the angle with respect to the direction of motion in an axially symmetric spherical coordinate system (`figure` 1). The Navier–Stokes equations are

$$\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}-\frac{v_\theta^2}{r}=-\frac{1}{\rho_f}\frac{\partial p}{\partial r}+\nu\left(\nabla^2 v_r-\frac{2v_r}{r^2}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}-\frac{2v_\theta cot\theta}{r^2}\right) \quad (6)$$

$$\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_r v_\theta}{r}=-\frac{1}{r\rho_f}\frac{\partial p}{\partial \theta}+\nu\left(\nabla^2 v_\theta+\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}-\frac{v_\theta}{r^2\sin^2\theta}\right) \quad (7)$$

Supposing the no-slip condition applies at the sphere surface, then

$$v_r\Big|_{r=a}=V_0\cos\theta, \quad v_\theta\Big|_{r=a}=-V_0\sin\theta \quad (8)$$

Provided the motion is symmetrical about the z axis, the components of the flow velocity along and perpendicular to the direction of r can be expressed in terms of the stream function as

$$v_r(t,r,\theta)=\frac{1}{r^2\sin\theta}\frac{\partial \psi(t,r,\theta)}{\partial \theta} \quad (9)$$

$$v_\theta(t,r,\theta)=-\frac{1}{r\sin\theta}\frac{\partial \psi(t,r,\theta)}{\partial r} \quad (10)$$

Therefore, in a stationary fluid, the conditions at an infinite distance from a moving sphere are

$$\frac{1}{r^2}\frac{\partial \psi(t,r,\theta)}{\partial \theta}\Big|_{r\to\infty}=0, \quad \frac{1}{r}\frac{\partial \psi(t,r,\theta)}{\partial r}\Big|_{r\to\infty}=0 \quad (11)$$

And from equations (9)-(10), the boundary conditions (8) at the sphere surface become

$$\frac{\partial \psi}{\partial \theta}\Big|_{r=a}=V_0a^2\sin\theta\ \cos\theta, \quad \frac{\partial \psi}{\partial r}\Big|_{r=a}=V_0a\ \sin^2\theta \quad (12)$$

Note that this last boundary condition implies that θ enters into ψ in the form of the factor $\sin^2\theta$. We can reduce the degree of differentiation and complexity of the Navier–Stokes equations defined in (9)-(10) by employing an operator

$$\mathrm{D}=\frac{\partial^2}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}-\frac{\cos\theta}{r^2\sin\theta}\frac{\partial}{\partial\theta} \tag{13}$$

Then, using (9)-(10), the Navier–Stokes equations (6)-(7) can be rewritten through operator D as

$$-\frac{1}{\rho_f}\frac{\partial p}{\partial r}=\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\Big(\frac{\partial\psi}{\partial t}-\nu\mathrm{D}\psi\Big) \tag{14}$$

$$\frac{1}{\rho_f}\frac{\partial p}{\partial\theta}=\frac{1}{\sin\theta}\frac{\partial}{\partial r}\Big(\frac{\partial\psi}{\partial t}-\nu\mathrm{D}\psi\Big) \tag{15}$$

Cancelling the pressure term, the complete Navier–Stokes equation in terms of the stream function becomes

$$\underbrace{\mathrm{D}\Big(\nu\mathrm{D}-\frac{\partial}{\partial t}\Big)\psi}_{\text{Eq.I}}+\sin\theta\underbrace{\left(\frac{\partial\psi}{\partial r}\frac{\partial}{\partial\theta}-\frac{\partial\psi}{\partial\theta}\frac{\partial}{\partial r}\right)\frac{\mathrm{D}\psi}{r^2\sin^2\theta}}_{\text{Eq.II}}=0 \tag{16}$$

This is the general equation for the stream function $\psi(t,r,\theta)$ around a moving sphere surrounded by a viscous and incompressible fluid. It has a linear component $\mathrm{D}\big(\nu\mathrm{D}-\frac{\partial}{\partial t}\big)\psi$ that describes laminar flow around a slow-moving sphere, and a non-linear component related to the squares and products of velocities, or the advection terms, in the surrounding flow in (6)-(7).

Stokes (1850) obtained the solution for the linear component at steady-state for a slowly moving sphere with constant velocity, that is $\mathrm{D}\big(\mathrm{D}\psi\big)=0$. In 1885, Boussinesq – and independently three years later Basset – solved the linear component for a slowly moving sphere with time-varying velocity, namely $\mathrm{D}\big(\mathrm{D}-\frac{1}{\nu}\frac{\partial}{\partial t}\big)\psi=0$. Oseen (1910) hypothetically ignored inertial forces closed to the sphere, but considered them far from the sphere, and solved $\mathrm{D}(\mathrm{D}\psi)+\frac{\sin\theta}{\nu}\big(\frac{\partial\psi}{\partial r}\frac{\partial}{\partial\theta}-\frac{\partial\psi}{\partial\theta}\frac{\partial}{\partial r}\big)\frac{\mathrm{D}\psi}{r^2\sin^2\theta}=0$. Note that far from the sphere, where the streamlines become uniform, what remains of the inertial forces in the Navier–Stokes equation are the advection terms.

III. A GENERALIZED SOLUTION

As outlined in section I, due to the consequent mathematical complexity when non-linear terms are retained, beyond approximate solutions at finite but small Reynolds numbers, a generalized analytical solution to (16) has not yet been found. We now introduce a generalized solution to the stream function around a moving sphere that includes the non-linear inertial terms in the Navier–Stokes equations (6)-(7) and satisfies (16). Using the change in variable $\mu = \cos\theta$, the operator D reduces to

$$\mathrm{D} = \frac{\partial^2}{\partial r^2} + \frac{1-\mu^2}{r^2}\frac{\partial^2}{\partial \mu^2} \tag{17}$$

Equation (16) becomes

$$\begin{aligned}\left(\frac{\partial^2}{\partial r^2} + \frac{1-\mu^2}{r^2}\frac{\partial^2}{\partial \mu^2}\right)\left[\frac{\partial^2\psi}{\partial r^2} + \frac{1-\mu^2}{r^2}\frac{\partial^2\psi}{\partial \mu^2} - \frac{1}{\nu}\frac{\partial\psi}{\partial t}\right] \\ - \frac{1}{\nu r^2}\frac{\partial\psi}{\partial r}\left[\frac{2\mu}{1-\mu^2}\frac{\partial^2\psi}{\partial r^2} + \frac{\partial^3\psi}{\partial\mu\partial r^2} + \frac{1-\mu^2}{r^2}\frac{\partial^3\psi}{\partial\mu^3}\right] \\ + \frac{1}{\nu r^2}\frac{\partial\psi}{\partial\mu}\left[-\frac{2}{r}\frac{\partial^2\psi}{\partial r^2} - \frac{4(1-\mu^2)}{r^3}\frac{\partial^2\psi}{\partial\mu^2} + \frac{\partial^3\psi}{\partial r^3} + \frac{1-\mu^2}{r^2}\frac{\partial^3\psi}{\partial r\partial\mu^2}\right] = 0\end{aligned} \tag{18}$$

We assume the solution for the stream function is amenable to a summation of a separation of variables

$$\psi(t,r,\mu) = \sum_\lambda R_\lambda(t,r)\cdot\Theta_\lambda(\mu) \tag{19}$$

where R_λ is a function of t and r, and Θ_λ is a solution to the differential equation

$$(1-\mu^2)\frac{d^2\Theta_\lambda}{d\mu^2} - \lambda(\lambda+1)\Theta_\lambda = 0 \tag{20}$$

where λ is a complex number. Specifying $m = -4\lambda(\lambda+1)+1$, the solution to (20) is

$$\begin{aligned}\Theta_m(\mu) = c_m\,{}_2\mathrm{F}_1\left(\left[-\frac{1}{4}-\frac{\sqrt{m}}{4}, -\frac{1}{4}+\frac{\sqrt{m}}{4}\right], \frac{1}{2}, \mu^2\right) \\ + d_m\,\mu\,{}_2\mathrm{F}_1\left(\left[\frac{1}{4}-\frac{\sqrt{m}}{4}, \frac{1}{4}+\frac{\sqrt{m}}{4}\right], \frac{3}{2}, \mu^2\right)\end{aligned} \tag{21}$$

Here c_m and d_m are solution coefficients. For the variable x, the hypergeometric function ${}_2\mathrm{F}_1([a,b],c,x)$ is a regular solution to the hypergeometric differential equation $x(1-x)\frac{d^2y}{dx^2} + [c-(a+b+1)x]\frac{dy}{dx} - ab\,y(x) = 0$ for all of non-negative values of c and $|x| < 1$. The hypergeometric function is represented by the following series

$$
{}_2\mathrm{F}_1([a,b],c,x) = 1 + \frac{a\,b}{1!\,c}\,x + \frac{a(a+1)\,b(b+1)}{2!\,c}\,x^2 + \ldots = \sum_{i=0}^{\infty} \frac{(a)_i\,(b)_i}{i!\,(c)_i}\,x^i
$$

where $(a)_i$ is a Pochhammer symbol. Assigning a solution in the form of (20) to $\Theta_\lambda(\mu)$ is possible because the operator D that has been defined in such a way as to reduce the complexity of the Navier–Stokes equations (6)-(7) to a compact form expressed by (16). Specifically, applying the operator D to $\Theta_\lambda(\mu)$ yields

$$
\begin{aligned}
\mathrm{D}(\Theta_\lambda) &= \frac{1-\mu^2}{r^2}\frac{d^2\Theta_\lambda}{d\mu^2} \\
&= \frac{\lambda(\lambda+1)}{r^2}\Theta_\lambda
\end{aligned} \tag{22}
$$

which significantly reduces the complexity related to higher order derivatives in (16).

Moving forward to solving for R_λ, by substituting (19) and (20) into (18), we obtain

$$
\begin{aligned}
&\left(\frac{\partial^2}{\partial r^2} + \frac{1-\mu^2}{r^2}\frac{\partial^2}{\partial \mu^2}\right)\left[\frac{\partial^2 R_\lambda}{\partial r^2}\Theta_\lambda + \frac{\lambda(\lambda+1)}{r^2}R_\lambda\Theta_\lambda - \frac{1}{\nu}\frac{\partial R_\lambda}{\partial t}\Theta_\lambda\right] \\
&\quad - \frac{1}{\nu r^2}\frac{\partial R_\lambda}{\partial r}\Theta_\lambda\left[(1-\mu^2)\frac{\partial^2 R_\lambda}{\partial r^2}\frac{d}{d\mu}\Big(\frac{\Theta_\lambda}{1-\mu^2}\Big) + (1-\mu^2)\frac{\lambda(\lambda+1)}{r^2}R_\lambda\frac{d}{d\mu}\Big(\frac{\Theta_\lambda}{1-\mu^2}\Big)\right] \\
&\quad + \frac{R_\lambda}{\nu r^2}\frac{d\Theta_\lambda}{d\mu}\left[-\frac{2}{r}\frac{\partial^2 R_\lambda}{\partial r^2}\Theta_\lambda - \frac{4\lambda(\lambda+1)}{r^3}R_\lambda\Theta_\lambda\right. \\
&\quad \left. + \frac{\partial^3 R_\lambda}{\partial r^3}\Theta_\lambda + \frac{\lambda(\lambda+1)}{r^2}\frac{\partial R_\lambda}{\partial r}\Theta_\lambda\right] = 0
\end{aligned} \tag{23}
$$

Dividing by Θ_λ

$$
\begin{aligned}
&\left(\frac{\partial^2}{\partial r^2} + \frac{1-\mu^2}{r^2}\frac{1}{\Theta_\lambda}\frac{d^2\Theta_\lambda}{d\mu^2}\right)\left[\frac{\partial^2 R_\lambda}{\partial r^2} + \frac{\lambda(\lambda+1)}{r^2}R_\lambda - \frac{1}{\nu}\frac{\partial R_\lambda}{\partial t}\right] \\
&\quad - \frac{1-\mu^2}{\nu r^2}\frac{\partial R_\lambda}{\partial r}\left[\frac{d}{d\mu}\Big(\frac{\Theta_\lambda}{1-\mu^2}\Big)\right]\left[\frac{\partial^2 R_\lambda}{\partial r^2} + \frac{\lambda(\lambda+1)}{r^2}R_\lambda\right] \\
&\quad + \frac{R_\lambda}{\nu r^2}\frac{d\Theta_\lambda}{d\mu}\left[-\frac{2}{r}\Big(\frac{\partial^2 R_\lambda}{\partial r^2} + \frac{\lambda(\lambda+1)}{r^2}R_\lambda\Big) + \frac{\partial}{\partial r}\Big(\frac{\partial^2 R_\lambda}{\partial r^2} + \frac{\lambda(\lambda+1)}{r^2}R_\lambda\Big)\right] = 0
\end{aligned} \tag{24}
$$

From (20)

$$\left(\frac{\partial^2}{\partial r^2}+\frac{\lambda(\lambda+1)}{r^2}\right)\left[\frac{\partial^2 R_\lambda}{\partial r^2}+\frac{\lambda(\lambda+1)}{r^2}R_\lambda-\frac{1}{\nu}\frac{\partial R_\lambda}{\partial t}\right]$$
$$-\frac{1-\mu^2}{\nu r^2}\frac{\partial R_\lambda}{\partial r}\left[\frac{d}{d\mu}\left(\frac{\Theta_\lambda}{1-\mu^2}\right)\right]\left[\frac{\partial^2 R_\lambda}{\partial r^2}+\frac{\lambda(\lambda+1)}{r^2}R_\lambda\right]$$
$$+\frac{R_\lambda}{\nu}\frac{d\Theta_\lambda}{d\mu}\frac{\partial}{\partial r}\left(\frac{1}{r^2}\left[\frac{\partial^2 R_\lambda}{\partial r^2}+\frac{\lambda(\lambda+1)}{r^2}R_\lambda\right]\right)=0 \quad (25)$$

The desired removal of dependence on Θ_λ can be obtained by requiring that R_λ satisfies

$$\left[\frac{\partial^2 R_\lambda}{\partial r^2}+\frac{\lambda(\lambda+1)}{r^2}R_\lambda\right]=0 \quad (26)$$

Then, the differential equation that determines $R_\lambda(t,r)$ is only a function of (t,r). Note that equation (26) is equivalent to stating $\mathrm{D}\psi(t,r,\mu)=0$. The consequence of this simplification is discussed later in section V. The solution to (25), satisfying the condition expressed by (26), and switching the index from λ to $m=-4\lambda(\lambda+1)+1$, is $R_m(t,r)=f_1(t)\,r^{\frac{1}{2}-\frac{\sqrt{m}}{2}}+f_2(t)\,r^{\frac{1}{2}+\frac{\sqrt{m}}{2}}$. This can be expressed more conveniently as

$$R_m(t,r)=\int_0^\infty\left(a_m(\rho)\,r^{\frac{1}{2}-\frac{\sqrt{m}}{2}}+b_m(\rho)\,r^{\frac{1}{2}+\frac{\sqrt{m}}{2}}\right)e^{-\rho^2/4\nu t}\,d\rho \quad (27)$$

where $a_m(\rho)$ and $b_m(\rho)$ remain to be determined. The variable ρ has units of length. It is zero at the particle surface and increases towards infinity far from the particle.

A generalized solution to the Navier–Stokes equation (16) for an arbitrary set of boundary conditions is now obtained by substituting the separated solutions (21) for Θ_λ and (27) for R_λ in the summation of separation of variables (19)

$$\psi(t,r,\mu)=\sum_m\int_0^\infty\left(a_m(\rho)\,r^{\frac{1}{2}-\frac{\sqrt{m}}{2}}+b_m(\rho)\,r^{\frac{1}{2}+\frac{\sqrt{m}}{2}}\right)\cdot$$
$$\left\{c_m(\rho)\cdot{}_2\mathrm{F}_1\left(\left[-\frac{1}{4}-\frac{\sqrt{m}}{4},-\frac{1}{4}+\frac{\sqrt{m}}{4}\right],\frac{1}{2},\mu^2\right)\right.$$
$$\left.+d_m(\rho)\cdot\mu\cdot{}_2\mathrm{F}_1\left(\left[\frac{1}{4}-\frac{\sqrt{m}}{4},\frac{1}{4}+\frac{\sqrt{m}}{4}\right],\frac{3}{2},\mu^2\right)\right\}e^{-\rho^2/4\nu t}\,d\rho \quad (28)$$

It is worth noting that although there is an assigned relation between m and λ, the value of λ can be any number, and therefore any complex or real value of m in (28) satisfies the Navier–Stokes equation (16). For particular values of m, the hypergeometric functions in (21) simplify to Legendre polynomials. For example

$$
\begin{cases}
m = 3^2 \to \ \Theta_9(\mu) = c_9(1-\mu^2)\frac{d\mathrm{P}_1}{d\mu} + \frac{d_9}{2}(1-\mu^2)\frac{d\mathrm{Q}_1}{d\mu}, \\
m = 5^2 \to \ \Theta_{25}(\mu) = -\frac{c_{25}}{2}(1-\mu^2)\frac{d\mathrm{Q}_2}{d\mu} + \frac{d_{25}}{3}(1-\mu^2)\frac{d\mathrm{P}_2}{d\mu}, \\
m = 7^2 \to \ \Theta_{49}(\mu) = \frac{2c_{49}}{3}(1-\mu^2)\frac{d\mathrm{P}_3}{d\mu} + \frac{d_{49}}{8}(1-\mu^2)\frac{d\mathrm{Q}_3}{d\mu}, \\
\dots
\end{cases}
\tag{29}
$$

In these cases, the hypergeometric functions given by (21) can be seen as a generalized form of derivatives of Legendre functions of the first and the second kinds P_n and Q_n, a solution to the Legendre differential equation $\frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right] + n(n+1)y = 0$. For other values of m in (28), other functional forms may apply.

Thus, we have obtained the generalized solution (28) by solving the Navier–Stokes equation in two-dimensional and the spherical coordinate system (6)-(7), employing an assumption that the stream function can be expressed as a function of a summation of separated variables (19), and assuming the condition (26). Note that assuming (19) does not mean that a simple separation of variables is physically or mathematically justified to solve the Navier–Stokes equation. Following Basset (1888), we assume that there exists a solution to the stream function that cannot be described by $\psi(t,r,\mu) = R(t,r)\cdot\Theta(\mu)$ but that does satisfy the summation of separation of variables $\psi(t,r,\mu) = R_1(t,r)\cdot\Theta_1(\mu) + R_2(t,r)\cdot\Theta_2(\mu) + \dots$

To find the stream function around a moving sphere surrounded by a viscous fluid, specific values for m in (28) must be determined from a particular set of boundary conditions. The problem thus reduces to determination of the proper boundary conditions for the solution. For each value of m, the solution contains four unknown coefficients to be determined. For example, when $m = 1$, the coefficients are a_1, b_1, c_1 and d_1. Given that the complete form of the solution has a summation over m, and there are only two no-slip conditions at the boundary of the sphere (12), and two conditions at infinity (11), the system is under-determined, with an infinite number of solutions.

IV. VISUALIZATION OF THE SOLUTION

For what follows, we illustrate the steady solution of the stream function (28) for a sphere moving at constant velocity V_0 in a stationary viscous fluid. Simple cases are considered assuming constant values for the coefficients. The summation over m is ignored so as to satisfy (11) at infinite r, and the values of m are limited to $|m| \leq 9$. Solutions are shown for the stream lines around a stationary sphere in a moving fluid with a constant velocity.

Representing the stream function around a sphere with radius $a = 1$, the assumed constant values of the coefficients are $a_m = -b_m = 1$ and $c_m = -d_m = 1$. These coefficients were chosen as they reproduce the desired behaviour for the flow around sphere. So, at this stage of mathematical development, the solutions do not in fact represent any realistic case as no specific boundary condition was applied to the determine the coefficients. Dimensionally the coefficients must be a function of sphere radius and velocity for the particle to move, and all that can be said is that they must be non-zero.

Adopting this approach, the stream function surrounding the sphere becomes

$$\psi_m(r,\mu) = \left(r^{\frac{1}{2}-\frac{\sqrt{m}}{2}} - r^{\frac{1}{2}+\frac{\sqrt{m}}{2}}\right) \cdot \Bigg\{ \mathrm{F}_1\left(\left[-\frac{1}{4}-\frac{\sqrt{m}}{4}, -\frac{1}{4}+\frac{\sqrt{m}}{4}\right], \frac{1}{2}, \mu^2\right) - \mu \cdot {}_2\mathrm{F}_1\left(\left[\frac{1}{4}-\frac{\sqrt{m}}{4}, \frac{1}{4}+\frac{\sqrt{m}}{4}\right], \frac{3}{2}, \mu^2\right) \Bigg\} \tag{30}$$

`Figure 2` displays the streamlines around a sphere for nine different values of m around a moving sphere in a stationary fluid. It shows a wholly laminar streamline about the sphere at $m = -2, -1, -0.5$. With increasing values of m, the streamlines behind the sphere broaden, and a stagnant region at the rear of sphere develops where the streamlines at the boundary layer detach from the sphere due to zero wall shear stress. The value $m = 1$ is the threshold value at which the laminar boundary layer separates from the rear surface of particle so that the flow and the particle no longer interact.

Behind the particle due to flow viscosity a flow reversal develops forming a wake. At $m = 3$, the wake exhibits an axisymmetric vortex contour structure. At $m = 5$ the wake broadens, and for $m = 7$, the vortex contours begins to split.

Negative values of m correspond with complex values of the stream function and only the absolute value is plotted. For $m < 1$, the solution produces laminar streamlines about the sphere, with two stagnation points, defined as points where the local velocity at the surface of particle becomes zero, located at the sphere front at $\theta = 0°$, and the rear at $\theta = 180°$. The formation of a wake shifts the second stagnation point behind the sphere to within the fluid (not shown). Terming the angle of separation as the angle relative to $\theta = 0°$ where a stagnant boundary layer separates from the main streamlines, for $m = 3$ the angle of separation is $\theta_s = 150°$, for $m = 5$ it is $\theta_s = 140°$, and for $m = 7$ it is $\theta_s = 135°$.

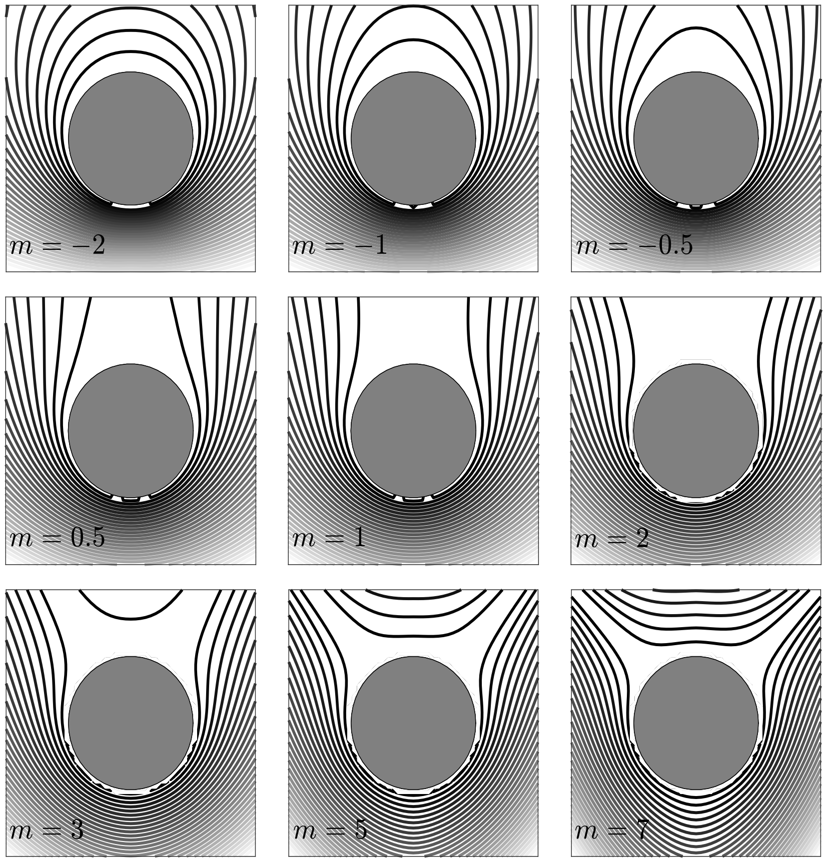

FIG. 2. Visualization of streamlines around a sphere moving with constant velocity in a stationary fluid for nine different values of m. The laminar streamline about the sphere are evident for $m < 1$. As m increases the laminar boundary layer starts to separate from behind the particle surface, and a wake forms behind the sphere. Higher values of m show deformation of the wake behind the sphere. In all the cases, the coefficients in (30) are assumed constant and equal to $a_m = -b_m = 1$ and $c_m = -d_m = 1$. A similar pattern is seen for constant m but increasing Reynolds numbers.

Similar behaviors to those shown in `figure 2` are obtained by changing the coefficients to $c_m = \frac{1}{2}\frac{\sqrt{\pi}}{\Gamma(\frac{5}{4}-\frac{m}{4})\Gamma(\frac{5}{4}+\frac{m}{4})}$, and $d_m = -V_0 a^2 \frac{\sqrt{\pi}}{\Gamma(\frac{3}{4}-\frac{m}{4})\Gamma(\frac{3}{4}+\frac{m}{4})}$, where Γ is a Gamma function, and holding m constant but changing the particle velocity or radius. What this suggests is some correspondence between m and the particle Reynolds number. The coefficients c_m and d_m, when multiplied by the hypergeometric functions in (28), yield an associated Legendre function of the first kind, as described by equation 8.1.4 in Abramowitz et al.[16] (1988). Expressed in this manner, the coefficients transform the solution from an expression in the form of hypergeometric functions to one in the form of Legendre functions.

The components of the flow velocity along and perpendicular to the direction of r around the sphere are obtained by substituting (28) in (9) and (10)

$$
\begin{aligned}
v_r(t,r,\theta) = \sum_m \int_0^\infty & \left(a_m(\rho)\, r^{-\frac{3}{2}-\frac{\sqrt{m}}{2}} + b_m(\rho)\, r^{-\frac{3}{2}+\frac{\sqrt{m}}{2}}\right)\cdot \\
& \Big\{ -d_m(\rho)\cdot {}_2F_1\Big(\Big[\frac{1}{4}-\frac{\sqrt{m}}{4},\frac{1}{4}+\frac{\sqrt{m}}{4}\Big],\frac{3}{2},\cos^2\theta\Big) \\
& + \frac{c_m(\rho)}{4}\cdot\big(m-1\big)\cdot\cos\theta\cdot {}_2F_1\Big(\Big[\frac{3}{4}-\frac{\sqrt{m}}{4},\frac{3}{4}+\frac{\sqrt{m}}{4}\Big],\frac{3}{2},\cos^2\theta\Big) \\
& + \frac{d_m(\rho)}{12}\cdot\big(m-1\big)\cdot\cos^2\theta\cdot {}_2F_1\Big(\Big[\frac{5}{4}-\frac{\sqrt{m}}{4},\frac{5}{4}+\frac{\sqrt{m}}{4}\Big],\frac{5}{2},\cos^2\theta\Big)\Big\}\, e^{-\rho^2/4\nu t}\, d\rho
\end{aligned}
\tag{31}
$$

$$
\begin{aligned}
v_\theta(t,r,\theta) = -\frac{1}{2}\sum_m \int_0^\infty & \left((1-\sqrt{m})\, a_m(\rho)\, r^{-\frac{3}{2}-\frac{\sqrt{m}}{2}} + (1+\sqrt{m})\, b_m(\rho)\, r^{-\frac{3}{2}+\frac{\sqrt{m}}{2}}\right)\cdot \\
& \sin\theta\cdot\Big\{c_m(\rho)\cdot {}_2\mathrm{F}_1\Big(\Big[\frac{3}{4}-\frac{\sqrt{m}}{4},\frac{3}{4}+\frac{\sqrt{m}}{4}\Big],\frac{1}{2},\cos^2\theta\Big) \\
& + d_m(\rho)\cdot\cos\theta\cdot {}_2\mathrm{F}_1\Big(\Big[\frac{5}{4}-\frac{\sqrt{m}}{4},\frac{5}{4}+\frac{\sqrt{m}}{4}\Big],\frac{3}{2},\cos^2\theta\Big)\Big\}\, e^{-\rho^2/4\nu t}\, d\rho
\end{aligned}
\tag{32}
$$

`Figure` 3 shows the steady flow velocity field around a stationary sphere obtained from (31) and (32) with $a_m = -b_m = 1$, $c_m = \frac{1}{2}\frac{\sqrt{\pi}}{\Gamma(\frac{5}{4}-\frac{m}{4})\Gamma(\frac{5}{4}+\frac{m}{4})}$, $d_m = V_0 a^2 \frac{\sqrt{\pi}}{\Gamma(\frac{3}{4}-\frac{m}{4})\Gamma(\frac{3}{4}+\frac{m}{4})}$, for $m = 18$. Note that the restriction on $|m| \leq 9$ only applies to the state where the fluid is stationary. Downstream of the sphere in the wake region, the velocity field is highly disturbed and exhibits curvature consistent with a vortex pair, although the vortices are not closed. For this case, the velocity field stagnates near the separation point $\theta_s = 107°$.

The characteristics of flow past a circular cylinder for Re > 1 has been widely studied experimentally and numerically, showing approximate linear growth of the standing vortex pair with Re[17–30]. `Figure` 4 shows experimental and numerical results for how the separation angle decreases with Reynolds number. Empirical relations have been proposed for the angle of separation and the Reynolds number in the range 10 to 10^5 (Wu et al. 2004; Jiang 2020), and as outlined before, a few theoretical studies have considered the flow and drag on a sphere at finite but small Reynolds numbers[6–8,11–13,22,31]. However, a simple formulation remains to be found for the angle of separation at the surface of a sphere at arbitrarily high Reynolds numbers.

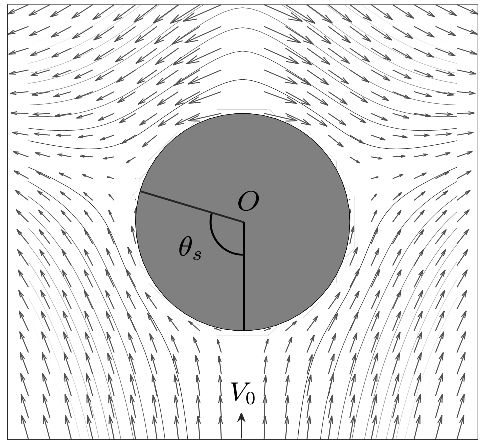

FIG. 3. Logarithmically scaled flow velocity field around a stationary sphere from (31) and (32). The separation point is near $\theta_s = 107°$.

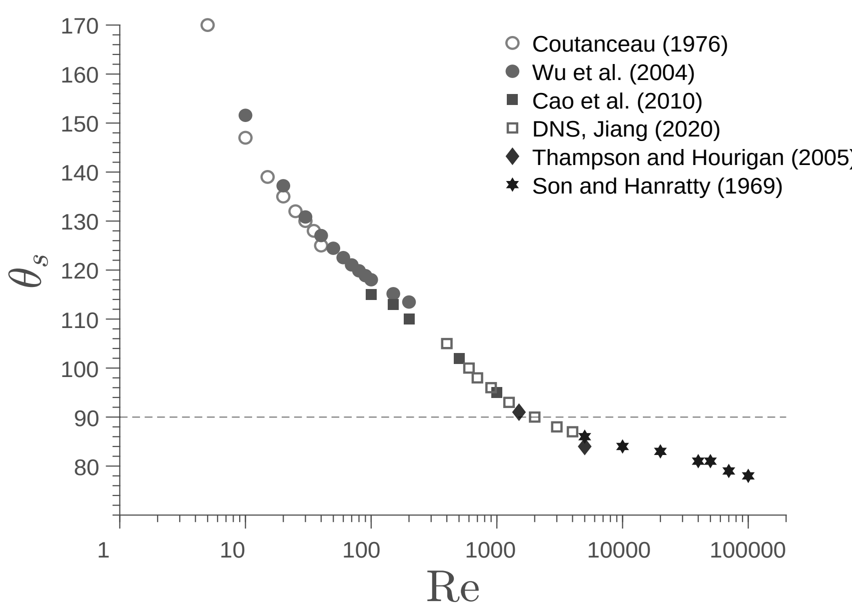

FIG. 4. Experimental (Son and Hanratty 1969; Coutanceau and Bouard 1977; Cao et al. 2010; Thompson and Hourigan 2005) and numerical (Wu et al. 2004; Jiang 2020) studies showing the angle of flow separation θ_s on a circular cylinder surface as a function of the Reynolds number.

In steady two-dimensional and axisymmetric flows, boundary-layer separation occurs at the point on the surface of sphere that shear stress becomes zero, i.e., $\frac{\partial v_\theta}{\partial r}\Big|_{r=a} = 0$. It corresponds to the angle where the $\mu = \cos\theta$ functionality of (32) becomes zero. Therefore, for each value of m, the equation that determines the angle of separation is

$$\left\{ c_m \cdot {}_2\mathrm{F}_1\left(\left[-\frac{1}{4}-\frac{\sqrt{m}}{4}, -\frac{1}{4}+\frac{\sqrt{m}}{4}\right], \frac{1}{2}, \mu^2\right) \right.$$
$$\left. + d_m \cdot \mu \cdot {}_2\mathrm{F}_1\left(\left[\frac{1}{4}-\frac{\sqrt{m}}{4}, \frac{1}{4}+\frac{\sqrt{m}}{4}\right], \frac{3}{2}, \mu^2\right) \right\}\Bigg|_{\theta=\theta_s} = 0 \quad (33)$$

The coefficients c_m and d_m remain to be determined, but for the case that $m = 3^2$ and assuming the boundary condition $c_m = -d_m$, equation (33) yields

$$(1-\mu^2) - \frac{1}{2}\Big[(1-\mu^2)\cdot\tanh^{-1}(\mu) + \mu\Big] = 0 \;\rightarrow\; \theta = 48^\circ \quad (34)$$

which provides the angle of separation with respect to the motion direction of $\theta_s = 132^\circ$ corresponding to $\mathrm{Re} = 25$ in `figure` 4. As outlined in Section III, the θ functionality of the stream function can be simplified to the summation of Legendre functions of the first and the second kinds for particular values of m. Therefore, from (29) one can assume that the simplest method for determination of the tendency of θ_s with m is to interpolate between those points in m where the derivative of the Legendre function of the first kind is equal to the Legendre function of the second kind.

For comparison with measurements in `figure` 4, `figure` 5 `(left)` shows the separation angle calculated from (33) for the case that $c_m = -d_m$ and plotted as a function of the order m, showing how θ_s declines with increasing m. For the range $1 < m < 3\times10^3$, the fit has the form $\theta_s = a\, e^{-b\cdot m^c} + 90$ where $a = 2\times10^4$, $b = 5.5$, and $c = 0.055$.

For each value of m, the angle of separation in `figure` 5 `(left)` can be directly compared with `figure` 4 to determine the implied relationship between m and Re as shown in `figure` 5 `(right)`. The data of Re are taken from the same data as in 4. For example, for $m = 60$ the analytically derived angle of separation is about $\theta_s = 110^\circ$ and compares with $\mathrm{Re} = 200$. For the range of Re shown in `figure` 5 `(right)`, the relation between the particle Reynolds number and the order of m suggests that each value of m corresponds with a state of the flow field around the sphere. For spheres moving with higher speeds, higher orders of m are required.

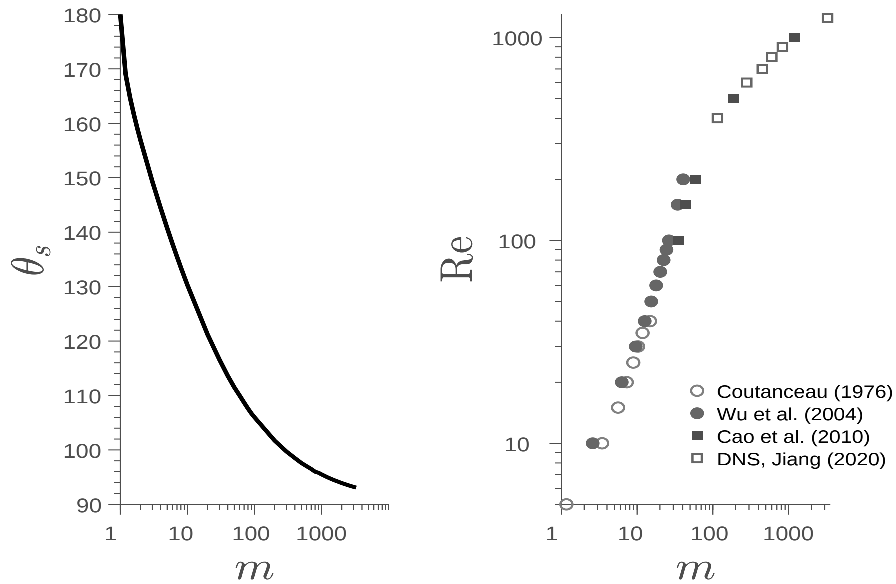

FIG. 5. Left) Separation angle θ_s calculated by (33) with the boundary condition $c_m = -d_m$, plotted as a function of m. Right) Reynolds number as a function of m implied from comparing from `figure` 4. The figure shows a relationship between Re and the order of m.

V. DISCUSSION

The solution described above is an extension of prior work that employed a summation of separation of variables technique, here in R_λ and Θ_λ, where $R_\lambda(t,r)$ is subject to the condition (26). This condition is equivalent to the restriction that $\mathrm{D}\psi(t,r,\mu) = 0$, the justification being that it simplifies (25) so that R_λ is independent of $\mu = \cos\theta$ as required. The full Navier–Stokes equation (16) is then satisfied because both the linear term given by Eq. I and the nonlinear advection term given by Eq. II, namely $\left(\frac{\partial\psi}{\partial r}\frac{\partial}{\partial\theta} - \frac{\partial\psi}{\partial\theta}\frac{\partial}{\partial r}\right)\frac{\mathrm{D}\psi}{r^2\sin^2\theta}$, are *individually* equal to zero. So, the approach does not provide a fully general solution. Even so, the resulting stream function (28) does provide such expected behaviors as the streamlines around a moving sphere and the angle of separation spanning a wide range of Reynolds numbers.

What was not reproduced is the expected behavior of closed vortices and eddies around a fast moving sphere. What is required is a solution that satisfies (16) without requiring the condition $\mathrm{D}\psi(t,r,\mu) = 0$. We proposed a change of variable $\eta = r\sin\theta$. In this case, the advection term given by Eq. II simplifies to zero

$$\left(\frac{\partial\psi}{\partial r}\frac{\partial}{\partial\theta}-\frac{\partial\psi}{\partial\theta}\frac{\partial}{\partial r}\right)\frac{\mathrm{D}\psi}{r^2\sin^2\theta}=\eta\cos\theta\left(\frac{\partial\psi}{\partial\eta}\frac{\partial}{\partial\eta}-\frac{\partial\psi}{\partial\eta}\frac{\partial}{\partial\eta}\right)\frac{\mathrm{D}\psi}{\eta^2}=0 \tag{35}$$

So, conveniently, since the two terms in parentheses cancel, what remains to be solved is only the linear component Eq. I of (16). Of course, this was achieved previously by Boussinesq (1885) and Basset (1888), but in terms of t, r and θ, in which case the advection term given by Eq. II remains non-zero and was simply neglected. In terms of t and η, the (16) becomes

$$\left(\frac{\partial^2}{\partial\eta^2}-\frac{1}{\eta}\frac{\partial}{\partial\eta}\right)\left[\frac{\partial^2\psi}{\partial\eta^2}-\frac{1}{\eta}\frac{\partial\psi}{\partial\eta}-\frac{1}{\nu}\frac{\partial\psi}{\partial t}\right]=0 \tag{36}$$

The general solution to (36) is

$$\psi(t,\eta)=\left[\alpha J_1\left(\sqrt{\frac{\mathbf{c}}{\nu}}\,\eta\right)+\beta Y_1\left(\sqrt{\frac{\mathbf{c}}{\nu}}\,\eta\right)\right]\eta\, e^{-\mathbf{c}t}+\gamma(t)+\delta(t)\,\eta^2 \tag{37}$$

where $J_1(x)$ and $Y_1(x)$ are Bessel functions of the first and the second kinds – the solutions to the Bessel differential equation $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-1)y=0$. The solution coefficients α, β, $\mathbf{c}$, $\gamma(t)$, and $\delta(t)$ are determined from a particular set of boundary conditions. Note that the Bessel functions $J_1(x)$ and $Y_1(x)$ are a quarter period out of phase, and that to be physical (11) requires that $\delta(t)=0$.

To visualize streamlines around a sphere moving with constant velocity in a stationary fluid, we ignore the time dependence in (37) and, regardless of the units, assume a particular set of boundary conditions with $\alpha=0$, $\beta=\frac{1}{Y_0(\sigma a\sin\theta)}$, and $\gamma(t)=0$, where Y_0 is the zero-order Bessel function of the second kind, $\sigma=\sqrt{\frac{\mathbf{c}}{\nu}}$, and $\sigma a\sin\theta$ is root of Y_1. `Figure` 6 shows the streamlines surrounding a moving sphere for nine different values of σ. For a high value of $\sigma=0.9$, it exhibits eddies in the wake region behind the sphere that are qualitatively consistent with those seen at high Reynolds numbers, indicating that σ maps on the Reynolds number. Dimensional analysis suggests that σ is proportional to $\sqrt{Re}/a$. The characteristic alternating vortex position behind the sphere is not reproduced because the two-dimensional stream function can only represent motion that is axially symmetric. Also, the problem remains that, while the boundary conditions (12) at the particle surface are time-independent, a solution of form (37) is a function of time at the particle surface $r=a$. Applying the no-slip boundary condition to (37), the coefficients α and β are not constant as desired. To obtain a particular solution where the time-dependency cancels at the particle surface, a procedure similar to that used by Basset (1888, chapter XXII, section 504) could be followed, where the solution is integrated with respect to the coefficients between the limits 0 and ∞, effectively considering all possible positive values, and then the boundary conditions are applied. The time dependency in the equation then cancels at $r = a$. Therefore, the solution (37) would be in the following form

$$\psi(t,r,\theta) = \frac{\sqrt{\pi} r \sin^2\theta}{\sqrt{\nu t}} \int_0^\infty \alpha(\rho)\, \frac{(r-a-\rho)}{4\nu t} \cdot \left\{ I_0\left[\frac{(r-a-\rho)^2 \sin^2\theta}{8\nu t}\right] - I_1\left[\frac{(r-a-\rho)^2 \sin^2\theta}{8\nu t}\right] \right\} e^{-\frac{(r-a-\rho)^2 \sin^2\theta}{8\nu t}}\, d\rho + \frac{\sin^2\theta}{\sqrt{\pi \nu t}} \int_0^\infty \beta(\rho)\, \frac{(r-a-\rho)^2}{4\nu t} \cdot \left\{ K_0\left[\frac{(r-a-\rho)^2 \sin^2\theta}{8\nu t}\right] + K_1\left[\frac{(r-a-\rho)^2 \sin^2\theta}{8\nu t}\right] \right\} e^{-\frac{(r-a-\rho)^2 \sin^2\theta}{8\nu t}}\, d\rho \tag{38}$$

where I_0 and I_1 are modified Bessel functions of the first kind, K_0 and K_1 of the second kind, and the coefficients $\alpha(\rho)$ and $\beta(\rho)$ are determined by applying the boundary conditions (12). The variable ρ has units of length. It is zero by definition at the particle surface and increases to infinity far from the particle.

VI. CONCLUSION

A complete description of a viscous fluid's response to a moving particle requires solving the Navier–Stokes equation. The first attempts by Stokes (1850), and later by Boussinesq (1885), and Basset (1888), were restricted to a sphere that moves slowly so that the advection terms could be omitted. In 1910, Oseen considered the advection and extended Stokes solution. Later, Oseen's solution was extended to higher approximations, but the solutions are limited to small Reynolds numbers. Starting from the Navier–Stokes equations and assuming axial symmetry of the flow field with respect to the direction of motion, we developed a form of the Navier–Stokes equation in two-dimensional and spherical coordinates that depends only on the stream function. The approach taken here was to solve the Navier–Stokes equation by assuming that the stream function can be expressed as a function of a summation of separation of variables in the radial and angular directions, assuming a specified angular functionality in the stream function and restricting a condition to the radial functionality of the stream function, yielding a solution that is a series in the form of hypergeometric functions.

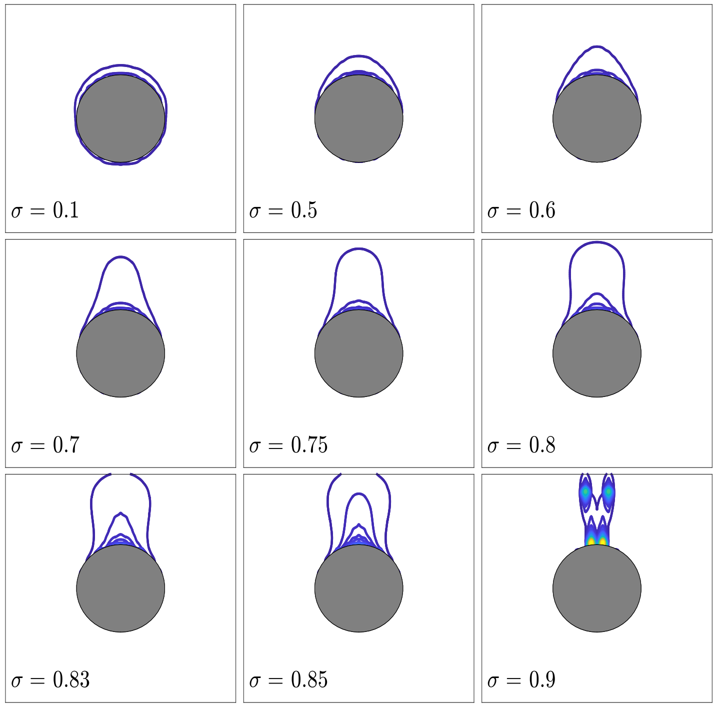

FIG. 6. Streamlines in the wake region satisfying (37) with $\beta = \frac{1}{Y_0(\sigma a \sin\theta)}$, and for σ ranging from 0.1 to 0.9.

The solution simplifies to derivatives of Legendre functions of the first and the second kinds for specific values of the index m, and to the associated Legendre function of the first kind for specific values of the coefficients in the summation expansion. The solution is found to simulate the stream function around a moving sphere, and to provide the angular locations of boundary layer separation points consistent with experiment. Graphical representation of the solution for specific real values of m suggests that m is proportional to the particle's Reynolds number. To obtain an analytical expression for the streamlines about a particle with a Reynolds number much greater than one, higher orders of m are required in the summation.

For the case of high Reynolds numbers, we showed that a change of variables leads to cancellation of the advection terms in the Navier–Stokes equation, and to a solution for the stream function expressible in terms of Bessel functions of the first and second kind. The solution successfully reproduces closed vortices in the wake region of the sphere. However, the solution expressed in terms of two-dimensional stream function can only represent vortices that are axially symmetric, but not asymmetry in their locations.

ACKNOWLEDGMENTS

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DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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