# Wavelet-based wavenumber spectral estimate of eddy kinetic energy: Application to the North Atlantic

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# Wavelet-based wavenumber spectral estimate of eddy kinetic energy: Application to the North Atlantic

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**Key Points:**

- Wavenumber spectra of eddies, defined as the fluctuations about an ensemble mean, are estimated for the North Atlantic basin.
- The wavenumber spectra and spectral flux of eddy kinetic energy and enstrophy are estimated using wavelet transform.
- We question the validity of quasi-geostrophic thinking for the Gulf Stream and offer a primitive equation extension.

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**Abstract**

An ensemble of eddy-rich North Atlantic simulations is analyzed, providing estimates of kinetic energy wavenumber spectra and spectral budget. A wavelet transform technique is used to estimate a localized 'pseudo-Fourier' spectrum, permitting comparisons to be made between spectra at different locations in a highly inhomogeneous and anisotropic environment. We find evidence of a Gulf Stream imprint on the near Gulf Stream eddy field appearing as enhanced levels of energy in the North-South direction relative to the East-West direction. Surprisingly, this signature holds into the quiescent interior. We detect forward cascades of energy and enstrophy but find no clear evidence of upscale energy cascades in the separated Gulf Stream region. The spectral slopes inferred from our analysis are significantly steeper than expected from quasi-geostrophic theory, but roughly in line with a primitive equation extension of the enstrophy inertial-range theory. Lastly, we propose that the spectral shapes are somewhat universal throughout our domain, over a broad wavenumber range. Deviations from this structure occur at high wavenumbers in locations characterized by strong surface fronts. A summary conclusion is that expectations built on quasi-geostrophy are at best only weakly supported in primitive equations.

## Plain language summary

Describing the statistical characteristics of the weather system of the ocean, known as eddies, has been a long standing problem in the field of ocean science. This has been motivated by the fact that eddies contribute significantly to the global heat and carbon transport. Here, in analysing numerical simulations of the North Atlantic that partially resolve the eddies, we apply a relatively novel diagnostic framework based on wavelet functions to characterize the statistical nature of eddies in a realistic setting of the ocean. We find that the signature of the Gulf Stream imprints itself onto the eddy statistics.

## 1 Introduction

The ocean is 'turbulent', implying the presence of energetic and widespread spatial and temporal 'eddies' (Stammer, 1998; Stammer & Wunsch, 1999). It is now commonly accepted in ocean modeling that resolving these features, at least at the mesoscale, leads to ocean simulations of a much more realistic nature (Chassignet & Marshall, 2008; Chassignet et al., 2020; Griffies et al., 2015; Constantinou & Hogg, 2021; G. Xu et al., 2022), which may have important implications for climate projections (Saba et al., 2016; Beech et al., 2022). This implies the eddy field is an integral part of the ocean structure, and necessary to include in some fashion if acceptable ocean models are to be constructed. The computational demands of eddy-resolving resolution have led to the search for eddy parameterizations that faithfully capture the dynamical role of eddies in the absence of their explicit presence (e.g. Redi, 1982; Gent & Mcwilliams, 1990; Gent, 2011; Jansen et al., 2019; Guillaumin & Zanna, 2021; Uchida, Deremble, & Popinet, 2022; Li et al., 2022, and references therein). It is essential therefore to understand the behavior of the eddy field in well-resolved models in order to ascertain the character eddy parameterizations should portray and to provide benchmarks for assessing the affects of any particular proposed parameterization. This paper attempts to serve these purposes by describing and applying a methodology that allows for spatial inhomogeneity in the mean flow to influence eddy characteristics. We analyze a recently developed ensemble of North Atlantic simulations (Jamet et al., 2019) and use two-dimensional wavelet analysis to diagnose the spectral structure.

Most available theoretical guidance on oceanic turbulence comes from quasi-geostrophic (QG) theory, where the combined conservations of energy and potential vorticity lead to predictions for specific shapes for wavenumber spectra. It is generally thought that the eddy field should display a so-called '$-5/3$' spectral slope as a result of an up-scale cascade of energy, and a '$-3$' slope due to a down-scale enstrophy cascade (Charney, 1971). Both predictions are based on the ideas of inertial ranges and involve a reasonable number of assumptions. Locality in spectral interactions, stationarity in time and homogeneity in space are amongst the most prominent assumptions; a thorough discussion appears in (Vallis, 2006). Numerical, observational and laboratory investigations in relevant settings tend to support the predictions (e.g. Gage & Nastrom, 1986; Yarom et al., 2013; Callies & Ferrari, 2013; Campagne et al., 2014).

The inertial-range ideas are usually adopted when venturing into the more dynamically complex settings of primitive equations and realistic ocean simulations (e.g. Y. Xu & Fu, 2011, 2012; Khatri et al., 2018; Vergara et al., 2019), although it is difficult to justify many of the assumptions. In particular, as will often be the focus of this paper, the presence of the Gulf Stream would seem to clearly violate spatial homogeneity in the field in which the eddies are viewed. In addition, and perhaps at an even more fundamental level, the mix of a coherent, large-scale mean with an incoherent, variable component renders the definition of what constitutes an 'eddy' somewhat vague. One then questions what features should be focused on when constructing a spectrum (cf. Uchida, Jamet, et al., 2021). This problem of identifying or defining ocean eddies is a well known one, with an early reference being (Wunsch, 1981).

Another problem facing the quantification of the eddy field in an inhomogeneous setting is a lack of available techniques for analyzing the data. A favorite, and classical, method for studying wavenumber spectra employs Fourier transforming momentum (e.g. Capet et al., 2008; Callies & Ferrari, 2013; Rocha et al., 2016; Uchida et al., 2017, 2019; Khatri et al., 2018, 2021). The connection between this measure and kinetic energy (KE) comes from Parseval's theorem, which equates the area integrated KE to the wavenumber integrated spectrum

$$\int_{\boldsymbol{x}} |\boldsymbol{u}(\boldsymbol{x})|^2 \, d\boldsymbol{x} = \int_{\boldsymbol{k}} \hat{\boldsymbol{u}} \cdot \hat{\boldsymbol{u}}^* \, d\boldsymbol{k} \tag{1}$$

where $\hat{\boldsymbol{u}}$ is the Fourier transform of $\boldsymbol{u} = (u, v)$. This permits the interpretation of the spectrum in terms of a wavenumber dependent energy density. However, this same equivalence then implies the resultant spectra are averages over the domain involved in the analysis. While this does not represent a conceptual problem if the domain is spatially homogeneous, the relation of the result to the local spectrum in an inhomogeneous setting is not clear.

Our primary numerical tool to tackle these questions is a recently developed eddying ensemble of partially air-sea coupled North Atlantic simulations. These simulations have been used before in studies of North Atlantic energetics (Jamet et al., 2020), the Atlantic Meridional Overturning Circulation (AMOC; Jamet et al., 2019), Empirical Orthogonal Function (EOF) analyses of eddies (Uchida, Jamet, et al., 2021), and the Thickness-Weighted Averaged (TWA) feedback of the eddies on the residual-mean flow (Uchida, Jamet, et al., 2022; Uchida, Balwada, et al., 2023). A full description of the simulations appears in Jamet et al. (2019). For our purposes, the ensemble consists of 48 members exposed to *small* initial-condition uncertainties (usually referred to as *micro* initial conditions; Stainforth et al., 2007) run at an 'eddy-rich' $1/12°$ resolution. A map of the surface eddy Ertel's potential vorticity (PV) appears in Fig. 1, displaying the expected activity around the Gulf Stream region, with a separation from the coastal U.S. around Cape Hatteras, and North Atlantic Current. Also shown are six marked locations which will be referred to later in the text.

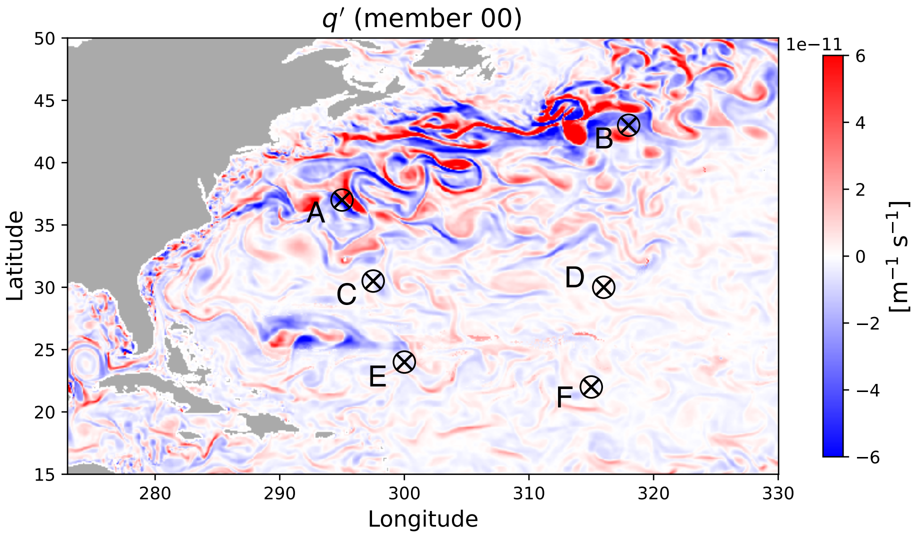


**Figure 1.** Surface eddy Ertel's PV from member 00 amongst the 48 ensemble members at 00:10, January 1, 1967 where buoyancy is defined as $b = -\delta/\rho_0$ and is dimensionless (cf. Section 2.3 and Appendix A). Land and coastlines are in grey; the Gulf Stream and its extension into the open Atlantic are visible. Locations within the Gulf Stream near to separation at Cape Hatteras and North Atlantic Current are marked, as are other locations in the North Atlantic interior and gyre retroflection with six regions in total named from A to F. These will be referred to later in the text.

We assert that such an ensemble leads to a clear identification of oceanic eddies, namely as fluctuations about the ensemble mean. Specifically, we can average our simulations at any space and time point across our ensembles to obtain an estimate of the classical ensemble mean. Then, we can revisit each individual ensemble member to compute its deviation from the ensemble mean at that same spatial and temporal location. Inasmuch as the ensemble mean represents that component of the solution common to all members, we identify it as the predictable part of the flow. The residuals, belonging to each individual realization, are the 'unpredictable' components of the flow and are identified as the eddies. An attempt to rationalise this in terms of integrated KE budgets has recently been proposed by Jamet et al. (2022). Note that this eddy definition is independent of any arbitrarily chosen spatial or temporal scale, a highly desirable feature not characteristic of most definitions reliant on some form of spatial or temporal filtering (Chen & Flierl, 2015; Uchida, Deremble, Dewar, & Penduff, 2021; Uchida, Jamet, et al., 2021). These eddies are the ones we propose to quantify.

As to spectral computations, we proceed using a wavelet-based analysis. To our knowledge, the wavelet approach to wavenumber spectra was initially examined by Daubechies (1992) and Perrier et al. (1995) and in an oceanographic context by Uchida, Jamet, et al. (2023). For our purposes, we will interpret the spectra computed using wavelets as an estimate of a localized 'pseudo-Fourier' spectrum. The locality of these estimates permits us to examine and compare the variability of the spectra throughout the domain.

Our eddy definition is reviewed briefly in the next section, along with a description of our wavelet-based analysis methods and a possible extensions of the quasi-geostrophic (QG) inertial range theory to primitive equation. Section 3 presents a comparison between wavelet-based spectral estimates and the canonical Fourier-based estimates within the North Atlantic gyre. The paper ends with a Discussion, speculations on the relevant dynamics and plans for further work.

## 2 Theory and techniques

In this section, we describe our definition of 'eddies' (Section 2.1) and provide an overview on wavelet analysis (Section 2.2). We also discuss possible extensions of the quasi-geostrophic (QG) inertial range theory to primitive equation in Section 2.3.

### 2.1 Eddy Definition

Due to the chaotic nature of the ocean (Poincaré, 1890; Lorenz, 1963), trajectories of eddying numerical simulations are sensitive to initial condition uncertainties (Sérazin et al., 2017; Leroux et al., 2018; Jamet et al., 2019; Zhao et al., 2021; Uchida, Deremble, & Penduff, 2021). This allows us to develop an ensemble of ocean simulations, differing only in small ways in their initial conditions; i.e. simulations based on initial states that have small differences well within current measurement uncertainties. It is a matter of experience that while gross characteristics of the resulting fully evolved states are similar (there will always be a Gulf Stream, for example), the mesoscale fields become incoherent. While each ensemble solution represents an equally valid and plausible simulation of the North Atlantic, none of them at any specified date will recreate the observed ocean state since the observed ocean is itself a single realization of the chaotic system.

From such an ensemble, one can take an 'ensemble mean', which we will denote by brackets, i.e. for any model variable $\psi(\boldsymbol{x}, t)$,

$$\langle \psi(\boldsymbol{x}, t) \rangle = \frac{1}{N} \sum_{i=1}^{N} \psi^i(\boldsymbol{x}, t) \,, \tag{2}$$

where $N$ is the total number of ensemble members and the superscript $i$ denotes the ensemble member. We interpret the ensemble mean as the 'forced' response of the ocean. That is, as the ensemble mean is common to all members, it reflects the common external conditions imposed at the boundaries of the system. In our case, these common conditions consist of the prescribed atmospheric states and the open ocean boundary conditions at the northern and southern domain boundaries and the Strait of Gibraltor (Jamet et al., 2019).

The eddy field is denoted by deviations of $\psi$ about the ensemble mean

$$\psi'^i(\boldsymbol{x}, t) = \psi^i(\boldsymbol{x}, t) - \langle \psi(\boldsymbol{x}, t) \rangle \,. \tag{3}$$

Each member, $i$, having its own eddy field thus identifies the eddies as an unpredictable component of the flow. Note that the ensemble mean in (2) is inherently a function of space and time, a feature which permits the examination of the non-stationary and inhomogeneous character of the statistics. It is a strength of the ensemble approach that these features of the statistics are preserved.

Finally, we note that the ensemble mean structure of the ocean is not independent of the eddies, rather the equations of motion in their non-linearity involve higher-order measures of the eddies as part of their balance. Each realization, in turn, is constrained by the lower-order statistics of the eddy contributions.

### 2.2 Spectral Considerations

We depart from the classical Fourier approach to compute wavenumber spectra for our non-periodic and inhomogenous settings, but do note the utility of wavenumber spectrum emerges largely from Parseval's equality (cf. Uchida, Jamet, et al., 2023, their Appendix A). We base our spectral analysis on wavelet decompositions. Here, we provide a brief overview.

Given a function of two spatial dimensions, $f(\boldsymbol{x})$, its continuous wavelet transform is given by

$$\tilde{f}(s, \phi, \boldsymbol{\gamma}) = \int_{\boldsymbol{x}} f(\boldsymbol{x}) \frac{1}{s} \xi^* (\mathsf{R}^{-1} \cdot \left( \frac{\boldsymbol{x} - \boldsymbol{\gamma}}{s} \right)) \, d\boldsymbol{x} \,, \tag{4}$$

where $\mathsf{R}^{-1}$ is the inverse of the rotation matrix

$$\mathsf{R}^{-1} = \begin{pmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{pmatrix}, \tag{5}$$

for rotation through an angle $\phi$. The quantity $s$ is referred to as the 'scale', $\boldsymbol{\gamma}\,(\in \mathbf{R}^2)$ is the two-dimensional coordinates of interest, $\xi(\boldsymbol{x})$ is the so-called 'mother' wavelet and $\xi(\mathsf{R}^{-1} \cdot (\boldsymbol{x} - \boldsymbol{\gamma})\,/s)$ in (4) are the daughter wavelets. The quantities $\tilde{f}$ are the wavelet coefficients. Subject to a few, relatively easy to meet conditions (Uchida, Jamet, et al., 2023), the original data can be reconstructed from the wavelet coefficients via an inverse wavelet transform

$$f(\boldsymbol{x}) = \mathscr{C} \int_{\boldsymbol{\gamma}} \int_{\phi} \int_{s} \frac{1}{s^4} \tilde{f}(s, \phi, \boldsymbol{\gamma}) \xi(\mathsf{R}^{-1} \cdot \left(\frac{\boldsymbol{x} - \boldsymbol{\gamma}}{s}\right))\, ds\, d\phi\, d\boldsymbol{\gamma} \tag{6}$$

where $\mathscr{C}$ is a constant, to be clarified below. Exploiting the properties of wavelets, it is possible to show they satisfy a generalized Parseval's equality

$$\int_{\boldsymbol{x}} f(\boldsymbol{x}) g(\boldsymbol{x}) d\boldsymbol{x} = \int_{\boldsymbol{\gamma}} \int_{\phi} \int_{s} \frac{\tilde{f}\tilde{g}^*}{s^3}\, ds\, d\phi\, d\boldsymbol{\gamma}, \tag{7}$$

with $\bullet^*$ the complex conjugate. Note, if $f = g$, (7) corresponds to the equality in (1).

We employ the so-called Morlet wavelet (Morlet et al., 1982; Gabor, 1946), i.e.

$$\xi(\boldsymbol{x}) = \left(e^{-2\pi i \boldsymbol{k}_0 \cdot \boldsymbol{x}} - c_0\right) e^{-\frac{\boldsymbol{x} \cdot \boldsymbol{x}}{2x_0^2}}, \tag{8}$$

where $c_0$ is a constant included to insure that the wavelet has zero mean $\int_{\boldsymbol{x}} \xi(\boldsymbol{x}) d\boldsymbol{x} = 0$. The central wavenumber $\boldsymbol{k}_0$ is taken to be $\boldsymbol{k}_0 = (k_0, 0)$ and the quantity $x_0$ is a reference length scale, here taken to be 50 km, viz. the length scale of the mother wavelet. We will choose $k_0 = 1/x_0$, in which case the constant $c_0$ is quite small and generally ignored (i.e. $c_0 = 0$), a convention adopted in this paper. Plots of (8) are found in Fig. 2. Note that the Morlet mother wavelet consists of a wave of wavelength $L = x_0$ inside a Gaussian envelope of decay scale $\sqrt{2}x_0$. Thus for $s = 1$ and $\phi = 0$, the wavelet coefficient produced by this transformation comments on the presence of the wavenumber $\boldsymbol{k}_0 = (k_0, 0)$ at location $\boldsymbol{\gamma}$ in the original data. Increasing the rotation angle $\phi$ and filtering returns information about the presence of the same wavelength at angle $\phi$. Finally allowing $s$ to vary modifies the filter so that the primary wavelength of the filter is $k = 1/(sx_0)$. The Morlet wavelet coefficient can thus be thought of as a 'local' Fourier transform at wavenumber $\boldsymbol{k}_0^{\mathsf{T}} \cdot \mathsf{R}^{-1}(\phi)/s$, where the superscript $\mathsf{T}$ denotes a transpose.

At this point, the scale factor in (4), $s$, is non-dimensional. It is more traditional in oceanography to discuss energy spectra in terms of wavenumber. As pointed out above,

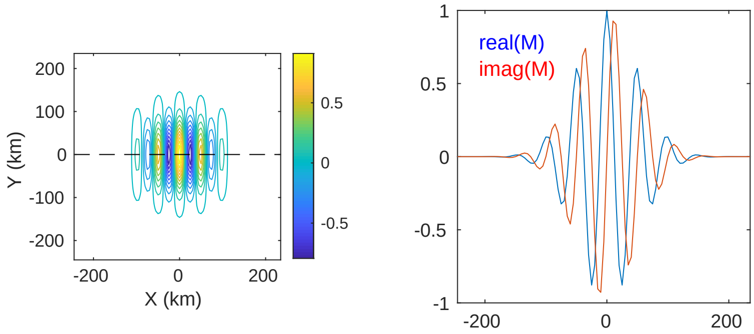


**Figure 2.** Structure of the Morlet wavelet. A contour plot of the real part of the mother Morlet wavelet is shown in the left panel. Transects of the real and imaginary parts along the dashed line appear in the right panel. The reference lengthscale is $x_0 = 50\,\text{km}$.

the effective wavenumber associated with $s$ is $k = 1/(sx_0) = 1/s_0$, where the quantity $s_0$ has units of length. Upon some algebra, one may transform (7) (with $f = g$) to wavenumber, $k = 1/s_0$, space, ending with

$$\int_{\boldsymbol{x}} f^2(\boldsymbol{x}) d\boldsymbol{x} = \frac{1}{C_\Xi} \int_\phi \int_k \int_{\boldsymbol{\gamma}} \tilde{f}^* \tilde{f} k x_0^2 \, d\boldsymbol{\gamma} \, dk \, d\phi \,, \tag{9}$$

where $C_\Xi = \int_{\boldsymbol{k}} \frac{\hat{\Xi}^* \hat{\Xi}}{\boldsymbol{k}\cdot\boldsymbol{k}} \, d\boldsymbol{k}$ and $\hat{\Xi}$ is the Fourier transform of the mother wavelet (cf. Uchida, Jamet, et al., 2023). Note, $\mathscr{C} = C_\Xi^{-1}$ in (7).

If we now produce wavelet coefficients for the zonal and meridional eddy velocities $u'^i$ and $v'^i$ from member $i$ of our ensemble, and manipulate them appropriately, we obtain

$$\widetilde{E}_K^i(\boldsymbol{\gamma}, \phi, k) = \frac{1}{C_\Xi} \frac{\widetilde{u'}^i \widetilde{u'}^{i*} + \widetilde{v'}^i \widetilde{v'}^{i*}}{2} x_0^2 k \,, \tag{10}$$

as a measure of energy density in wavelet transform space. Each value of $\widetilde{E}_K^i$ is a random number as each ensemble member possesses a 'random' eddy field emerging from the non-linearities in the system. Ensemble averaging those values returns an estimate of the ensemble-mean energy spectrum as a function of wavenumber $k$ in direction $\phi$. The spatial locality of the mother wavelet permits the interpretation of $\widetilde{E}_K(s, \phi, \boldsymbol{\gamma}) = \langle \widetilde{E}_K^i(s, \phi, \boldsymbol{\gamma}) \rangle$ as the local energy spectrum at location $\boldsymbol{\gamma}$.

In calculating the wavelet coefficients, we spatially interpolate each $10^\circ \times 10^\circ$ domain centered around each $\otimes$ in Fig. 1 onto a uniform grid (cf. section 3). The wavelet transform appropriate to the scale factor $s$ was then taken between $[k_F^{\text{min}}, k_F^{\text{max}}]$ with 40 monotonic increments where $k_F^{\min}$ and $k_F^{\max}$ are the minimum and maximum Fourier wavenumbers respectively leaving us with 47 increments, and angle $\phi$ with the resolution of $\pi/18$ radian (= 10°) between $[0, \pi)$. The scaling was then truncated at scales below 50 km and appended with scales corresponding to the Fourier wavenumbers to increase the wavenumber resolution at higher wavenumbers. The spatial integration of the product of the wavelet and the data is the wavelet coefficient for each location.

### 2.3 Dimensional analysis of isotropic spectral slopes

To interpret our spectral slope estimates and recast them within the inertial-range theory, we provide here a scaling for the Ertel PV by extending usual QG thinking for primitive equations. It is interesting to consider inertial range arguments in terms of $\widetilde{E}_K$. We assume for convenience that the mother wavelet is a dimensionless function, in which case the wavelet transform $\tilde{u}$ carries the dimensions $L^3T^{-1}$ and $C_\Xi$ the units $L^4$. Thus, the units of $\widetilde{E}_K$ are $L^3T^{-2}$. A cross-scale energy flux $\varepsilon$ must have dimensions of $L^2T^{-3}$ and the usual inertial range arguments lead to

$$\widetilde{E}_K \propto \varepsilon^{2/3} k^{-5/3} \,, \tag{11}$$

in the energy cascade range (Vallis, 2006). In quasi geostrophy, a materially conserved quantity is QG potential vorticity (PV) and its enstrophy flux ($\eta_{\mathrm{QG}}$) has the dimensions of $T^{-3}$. If we assume a so-called inertial enstrophy range, characterized by constant enstrophy flux, similar dimensional arguments yield the classical

$$\widetilde{E}_K \propto \eta_{\mathrm{QG}}^{2/3} k^{-3} \,. \tag{12}$$

spectral shape. Accepting the usual QG idea that the enstrophy spectrum is given by $\widetilde{Z}_K = k^2 \widetilde{E}_K(k)$, one obtains a $-1$ law for the enstrophy spectra.

Here, we consider an extension of these ideas for primitive equations. In the richer dynamics of primitive equations, we can write a conservation equation

$$\frac{Dq}{Dt} \approx 0 \,, \tag{13}$$

for Ertel's PV $q = \boldsymbol{\omega} \cdot \nabla b = (f + \zeta) b_z + (\boldsymbol{k} \times \boldsymbol{u}_z) \cdot \nabla_{\mathrm{h}} b$ where $\boldsymbol{\omega}$ is the absolute vorticity, $b$ buoyancy (detailed in Appendix A), and $\boldsymbol{k}$ the vertical unit vector. The horizontal gradients of vertical velocity are neglected consistent with the hydrostatic approximation of our simulation (Vallis, 2006).

From a dimensional perspective, we argue the units of $b$ are immaterial; it is possible to write a PV equation as (13) where '$b$' is replaced by any thermodynamic variable, $\Theta$, such that

$$\frac{D\Theta}{Dt} \approx 0\,, \tag{14}$$

so that surfaces of $\Theta$ are nearly material. In the following analysis, we take $b$ to be dimensionless, which is equivalent to buoyancy divided by gravity. Thus, the relevant dimensions of $q$ become $L^{-1}T^{-1}$ and the enstrophy flux ($\eta_{\mathrm{PE}}$) has dimensions $L^{-2}T^{-3}$. In an inertial range where the time scale is set by a constant PV flux, standard dimensional arguments imply that the energy spectrum should scale as

$$\widetilde{E}_K \propto \eta_{\mathrm{PE}}^{2/3} k^{-13/3}\,, \tag{15}$$

a spectral slope close to $-4.3$. Due to the richer definition of Ertel's PV compared to QGPV, a simple spectral relation between the KE and enstrophy spectrum does not exist.

## 3 Results

In this section, we examine the kinetic energy (KE) and enstrophy spectra and spectral flux from the various locations in Fig. 1; the location of the panels correspond to the locations on the map. We remind the reader that enstrophy here is defined by Ertel's potential vorticity (PV). The depth of 452 m was chosen to be within the general wind-driven circulation but beneath the mixed layer in order to avoid KE input from convective events, in our case parametrized by the K-profile parametrization (KPP; Large et al., 1994). Prior to taking the wavelet transforms, the fields were linearly interpolated onto a uniform grid. In order to account for the finite-volume discretization of MITgcm, we first weighted the velocity fields by the grid area. The velocities were then linearly interpolated onto the uniform grid and divided by the area also interpolated onto the uniform grid. The uniform grid spacings were taken as the minimum spacing per $10^\circ \times 10^\circ$ domain centered around each location in Fig. 1. The wavelet transforms are taken at the center of the $10^\circ \times 10^\circ$ domain while the Fourier transforms are taken over the $10^\circ \times 10^\circ$ domain. The 48-member ensemble outputs used in this study are instantaneous snapshots at 00:10, January 1, 1967, viz. there is no temporal averaging applied.

### 3.1 The wavelet and Fourier approach

One of the major differences between quasi geostrophy and primitive equations is that advection is two-dimensional (2D) in the former and three-dimensional (3D) for the latter. It can be argued that for primitive equations, the eddy velocity defined about the thickness-weighted averaged residual mean, which reduces to 2D under adiabatic conditions (Young, 2012; Marshall et al., 2012; Aoki, 2014; Loose et al., 2022; Uchida, Balwada, et al., 2023), corresponds to the QG eddy velocities under order-Rossby number fluctuations in the layer thickness. Nonetheless, the spectral flux of KE and enstrophy have commonly been examined in geopotential coordinates (e.g. Capet et al., 2008; Arbic et al., 2013; Khatri et al., 2018, 2021; Ajayi et al., 2021). Due to the discrepancies between quasi geostrophy and primitive equations in geopotential coordinates, there is no guarantee that the inertial-range theory should hold for the latter. In this section, we examine the agreement between the wavelet and Fourier approach, and to what extent the spectra and spectral fluxes in geopotential coordinates are consistent with QG predictions. We also include contributions from vertical advection unlike studies using satellite observations where only the horizontal velocities are available (Scott & Wang, 2005).

We start by examining the wavenumber spectrum derived from the wavelet and Fourier method; the agreement between the two is impressive with the black and red curves overlying with each other (Fig. 3a). Such a similarity between Fourier and wavelet estimates have also been identified in doubly periodic homogeneous quasi-geostrophic simulations where Fourier modes are best suited (Uchida, Jamet, et al., 2023). Prior to taking the Fourier transform, land cells surrounded by ocean were linearly interpolated over and filled in with zeros otherwise, after which a Hann window was applied to make the data doubly periodic. While neither procedure is necessary in the wavelet method, in order to make a fair comparison, we also take the wavelet transform over the fields with land cells filled in but without the windowing. The two approaches agree well in their spectral estimates being within the 95% bootstrap confidence interval of each other. The bootstrapping was done by randomly resampling (with replacement) the 48 ensemble member energy densities 9999 times. The overall spectral slopes are relatively steep. A best fit to the spectra between roughly 250 km and 40 km suggests a $-4.13$ power law, which is considerably steeper than either the $-3$ or $-5/3$ energy and enstrophy inertial range laws emerging from quasi geostrophy.

In the ocean, it is unlikely that the sources and sinks of energy are localized in wavenumber as assumed by standard, idealized inertial-range theories. We explicitly examine this by computing the 'dynamics', i.e. computing all the terms in the eddy KE spectral budget

$$T_K = P_K + A_K + \mathrm{MtE}_K + \mathcal{K}_K \tag{16}$$

where the tendency of KE, $T_K$, equals the sum of pressure work $P_K$, advection $A_K$, KE exchange with the mean flow $\mathrm{MtE}_K$, and non-conservative terms $\mathcal{K}_K$ respectively. Detailed notations of each term are given in (B6). Our form of pressure work consists only of the wavelet transforms related to $-\langle \boldsymbol{u'} \cdot \nabla_\mathrm{h} \phi' \rangle$. Adding and subtracting $\langle w'b' \rangle$ and using the hydrostatic relationship demonstrates that exchanges between potential and kinetic energies are contained in this term. We do not consider potential energy explicitly here, leaving this as a topic for consideration elsewhere.

The relative contributions of terms in the spectral budget computed at location A are shown in Fig. 3b where the residual (grey dashed line) is seen to be negligible. Positive values indicate a source for the eddy KE reservoir and negative values a sink at a given wavenumber. The largest values from the dynamics belong to pressure work, KE tendency and advection. However, all the quantities, except for advection, are not distinguishable from zero at the 95% confidence level. The advection $A_K$ is positive across all wavenumbers, which would imply a forward cascade of energy. We have examined the spatial stability of these quantities by evaluating the dynamics in the vicinity of location A, up to 70 km away. The tendencies seen in the plot hold over the area, most importantly the advective effects are positive and significant, consistent with a forward cascade.

We have also computed Ertel's PV spectra at this depth (Fig. 3c). QG reasoning in a region of forward enstrophy cascade would argue for a $-1$ slope, which is shown by the dashed line. Clearly the spectral drop-off is much greater than this, consistent with the steep KE spectra, but shoals towards $-1$ at scales larger than $\sim 250$ km potentially indicating a QG scaling.

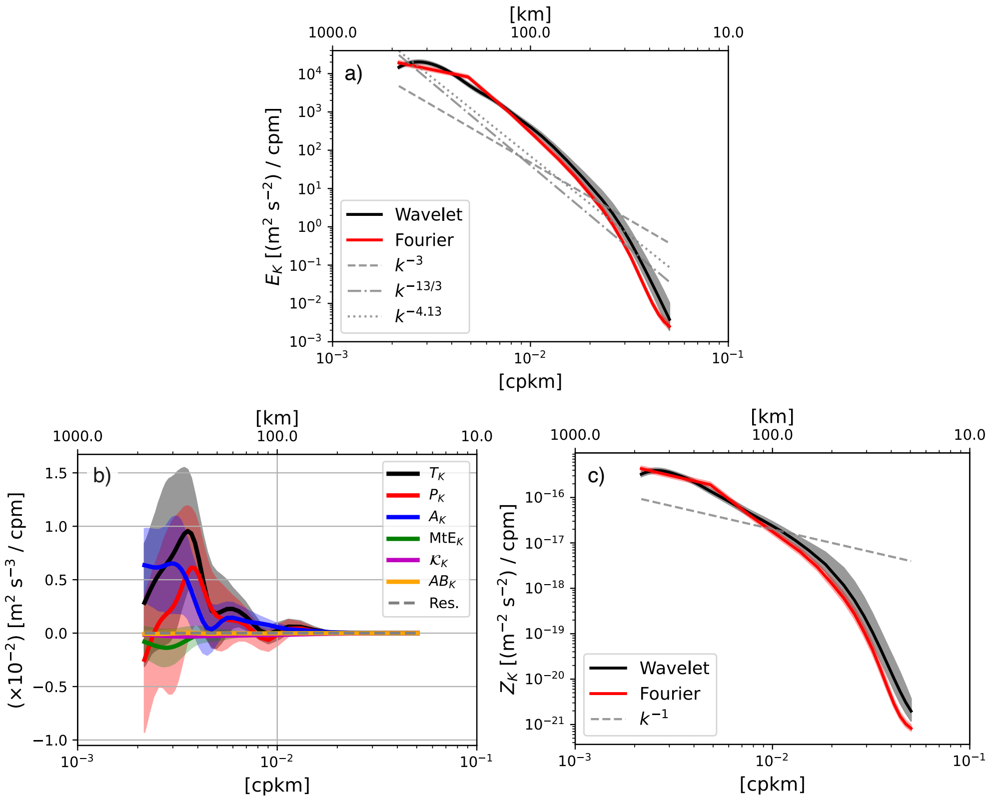


**Figure 3.** Isotropic (azimuthally-integrated) KE spectrum $E_K(k)$ using the wavelet and Fourier approach from $z = -452$ m at location A (indicated in Fig. 1) shown as black and red curves respectively on January 1, 1967 (a). The isotropic wavelet spectral KE budget (B6) is shown in panel (b) with the AB$_K$ term stemming from the Adam-Bashforth time stepping. The residual in the spectral budget is shown as the grey dashed curve being negligible. The isotropic enstrophy spectrum $Z_K(k)$ where buoyancy is dimensionless (cf. Section 2.3) (c). The land cells are interpolated over except for the budget. The colored shadings indicate the 95% bootstrap confidence interval.

Using the spectral transfers, we can diagnose the KE and enstrophy spectral flux given respectively as

$$\tilde{\varepsilon}_K(\boldsymbol{\gamma}, \phi, k) = -\frac{1}{C_\Xi}\int_{k>\kappa} \mathcal{R}\left[\langle \tilde{u'}^{*} (\widetilde{\boldsymbol{v}\cdot\nabla u})'\rangle + \langle \tilde{v'}^{*} (\widetilde{\boldsymbol{v}\cdot\nabla v})'\rangle + \widetilde{\langle u\rangle}^{*} \nabla\cdot\widetilde{\langle \boldsymbol{v}'u'\rangle} + \widetilde{\langle v\rangle}^{*} \nabla\cdot\widetilde{\langle \boldsymbol{v}'v'\rangle}\right] x_0^2 \kappa \, d\kappa\,, \tag{17}$$

$$\tilde{\eta}_K(\boldsymbol{\gamma}, \phi, k) = -\frac{1}{C_\Xi}\int_{k>\kappa} \mathcal{R}\left[\langle \tilde{q'}^{*} (\widetilde{\nabla\cdot\boldsymbol{v}q})'\rangle + \widetilde{\langle q\rangle}^{*} \nabla\cdot\widetilde{\langle \boldsymbol{v}'q'\rangle}\right] x_0^2 \kappa \, d\kappa\,, \tag{18}$$

where $\mathcal{R}[\cdot]$ indicates the real part (Appendix B). Positive values indicate a forward cascade towards smaller scales and negative values an inverse cascade towards larger scales.

The Fourier equivalent of (17) corresponds to the kinetic energy spectral fluxes often examined by others. The azimuthally-integrated KE spectral flux $\tilde{\varepsilon}_K$ displays a forward cascade at all scales, consistent with the Fourier-based estimate (Fig. 5a). The levels of kinetic energy exchange with the mean flow are quite low, and insignificant (green curve in Fig. 3b). The pressure work term, while noisy, tends to peak at around 250 km (red curve in Fig. 3b), so QG theory might argue for an upscale energy cascade at smaller wavenumbers (Vallis, 2006). This is not what we find, however, arguing for a deviation from quasi geostrophy in our results. We have also investigated the temporal stability of these results by looking at the following Jan. 6 and 11 outputs. The lack of significance for most of the sizeable quantities, like pressure work and kinetic energy tendency manifest in greatly different values for these quantities on those dates. Clearly, they are not stable in sign (Fig. 4). In contrast, advection is persistently positive and significant at the 95% level and scales above $\sim 300$ km at point A. The deviation from quasi geostrophy in the separated Gulf Stream region is consistent with a recent study which examined the energetics in the North Atlantic subtropical gyre (Jamet et al., 2020).

In contrast to the spectral energy flux, the isotropic spectral flux of Ertel's enstrophy at location A (Fig. 5b) is relatively scale independent and positive in the wavelet analysis, albeit with sizeable uncertainties. As in the KE flux, the wavelet approach shows reduced uncertainty at the largest scales where the Fourier flux is likely affected by the windowing (cf. Uchida, Jamet, et al., 2023). The presence of an extended range of scales with constant $\eta_K$, as discussed in Sec 2.3, is consistent with the observations of spectral slopes steeper than standard QG estimates.

Given that the wavelet technique agrees well with the Fourier method with the additional strengths of: i) negating the necessity for the data to be periodic, ii) flexibility in defining the wavenumber resolution via the scaling $s$, and iii) being able to extract the anisotropy in the flow through the rotational matrix $\mathsf{R}$, we apply our method to five other locations in the North Atlantic subtropical gyre.

### 3.2 Wavelet spectra of the North Atlantic domain

We take advantage of our wavelet approach in this section; we do not interpolate over land and take the spatial integration (4) by treating them as missing data and plot the orientations with the maximum and minimum integrated power over wavenumber,

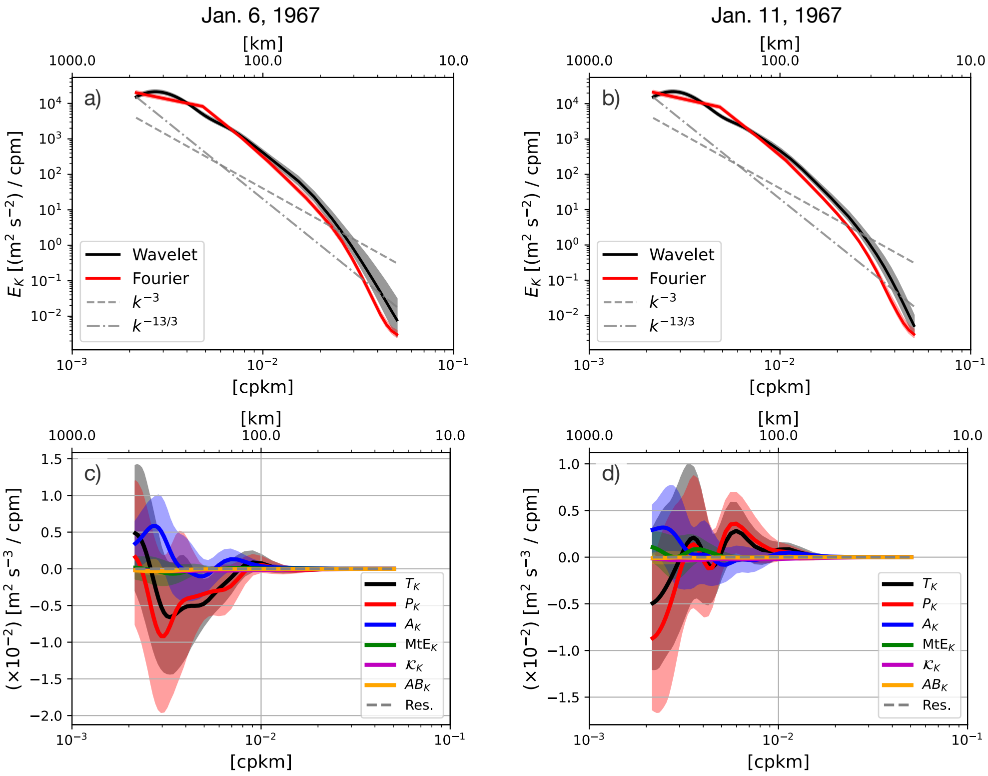


**Figure 4.** The KE spectra and spectral budget at location A on January 6 and 11, 1967 from $z = -452$ m. The land cells are interpolated over for the spectra to make the Fourier and wavelet approach consistent with each other but are not interpolated for the budgets.

i.e. energy. The maximum and minimum tend to be oriented perpendicular to one another (Fig. 6). We first examine a location close to the Gulf stream separation point, as seen in Fig. 1 (location A; Fig. 6a), which exhibits the highest energy levels (close to $10^3$ (m$^2$ s$^{-2}$)/cpm) among all analysed regions. A dashed line indicating a $-3$ slope appears in grey; the spectrum aligns reasonably well with this slope for a small range at the lowest wavenumbers, and then transitions to steeper decays for higher wavenumbers as already observed in Fig. 3a. The $-3$ slope is consistent with that expected from a forward QG enstrophy cascade. A statistically significant signal of anisotropy is apparent, characterized by enhanced energy in the meridional direction relative to the zonal direction. This is likely an imprint of the Gulf Stream on the eddy field due to the roughly zonal orientation of the separated Gulf Stream.

Moving downstream in the North Atlantic Current region (location B; Fig. 6b), the spectral slopes are similar to location A across a wide range except for the smallest wavenum-

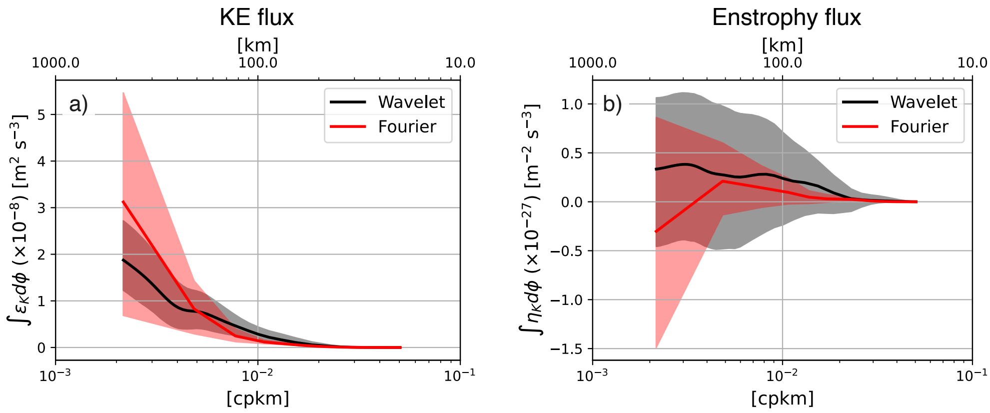


**Figure 5.** Isotropic (azimuthally-integrated) spectral KE flux $\varepsilon_K$ (a), equivalent to $A_K$ integrated in wavenumber, and spectral flux of Ertel's enstrophy $\eta_K$ at location A from $z = -452\,\mathrm{m}$ on January 1, 1967 (b). The Fourier approach has the land cells interpolated over and is windowed while neither are applied for the wavelet approach. The colored shadings indicate the 95% bootstrap confidence interval.

bers. In stark contrast to A, no statistically significant evidence of anisotropy is seen. The spectral amplitudes have dropped from A by about a factor of three. Location C is roughly from a location on the edge of the so-called 'inertial recirculation' (Fig. 6c). A broad wavenumber band exhibits a steep slope, with best fit line of $-4.2$, and similarly to B exhibits little to no evidence of anisotropy. Spectral amplitudes are comparable to those at B.

The remaining three regions (locations D-F; Fig. 6d-f) come from locations that are ostensibly in the interior of the general circulation, at locations where one might anticipate QG dynamics would govern. Mean flows are weak and do not exhibit much structure on the deformation scale, generating conditions in which isotropy might be anticipated. In accord with these expectations, all three regions have the weakest energy levels, and all are comparable in amplitude. Beyond this, however, the results are quite surprising. Most unexpectedly, all three locations exhibit statistically significant anisotropy, in a sense similar to that at location A. Namely, North-South (nominally) spectra are more energetic than East-West spectra. The spectral slopes are also all steep, and similar to those seen in locations A-C. This is difficult to ascribe to QG dynamics. In short, our quantitative measures of the eddy field in the ocean interior do not meet with our expectations.

Along with the spectra, we exhibit the eddy anisotropy angles defined as (Waterman & Lilly, 2015)

$$\vartheta = \frac{1}{2} \arctan \left( \frac{2\langle u'v' \rangle}{\langle u'^2 - v'^2 \rangle} \right) . \tag{19}$$

The angles north of 30°N show no coherent patterns while there is some indication of a slight north-eastward self-organization of angular patterns south of 30°N (Fig. 7), which may be associated to the anisotropy observed in the spectra. We do not exhibit the angle dependency of the spectral flux due to large confidence intervals.

## 4 Conclusions and discussion

Using a relatively novel wavelet approach applied to an ensemble of eddy-rich North Atlantic simulations, we claim we can compare local spectra from several spots within the general circulation characterized by vastly different dynamics. Specifically, we compare spectra within the recently separated Gulf Stream to those found further downstream, in the inertial recirculation and the gyre interior. The motivation for these comparisons arise from a parameter free definition of 'eddy' and interest in clarifying the description of eddies in this heterogeneous field dominated by an ensemble-mean Gulf Stream and relatively quiescent interior. We anticipated that the Gulf Stream would imprint the eddy field with an anisotropic structure, but that the gyre interior would be much simpler and isotropic (Pedlosky et al., 1987). Although an earlier study had warned that the separated Gulf Stream might not be quasi-geostrophic (QG; Jamet et al., 2020), we nonetheless expected to see evidences of up-scale energy cascades at scales beyond the deformation radius, and down-scale cascades at shorter length scales, and that spectral slopes would follow QG expectations.

Several relatively robust characteristics emerge from our calculations, almost none of which aligned with our hypotheses. As expected, the near separation Gulf Stream was found to be anisotropic at the 95% confidence level. However, beyond this, our analysis yielded surprising results. An examination of spectral flux in the near Gulf Stream argued to down scale energy cascades across the spectrum and yielded essentially no evidence for an up-scale flux. Consistent with this, with the caveat of large 95% confidence intervals, was a forward flux of enstrophy, although our spectral shape was far steeper than the quasi-geostrophically motivated value of $-3$. What was missing was any clear evidence of an upscale cascade. This exceptionally steep ($\sim -4.2$) slope was found across all our spectra, including those in the gyre interior where parameterically QG reasoning is expected. Another unexpected result was the persistence of anisotropy throughout the interior, with exceptions appearing in the far Gulf Stream and the inertial recirculation (Fig. 6).

Perhaps the most surprising is the gross similarity of the kinetic energy (KE) spectral structures throughout the North Atlantic gyre, despite the hugely different dynamical regimes in which those spectra are embedded. The local spectra do differ in amplitude, with distance from the Gulf Stream associated with a decrease in eddy intensity. This is an expected result. The similarity in spectral slopes we find surprising. We have thus been motivated to compared the gross spectral structures at these locations by normalizing the spectra by their locally integrated values over wavenumber (i.e. energy), with the results appearing in Fig. 8. By gross structure, we are referring to the integrated spectra over azimuthal angle prior to normalization. An impression left by this comparison is that the spectral shapes are all the same at leading order which, given the zones generating those spectra, is unexpected. The range over which this comparison emerges is that from roughly 300 km to 80 km. At smaller scales, the results diverge, but in view of this result we suggest that the eddy field is occupying a universal shape over a significant bandwidth. This is a result we had not anticipated but is consistent with Storer et al. (2022) who recently proposed a global universal KE spectrum.

While we remain largely unable to offer explanations for our results, we have had some possible success in generalizing the enstrophy inertial-range theory. We can write a potential vorticity (PV) conservation equation for primitive equations, as can be done for a QG system. An unavoidable difference is in the scaling of PV between the two. We argue the dynamically significant difference involves a length scale, which results naturally in a steeper slope. To the extent we can assign dynamics to this, the importance of leading order vortex tube stretching is emphasized, and this is a phenomenon easily present in the highly stratified Gulf Stream region. Its relevance there also underscores, and supports, our earlier analysis suggesting QG dynamics do not adequately describe the separated Gulf Stream (Jamet et al., 2020). What we are unable to explain is the appearance of such steep slopes in locations where QG dynamics are expected to dominate.

In summary, we argue the North Atlantic eddy field is found in an unavoidably inhomogeneous environment, and exhibits characteristics that we currently have little theoretical guidance to interpret. Amongst the most confusing of our results is the inconsistent appearance and disappearance of anisotropy in our spectra.

The goals of this paper were to apply the wavelet-based technique for estimating the KE spectra and its spectral flux in realistic simulations where the usual assumptions of homogeneity and isotropy are clearly suspect. We have demonstrated that the wavelet method agrees well with the canonical Fourier approach but with the additional strengths of: i) negating the necessity for the data to be periodic, ii) flexibility in defining the wavenumber resolution via the scaling $s$, and iii) being able to extract the anisotropy in the flow through the rotational matrix $\mathsf{R}$ (cf. Uchida, Jamet, et al., 2023). It is also true that the eddy field is not expected to be stationary, although this is a topic that has not received any serious attention in this paper. Based on characteristic time scale arguments $\tau = \widetilde{E}_K/T_K$, one might expect the spectra at scales above 100 km to vary on the timescales of $\sim 10^6$ seconds $\simeq$ 10 days looking at Fig. 3a,c. Interestingly, the KE spectra seem remarkably stable over time whereas its tendency $T_K$ fluctuates rapidly with time (Fig. 4). While the ensemble technique permits the examination of the time dependence of eddy spectra, we have only touched upon it here. A more complete examination of the cross-scale eddy energy transfers is also desirable and possible within the ensemble framework. And with it, one can examine in more detail the eddy dynamics to address the question of anisotropic up and down-scale energy transfers. These are amongst the next set of items we intend to address.

A highly related and separate issue involves the examination of potential energy fluxes. We have here looked solely at the KE spectra. QG theory in its predictions for up and down scale cascades involves the combined kinetic and potential energies of the flow. However, in contrast to QG theory, where the resulting total energy is quadratic and positive definite, primitive equation settings in geopotential coordinates bring no such guarantees as the eddy dynamic enthalpy is a linear term ($h' \stackrel{\text{def}}{=} h - \langle h \rangle$ following the notation by Young, 2010); the TWA framework, on the other hand, suggests a (quadratic) positive-definite total eddy energy when the equation of state for density is linear (cf. Loose et al., 2022; Uchida, Jamet, et al., 2022, their Appendix A). How to address the role of potential energy in non-linear cascades and its impact on KE anisotropy is left for future work.

## Open Research

The open-source Fourier and wavelet Python packages are available via Github (Uchida, Rokem, et al., 2021; Uchida & Dewar, 2022). Jupyter notebooks used to conduct the analysis are available via Github (`https://github.com/roxyboy/NA-wavelet-notes/tree/master/Snapshots`; a DOI will be added upon acceptance of the manuscript). The simulation outputs are available on the Florida State University cluster (`http://ocean.fsu.edu/~qjamet/share/data/Uchida2021/`).

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## Appendix A A dynamically consistent buoyancy

For primitive equation models employing a non-linear equation of state (Jackett & McDougall, 1995), the choice of a materially conserved buoyancy variable is non-trivial and has been a subject of debate (e.g. Montgomery, 1937; Jackett & McDougall, 1997; McDougall & Jackett, 2005; de Szoeke & Springer, 2009; Klocker et al., 2009; Tailleux, 2016a, 2016b, 2017, 2021; McDougall et al., 2017; Lang et al., 2020; Stanley et al., 2021). Following Stanley (2019), Stanley and Marshall (2022) and Uchida, Jamet, et al. (2022), we opt for in-situ density anomaly to define the buoyancy $b = -g\delta/\rho_0$ where $\rho_0 = 999.8\,\text{kg}\,\text{m}^{-3}$ is the Boussinesq reference density prescribed in MITgcm. The in-situ density anomaly $\delta\,(= \rho - \breve{\rho})$ is defined by removing the effect of compressibility while retaining a straightforward dynamical relation to the horizontal gradients of hydrostatic pressure in Boussinesq fluids; this relation is crucial for dynamical consistency in how buoyancy relates to momentum. Taking $\mathcal{C}_s(z)$ as the maximum sound speed at each depth over the entire model domain and ensemble, we define $\breve{\rho}$ as:

$$\breve{\rho}(z) = -\int_z^0 \frac{\rho_0 g}{\mathcal{C}_s} dz + \rho_0, \tag{A1}$$

which reduces to $\breve{\rho}\big|_{z=0} = \rho_0$. $\delta$ is subsequently diagnosed as the difference between the in-situ density and $\breve{\rho}$. The interested reader is referred to Uchida, Jamet, et al. (2022) for further details. While more elaborate techniques may improve the material conservation of $\delta$ (and hence $b$), the relation to the dynamics is non-trivial for other density variables such as omega, neutral, orthobaric and topological density surfaces (Jackett & McDougall, 1997; McDougall & Jackett, 2005; Klocker et al., 2009; Stanley, 2019).

# Appendix B Spectral budget of the eddy flow

One of the desirable properties of taking the averaging over the ensemble dimension is that the wavelet transform and averaging operator commute with each other, i.e. $\langle\tilde{\cdot}\rangle = \widetilde{\langle\cdot\rangle}$, owing to the ensemble dimension being orthogonal to the spatiotemporal dimensions.

## B1 Eddy kinetic energy

The ensemble mean kinetic energy (KE; $K^{\#} = |\langle \boldsymbol{u} \rangle|^2/2$) equation is given as

$$K_t^{\#} + \langle \boldsymbol{v} \rangle \cdot \nabla K^{\#} = -\langle \boldsymbol{u} \rangle \cdot \nabla_{\mathrm{h}} \langle \phi \rangle - \langle u \rangle \nabla \cdot \langle \boldsymbol{v}' u' \rangle - \langle v \rangle \nabla \cdot \langle \boldsymbol{v}' v' \rangle + \langle \boldsymbol{u} \rangle \cdot \langle \boldsymbol{\mathcal{K}} \rangle \tag{B1}$$

where $\boldsymbol{v} = \boldsymbol{u} + w\mathbf{k}$ is the three-dimensional velocity, and $\boldsymbol{\mathcal{K}}$ is the non-conservative term consisting of dissipation and contribution from KPP. The total KE, on the other hand, is

$$K_t + \boldsymbol{v} \cdot \nabla K = -\boldsymbol{u} \cdot \nabla_{\mathrm{h}} \phi + \boldsymbol{u} \cdot \boldsymbol{\mathcal{K}} \,. \tag{B2}$$

Now, the total KE can be expanded as

$$\begin{aligned} K &= \frac{1}{2} |\langle \boldsymbol{u} \rangle + \boldsymbol{u}'|^2 \\ &= K^{\#} + \mathscr{K} + \langle \boldsymbol{u} \rangle \cdot \boldsymbol{u}' \,, \end{aligned} \tag{B3}$$

where $\mathscr{K} = |\boldsymbol{u}'|^2/2$ so

$$\begin{aligned} \langle \boldsymbol{v} \cdot \nabla K \rangle &= \big\langle (\langle \boldsymbol{v} \rangle + \boldsymbol{v}') \cdot \nabla (K^{\#} + \mathscr{K} + \langle \boldsymbol{u} \rangle \cdot \boldsymbol{u}') \big\rangle \\ &= \langle \boldsymbol{v} \rangle \cdot \nabla K^{\#} + \langle \boldsymbol{v}' \cdot \nabla \mathscr{K} \rangle + \langle \boldsymbol{v} \rangle \cdot \nabla \langle \mathscr{K} \rangle + \langle \boldsymbol{v}' \cdot \nabla (\langle \boldsymbol{u} \rangle \cdot \boldsymbol{u}') \rangle \,. \end{aligned} \tag{B4}$$

Hence, subtracting (B1) from the ensemble mean of (B2) yields

$$\langle \mathscr{K} \rangle_t = -\langle \boldsymbol{u}' \cdot \nabla_\mathrm{h} \phi' \rangle - \langle \boldsymbol{v} \cdot \nabla \mathscr{K} \rangle - \langle \boldsymbol{v}' \cdot \nabla (\langle \boldsymbol{u} \rangle \cdot \boldsymbol{u}') \rangle + \langle u \rangle \nabla \cdot \langle \boldsymbol{v}' u' \rangle + \langle v \rangle \nabla \cdot \langle \boldsymbol{v}' v' \rangle + \langle \boldsymbol{u}' \cdot \boldsymbol{\mathcal{K}}' \rangle \,, \quad \text{(B5)}$$

where we see the mean flow and eddies exchanging energy via the term $\langle u \rangle \nabla \cdot \langle \boldsymbol{v}' u' \rangle + \langle v \rangle \nabla \cdot \langle \boldsymbol{v}' v' \rangle$, which can be interpreted as a eddy forcing onto the mean flow.

In order to achieve machine precision in closing the budget using the MITgcm `diagnostics` package outputs, we rearrange Equation (B5) as

$$\begin{aligned}
\langle \mathscr{K} \rangle_t &= -\langle \boldsymbol{u}' \cdot \nabla_\mathrm{h} \phi' \rangle - \langle \boldsymbol{v} \cdot \nabla \mathscr{K} \rangle - \underbrace{\left( \langle u' \boldsymbol{v}' \rangle \cdot \nabla \langle u \rangle + \langle v' \boldsymbol{v}' \rangle \cdot \nabla \langle v \rangle + \langle u \rangle \nabla \cdot \langle \boldsymbol{v}' u' \rangle + \langle v \rangle \nabla \cdot \langle \boldsymbol{v}' v' \rangle \right)}_{= \langle \boldsymbol{v}' \cdot \nabla (\langle \boldsymbol{u} \rangle \cdot \boldsymbol{u}') \rangle} \\
&\qquad + \langle u \rangle \nabla \cdot \langle \boldsymbol{v}' u' \rangle + \langle v \rangle \nabla \cdot \langle \boldsymbol{v}' v' \rangle + \langle \boldsymbol{u}' \cdot \boldsymbol{\mathcal{K}}' \rangle \\
&= -\langle \boldsymbol{u}' \cdot \nabla_\mathrm{h} \phi' \rangle - \underbrace{\left( \langle u' (\boldsymbol{v} \cdot \nabla u)' \rangle + \langle v' (\boldsymbol{v} \cdot \nabla v)' \rangle + \langle u \rangle \nabla \cdot \langle \boldsymbol{v}' u' \rangle + \langle v \rangle \nabla \cdot \langle \boldsymbol{v}' v' \rangle \right)}_{= \langle \boldsymbol{v} \cdot \nabla \mathscr{K} \rangle + \langle \boldsymbol{v}' \cdot \nabla (\langle \boldsymbol{u} \rangle \cdot \boldsymbol{u}') \rangle} \\
&\qquad + \langle u \rangle \nabla \cdot \langle \boldsymbol{v}' u' \rangle + \langle v \rangle \nabla \cdot \langle \boldsymbol{v}' v' \rangle + \langle \boldsymbol{u}' \cdot \boldsymbol{\mathcal{K}}' \rangle \,,
\end{aligned}$$

where we have also grouped all the divergence terms together as they are neither a source nor sink of energy and only redistribute it. The spectral budget of eddy KE, therefore, becomes

$$\begin{aligned}
\underbrace{\frac{1}{C_\Xi} \langle \tilde{\boldsymbol{u}'}^* \cdot \tilde{\boldsymbol{u}'_t} \rangle \, x_0^2 \kappa}_{T_K} = & \underbrace{- \frac{1}{C_\Xi} \langle \tilde{\boldsymbol{u}'}^* \cdot \widetilde{\nabla_\mathrm{h} \phi'} \rangle \, x_0^2 \kappa}_{P_K} \\
& \underbrace{- \frac{1}{C_\Xi} \left( \langle \tilde{u'}^* \widetilde{(\boldsymbol{v} \cdot \nabla u)'} \rangle + \langle \tilde{v'}^* \widetilde{(\boldsymbol{v} \cdot \nabla v)'} \rangle + \widetilde{\langle u \rangle}^* \nabla \cdot \widetilde{\langle \boldsymbol{v}' u' \rangle} + \widetilde{\langle v \rangle}^* \nabla \cdot \widetilde{\langle \boldsymbol{v}' v' \rangle} \right) x_0^2 \kappa}_{A_K} \\
& + \underbrace{\frac{1}{C_\Xi} \left( \widetilde{\langle u \rangle}^* \nabla \cdot \widetilde{\langle \boldsymbol{v}' u' \rangle} + \widetilde{\langle v \rangle}^* \nabla \cdot \widetilde{\langle \boldsymbol{v}' v' \rangle} \right) x_0^2 \kappa}_{\mathrm{MtE}_K} + \underbrace{\frac{1}{C_\Xi} \langle \tilde{\boldsymbol{u}'}^* \cdot \widetilde{\boldsymbol{\mathcal{K}}'} \rangle \, x_0^2 \kappa}_{\mathcal{K}_K} \,, \quad \text{(B6)}
\end{aligned}$$

(cf. (10)) where $\mathrm{MtE}_K$ is the KE exchange between the mean and eddy flow. $C_\Xi$ is computed using the `xrft` Python package (Uchida, Rokem, et al., 2021). The horizontal KE spectral flux often examined by other studies is encapsulated in $A_K$ of (B6).

### B2 Eddy enstrophy

The enstrophy equation is slightly more tractable than the KE equation so we start off with the mean and eddy Ertel's PV equations neglecting the non-conservative terms

$$\langle q \rangle_t + \nabla \cdot (\langle \boldsymbol{v} \rangle \langle q \rangle) = -\nabla \cdot \langle \boldsymbol{v}' q' \rangle \,, \quad \text{(B7)}$$

$$q'_t + \nabla \cdot (\boldsymbol{v} q') + \nabla \cdot (\boldsymbol{v}' \langle q \rangle) = \nabla \cdot \langle \boldsymbol{v}' q' \rangle \,, \quad \text{(B8)}$$

where the mean flow and eddies exchange PV via the term $\nabla\cdot\langle \boldsymbol{v}'q'\rangle$. Multiplying each by $\langle q\rangle$ and $q'$ and taking the ensemble mean gives the mean and eddy enstrophy equations

$$\langle q\rangle\langle q\rangle_t + \langle q\rangle\nabla\cdot(\langle \boldsymbol{v}\rangle\langle q\rangle) = -\langle q\rangle\nabla\cdot\langle \boldsymbol{v}'q'\rangle\,, \tag{B9}$$

$$\langle q'q'_t\rangle + \langle q'\nabla\cdot(\boldsymbol{v}q')\rangle + \underbrace{\langle q'\nabla\cdot(\boldsymbol{v}'\langle q\rangle)\rangle}_{=\langle \boldsymbol{v}'\cdot\nabla(\langle q\rangle q')\rangle - \langle q\rangle\nabla\cdot\langle \boldsymbol{v}'q'\rangle} = 0\,. \tag{B10}$$

In a similar manner to the EKE equation, the eddy enstrophy equation can also be rearranged as

$$\langle q'q'_t\rangle + \underbrace{\langle q'(\boldsymbol{v}\cdot\nabla q)'\rangle + \langle q\rangle\nabla\cdot\langle \boldsymbol{v}'q'\rangle}_{=\langle q'\nabla\cdot(\boldsymbol{v}q')\rangle + \langle \boldsymbol{v}'\cdot\nabla(\langle q\rangle q')\rangle} = \langle q\rangle\nabla\cdot\langle \boldsymbol{v}'q'\rangle\,. \tag{B11}$$

Thus, the eddy enstrophy budget in wavelet domain becomes

$$\frac{1}{C_\Xi}\langle \tilde{q'}^*\tilde{q'_t}\rangle x_0^2\kappa + \frac{1}{C_\Xi}\left[\langle \tilde{q'}^*\widetilde{(\nabla\cdot\boldsymbol{v}q)'}\rangle + \widetilde{\langle q\rangle}^*\nabla\cdot\widetilde{\langle \boldsymbol{v}'q'\rangle}\right]x_0^2\kappa = \frac{1}{C_\Xi}\widetilde{\langle q\rangle}^*\nabla\cdot\widetilde{\langle \boldsymbol{v}'q'\rangle}x_0^2\kappa\,. \tag{B12}$$

We note that the enstrophy budget does not close to machine precision due to the lack of `diagnostics` outputs.

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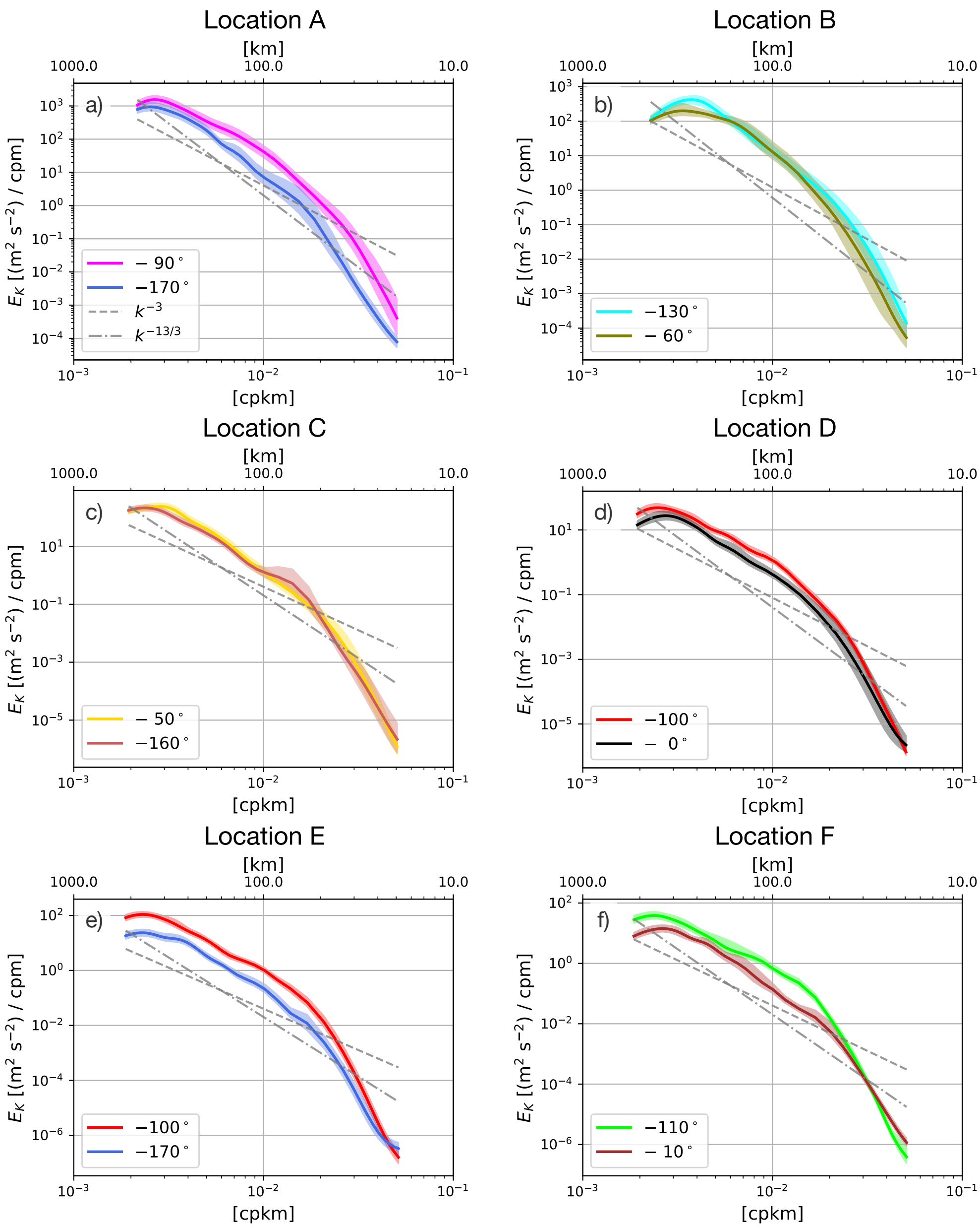


**Figure 6.** KE spectra $\widetilde{E}_K(\phi, k)$ plotted along the orientation of maximum and minimum energy from $z = -452$ m at all six locations (A-F) on January 1, 1967. The angles are color coded. The land cells are not interpolated over and data are not windowed prior to taking the wavelet transforms, differing from Fig. 3a. The 95% bootstrap confidence intervals are shown in colored shadings. Power laws with the slope of −3 and −13/3 are indicated with the grey dashed and dotted-dashed curves respectively.

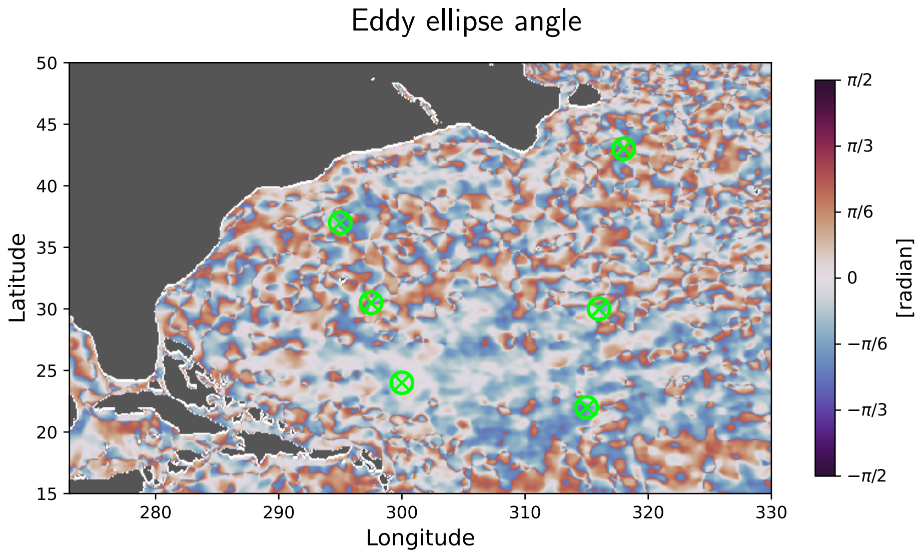


**Figure 7.** The eddy ellipse angle $\vartheta$ at $z = -452\,\mathrm{m}$ on January 1, 1967. The lime-colored $\otimes$ indicate locations A–F.

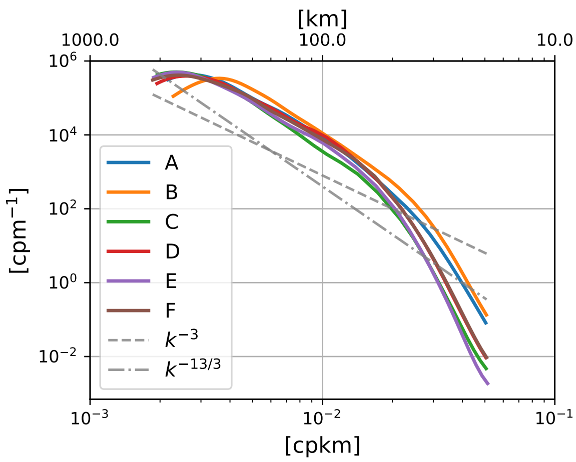


**Figure 8.** Isotropic KE spectra normalized by their respective energy for locations A–F on January 1, 1967.