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No new data are presented in this study. The code is openly available at https://github.com/agstub/linear-shelf-melt (see Jupyter notebooks) and will be archived with Zenodo prior to publication.

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Linear analysis of ice-shelf topography response to basal melting and freezing

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Floating ice shelves in Antarctica and Greenland limit land-ice contributions to sea level rise by resisting the flow of grounded ice. Melting at the surface and base of ice shelves can lead to destabilisation by promoting thinning and fracturing. Basal melting often results in channelised features that manifest as surface topography due to buoyancy. The assumption of hydrostatic flotation commonly underlies estimates of basal melting rates. However, numerical simulations and ice-penetrating radar data have shown that narrow topographic features do not necessarily satisfy the local flotation condition. Here, we introduce a linearised model for ice-shelf topographic response to basal melting perturbations to quantify deviations from hydrostatic flotation and the stability of topography. While hydrostatic flotation is the dominant behaviour at wavelengths greater than the ice thickness, ice elevation anomalies diminish relative to the perfect flotation condition at smaller wavelengths. The linearised analysis shows that steady solutions are stable. However, we do not consider physical effects such as fracturing, which could lead to instability. We find a Green's function for one-dimensional steady-state solutions in the absence of background advection. We validate the linearised analysis by comparing numerical solutions to a nonlinear ice-flow model with steady-state solutions obtained via the Green's function.

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1. Introduction

Ice shelves and floating ice tongues in Greenland and Antarctica slow ice-mass loss and land-ice contributions to sea level rise by buttressing the flow of grounded ice [1–4]. Subglacial outflow near grounding lines and warm seawater intrusions can cause localised melting at the base of ice shelves [5–8]. Topographic features that are elongated in one direction, commonly called ice-shelf channels, have been found on many ice shelves and arise from localised melting at the base [7,9]. These anomalous topographic features manifest at the ice-shelf surface due to buoyancy. Ice-shelf channels have been associated with a variety of features including rifts, subglacial outflow, persistent polynyas, and terracing of the basal surface [9–11]. While channel formation is driven by persistent plumes of buoyant water, the dynamics vary and are influenced by the plume source (i.e., oceanic or subglacial), the presence of heterogeneous surfaces, and ice deformation [7,12–15].

Although melting dominates the basal mass balance of most Antarctic ice shelves [16], altimetry, ice-penetrating radar, and numerical models have shown that many ice shelves have regions where water is freezing to the base [17–22]. Borehole drilling has shown that thick units of frozen-on marine ice exist beneath the Amery Ice Shelf [23–25] and the Ronne-Filchner Ice Shelf [26,27]. Marine ice can have different rheological properties than meteoric ice due to differences in crystal size and orientation as well as the presence of interstitial brine [28]. Marine ice has also been found within suture zones, boundaries within ice shelves that separate different flow units, and may contribute to the structural stability of ice shelves by impeding fracture propagation [29–33]. For these reasons, quantifying the mechanical effects of marine ice accretion and refining inferences from ice-shelf topography remain important areas of investigation.

The melting or freezing rate at the base of ice shelves can be estimated from surface observations under the assumptions that the ice is undergoing hydrostatic flotation and that the horizontal velocity is depth-independent. In this limit, the surface elevation of the ice shelf relative to sea level only depends on the ice thickness and the densities of ice and water. However, since ice flows like a viscous fluid over long time scales [34–36], deviatoric (bridging) stresses in the ice can lead to a non-hydrostatic state where the ice-surface elevation is diminished relative to the hydrostatic elevation [37–39]. Ice-penetrating radar data from the Fimbul Ice Shelf, East Antarctica, shows that narrow basal channels can be associated with diminished topographic anomalies [14]. Similarly, hydrostatic imbalance was found in the vicinity of a channel on the Getz Ice Shelf, West Antarctica, by comparing hydrostatic flotation thickness estimates with ice thickness obtained from radar profiles along NASA Operation IceBridge transects [40]. When hydrostatic balance is not satisfied exactly, the velocity is not depth-independent due to the presence of a secondary flow that balances the formation of basal channels. Deviation of the surface velocity and elevation from hydrostatic balance produces errors in melting-rate estimates that are based solely on surface observations [39,40].

Previous models of subglacial channels have been used to explore deviations from hydrostatic flotation [38,39], coupling with ocean physics [13,41–43], inclusion of viscoelastic effects [44, 45], and evolution of ice-shelf stratigraphy [46]. In a similar setting, a small-perturbation approximation of the Stokes equations has been used to model the evolution of crevasses in ice shelves over long time scales in relation to oceanic forcing [47,48]. While detailed analyses of Stokes-based perturbation models have been developed previously for grounded ice [49–52], similar analyses for floating ice shelves have not been developed beyond the model for crevasse evolution in a floating viscous medium that is undergoing extension [47].

Here, we develop a detailed linearised analysis of the ice-shelf topographic response to melting or freezing perturbations at the base. We derive a linearised model based on a small perturbation approximation of a fully nonlinear Stokes-flow model. We analyse the problem via a spectral decomposition, discussing in detail long-wavelength singularities that are similar to those found previously [47]. In particular, we show that solutions are well-defined despite the singularities. We quantify conditions under which ice-elevation anomalies approach perfect flotation as well as the stability of steady-state solutions. To test the validity of the small-perturbation approach,

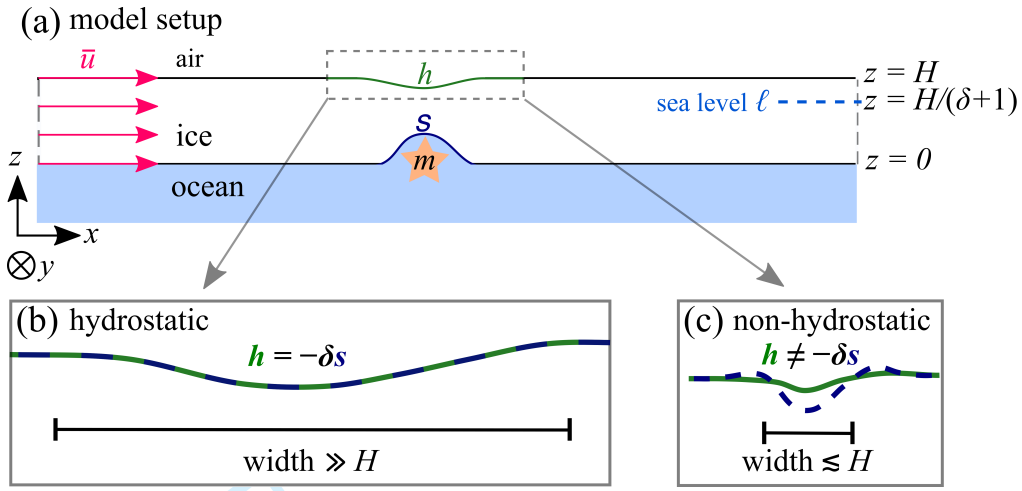

Figure 1. (a) Sketch of model variables. The ice-surface elevation perturbation is h, the elevation perturbation at the ice-water interface is s, the melting rate is m, the ice thickness is H, the sea level is ℓ, the flotation factor is $\delta = \rho_w/\rho_i - 1$, and advection from the background horizontal velocity is $\bar{u}$. The background flow (red arrows) is in the x direction, and the y direction points into the page. (b) Example of a topographic anomaly that is wide relative to the ice thickness and satisfies the perfect flotation condition ($h = -\delta s$) locally. (c) Example of a narrow topographic anomaly that does not satisfy the perfect flotation condition locally.

we compare numerical solutions obtained by solving the fully nonlinear ice-flow problem with a finite element method to semi-analytical steady-state solutions that we obtain with a Green's function. We conclude by discussing implications and avenues for further research.

2. Model derivation

(a) Governing equations

First, we outline a general Stokes model for ice-shelf flow. We assume that the domain is an ice shelf of finite thickness H and infinite horizontal extent (Figure 1a). The domain is defined by $|x| < \infty$, $|y| < \infty$, and $s \le z \le h$, where h and s are the upper and lower surfaces of the ice shelf, respectively. We assume that ice deforms according to the incompressible Stokes equations

$$-\boldsymbol{\nabla} p + \boldsymbol{\nabla}\cdot(2\eta\dot{\boldsymbol{\varepsilon}}) = \rho_i \boldsymbol{g} \tag{2.1}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0, \tag{2.2}$$

where $\mathbf{u} = [u, v, w]^T$ is the velocity, $\dot{\boldsymbol{\varepsilon}} = \frac{1}{2}\left(\boldsymbol{\nabla}\boldsymbol{u} + \boldsymbol{\nabla}\boldsymbol{u}^T\right)$ is the strain rate, η is the ice viscosity, p is the pressure, ρ_i is the ice density, and $\boldsymbol{g} = [0, 0, -g]^T$ is gravitational acceleration with magnitude g. We relate the ice viscosity to strain rate through a regularised Glen's flow law [34–36]

$$\eta = \frac{1}{2} B(|\dot{\boldsymbol{\varepsilon}}|^2 + \nu)^{\frac{1-n}{2n}}, \tag{2.3}$$

where n is the stress exponent, B is the ice hardness parameter, $|\dot{\boldsymbol{\varepsilon}}| = \sqrt{\frac{1}{2}\dot{\boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}}}$ is the second invariant, and ν is a regularization parameter that ensures finite viscosity in the limit of zero strain rate.

We assume a stress-free condition at the upper surface ($z = h$), which is equivalent to

$$[2\eta\dot{\boldsymbol{\varepsilon}} - p\mathbf{I}] \cdot \boldsymbol{n} = \mathbf{0}, \tag{2.4}$$

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where $\boldsymbol{n}$ is an upward-pointing unit normal to the ice boundary and $\mathbf{I}$ is the identity tensor. The normal stress at the ice-water interface ($z = s$) is hydrostatic and the shear stress vanishes,

$$[2\eta\dot{\boldsymbol{\varepsilon}} - p\mathbf{I}] \cdot \boldsymbol{n} = \rho_{\mathrm{w}} g(\ell - s)\boldsymbol{n}, \tag{2.5}$$

where ρ_{w} is the water density and $z = \ell$ is the (far-field) sea level. Below, we set the sea level to $\ell = (\rho_{\mathrm{i}}/\rho_{\mathrm{w}})H$, where H is the ice thickness in the uniform reference state.

The upper and lower surfaces of the ice shelf evolve according to kinematic equations that are coupled to the Stokes flow equations. The upper surface, $z = h(x, y, t)$, evolves according to

$$\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y} = w + a, \tag{2.6}$$

where a denotes accumulation or ablation at the ice-shelf surface. Similarly, the lower surface, $z = s(x, y, t)$, evolves according to

$$\frac{\partial s}{\partial t} + u\frac{\partial s}{\partial x} + v\frac{\partial s}{\partial y} = w + m, \tag{2.7}$$

where m is the basal melting or freezing rate (Figure 1). We also require that all fields approach appropriate far-field reference states in the limits $|x| \to \infty$ and $|y| \to \infty$, which are naturally incorporated in the perturbation-based framework described below. While our focus is on a spectral analysis of the linearised problem, we also compare the results to solutions obtained by solving the fully nonlinear problem (2.1)-(2.7) with a finite element method, subject to additional boundary conditions on the side-walls of the computational domain.

(b) Perturbation equations

Next, we derive a linearisation of the problem (2.1)-(2.7). We take perturbations relative to a reference state characterised by uniform flow in the x direction, uniform ice thickness, cryostatic pressure, perfect flotation, and no surface accumulation or basal melting (Figure 1). Denoting the reference states with bars, these conditions are given by

$$\begin{aligned} &\bar{u} \geq 0, \quad \bar{v} = 0, \quad \bar{w} = 0, \quad \bar{p} = \rho_{\mathrm{i}} g(H - z), \quad \bar{\eta} = \frac{1}{2} B \nu^{\frac{1-n}{2n}}, \\ &\bar{h} = H, \quad \bar{s} = 0, \quad \bar{\ell} = \frac{\rho_{\mathrm{i}}}{\rho_{\mathrm{w}}} H, \quad \bar{a} = 0, \quad \bar{m} = 0. \end{aligned} \tag{2.8}$$

We introduce small, localised perturbations (denoted by asterisks) to the reference states via

$$\begin{aligned} &u = \bar{u} + \epsilon u_*, \quad v = \bar{v} + \epsilon v_*, \quad w = \bar{w} + \epsilon w_* \\ &p = \bar{p} + \epsilon p_*, \quad s = \bar{s} + \epsilon s_*, \quad h = \bar{h} + \epsilon h_*, \quad m = \bar{m} + \epsilon m_*, \end{aligned} \tag{2.9}$$

where ϵ is a small parameter. For simplicity, we do not consider perturbations to the surface accumulation (or ablation) rate. We revisit possible sources of surface-mass balance perturbations and related model extensions in the discussion. We insert the perturbations (2.9) into (2.1)-(2.2) and discard the $O(\epsilon^2)$ terms to obtain a homogeneous, linear Stokes problem for the perturbed fields,

$$-\boldsymbol{\nabla} p_* + \bar{\eta}\nabla^2 \boldsymbol{u}_* = \mathbf{0} \tag{2.10}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{u}_* = 0. \tag{2.11}$$

Viscosity variations associated with Glen's flow law do not contribute to the first-order perturbation equations because the uniform-flow reference states have a strain rate of zero. Although not explored here, an extensional reference state results in a perturbation problem that depends on the stress exponent and background strain rates [47].

The surface kinematic equations (2.6) and (2.7) reduce to

$$\frac{\partial h_*}{\partial t} + \bar{u}\frac{\partial h_*}{\partial x} = w_* \tag{2.12}$$

$$\frac{\partial s_*}{\partial t} + \bar{u}\frac{\partial s_*}{\partial x} = w_* + m_*. \tag{2.13}$$

To account for changes in ice geometry, we linearise the upper and lower surface boundary conditions at $z = H + h_*$ and $z = s_*$ onto $z = H$ and $z = 0$, respectively. The vanishing normal stress condition at $z = H + h_*$ in (2.4) is approximated at $z = H$ by

$$p_* - 2\bar{\eta}\frac{\partial w_*}{\partial z} = h_* \rho_\mathrm{i}\, g. \tag{2.14}$$

Similarly, the hydrostatic normal stress condition at the base in (2.5) becomes

$$p_* - 2\bar{\eta}\frac{\partial w_*}{\partial z} = -s_* \Delta\rho\, g \tag{2.15}$$

where $\Delta\rho = \rho_\mathrm{w} - \rho_\mathrm{i}$ is the density difference between ice and water. The vanishing shear-stress conditions in (2.4) and (2.5) take the form

$$\begin{aligned} \frac{\partial u_*}{\partial z} + \frac{\partial w_*}{\partial x} &= 0 \\ \frac{\partial v_*}{\partial z} + \frac{\partial w_*}{\partial y} &= 0 \end{aligned} \tag{2.16}$$

which hold at both the surface ($z = H$) and the base ($z = 0$). We drop the asterisk subscripts on the perturbed fields below.

A primary goal in the analysis below is to determine when the perfect flotation condition

$$h = -\delta s \tag{2.17}$$

is satisfied locally, where we have defined the flotation factor

$$\delta \equiv \frac{\rho_\mathrm{w}}{\rho_\mathrm{i}} - 1. \tag{2.18}$$

The perfect flotation condition (2.17) corresponds to a purely hydrostatic state (Figure 1b). However, deviatoric (bridging) stresses within the ice can support departures from this state [37–39](Figure 1c). While our analysis is focused on ice-shelf topography, testing whether or not the horizontal velocity is depth-independent is also possible in this framework.

(c) Fourier transform approach

We apply Fourier transforms with respect to the horizontal coordinates (x, y) to simplify the system (2.10)-(2.16). The Stokes equations (2.10)-(2.11), written component-wise, transform to

$$-ik_x\widehat{p} + \bar{\eta}\left(-k^2\widehat{u} + \frac{\partial^2\widehat{u}}{\partial z^2}\right) = 0 \tag{2.19}$$

$$-ik_y\widehat{p} + \bar{\eta}\left(-k^2\widehat{v} + \frac{\partial^2\widehat{v}}{\partial z^2}\right) = 0 \tag{2.20}$$

$$-\frac{\partial\widehat{p}}{\partial z} + \bar{\eta}\left(-k^2\widehat{w} + \frac{\partial^2\widehat{w}}{\partial z^2}\right) = 0 \tag{2.21}$$

$$ik_x\widehat{u} + ik_y\widehat{v} + \frac{\partial\widehat{w}}{\partial z} = 0, \tag{2.22}$$

where $\boldsymbol{k} = [k_x, k_y]^\mathrm{T}$ is the horizontal wavevector with magnitude $k = \sqrt{k_x^2 + k_y^2}$. Equations (2.19)-(2.22) reduce to a fourth-order equation for the transformed vertical velocity [52],

$$\frac{\partial^4\widehat{w}}{\partial z^4} - 2k^2\frac{\partial^2\widehat{w}}{\partial z^2} + k^4\widehat{w} = 0. \tag{2.23}$$

The general solution to (2.23) is

$$\widehat{w} = \frac{c_1}{k} e^{kz} + \frac{c_2}{k} e^{-kz} + c_3 z e^{kz} + c_4 z e^{-kz}, \tag{2.24}$$

where the coefficients c_j depend on k. To determine the coefficients in (2.23), we rewrite all of the boundary conditions in terms of $\widehat{w}$. The vanishing shear-stress conditions (2.16) along with the transformed incompressibility condition (2.22) imply that

$$\frac{\partial^2 \widehat{w}}{\partial z^2} + k^2 \widehat{w} = 0 \tag{2.25}$$

at $z = 0$ and $z = H$. The transformed normal-stress condition at the upper surface can be rewritten using (2.19)-(2.22) as

$$\bar{\eta} \left(3k^2 \frac{\partial \widehat{w}}{\partial z} - \frac{\partial^3 \widehat{w}}{\partial z^3} \right) = -k^2 \rho_{\mathrm{i}} g \widehat{h}, \tag{2.26}$$

and, similarly, the transformed normal-stress condition at the base becomes

$$\bar{\eta} \left(3k^2 \frac{\partial \widehat{w}}{\partial z} - \frac{\partial^3 \widehat{w}}{\partial z^3} \right) = k^2 \Delta\rho g \widehat{s}. \tag{2.27}$$

We insert the formula for $\widehat{w}$ (2.24) into the boundary conditions (2.25)-(2.27) to obtain a linear system for the coefficients c_j,

$$\begin{bmatrix} e^{k'} & -e^{-k'} & k' e^{k'} & -k' e^{-k'} \\ e^{k'} & e^{-k'} & (k'+1) e^{k'} & (k'-1) e^{-k'} \\ 1 & -1 & 0 & 0 \\ 1 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \frac{\rho_{\mathrm{i}} g}{2\bar{\eta}} \begin{bmatrix} -\widehat{h} \\ 0 \\ \delta \widehat{s} \\ 0 \end{bmatrix}, \tag{2.28}$$

where the scaled wavevector magnitude,

$$k' = kH, \tag{2.29}$$

appears due to evaluation of the boundary conditions at $z = H$.

We solve the linear system (2.28) for the coefficients c_j for all $k' > 0$. In the limit $k' \to 0$, the system is singular, meaning that the long-wavelength component of $\widehat{w}$ is not uniquely determined by the momentum balance alone. We show that solutions to the coupled problem are well-defined in the limit $k' \to 0$ because the surface evolution equations (2.12) and (2.13) impose additional constraints on the vertical velocity in relation to the surface elevations and melting rate. A necessary condition for the existence of solutions to (2.28) is that $\widehat{h} \to -\delta \widehat{s}$ as $k' \to 0$, or, equivalently,

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} h \, \mathrm{d}x \, \mathrm{d}y = -\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \delta s \, \mathrm{d}x \, \mathrm{d}y. \tag{2.30}$$

The condition (2.30) implies that that the perturbations will satisfy the perfect flotation condition, $h = -\delta s$, on average, but not necessarily locally.

Substituting the coefficients c_j into (2.24), we obtain an expression for the vertical velocity at the upper surface,

$$\widehat{w}|_{z=H} = -\mathsf{R} \widehat{h} - \mathsf{B} \delta \widehat{s}, \tag{2.31}$$

where the surface relaxation function R is given by

$$\mathsf{R} = \frac{1}{t_{\mathrm{r}}} \frac{e^{4k'} + 4k' e^{2k'} - 1}{k' [e^{4k'} - 2(1 + 2k'^2) e^{2k'} + 1]}, \tag{2.32}$$

and the buoyancy transfer function B is given by

$$\mathsf{B} = \frac{1}{t_{\mathrm{r}}} \frac{2(k'+1) e^{3k'} + 2(k'-1) e^{k'}}{k' [e^{4k'} - 2(1 + 2k'^2) e^{2k'} + 1]}. \tag{2.33}$$

In equations (2.32) and (2.33), we have introduced

$$t_\mathrm{r} \equiv \frac{2\bar{\eta}}{\rho_\mathrm{i} g H}, \tag{2.34}$$

which is the characteristic timescale for decay of surface topography in an infinitely thick viscous medium without buoyancy forcing [53]. In the analysis below, we compare the timescale t_r to a natural timescale that arises from competition between topographic relaxation and buoyancy.

Combining (2.12) and (2.31), the upper surface evolves according to

$$\frac{\partial \widehat{h}}{\partial t} + [ik_x\bar{u} + \mathsf{R}]\,\widehat{h} = -\mathsf{B}\delta\widehat{s}. \tag{2.35}$$

Similarly, the vertical velocity at the lower surface is given by

$$\widehat{w}|_{z=0} = -\mathsf{R}\delta\widehat{s} - \mathsf{B}\widehat{h}, \tag{2.36}$$

which, along with (2.13), leads to the evolution equation

$$\frac{\partial \widehat{s}}{\partial t} + [ik_x\bar{u} + \delta\mathsf{R}]\widehat{s} = \widehat{m} - \mathsf{B}\widehat{h}. \tag{2.37}$$

The relaxation function (2.32) and buoyancy function (2.33) are singular in the limit $k \to 0$ (Figure 2). Physically, these singularities imply that topographic relaxation and buoyancy adjustments occur instantaneously at long wavelengths under the assumption of an infinitely expansive ice shelf. Laurent expansions of R and B indicate that they grow like k^{-4} as $k \to 0$ in accordance with the similar crevasse evolution model [47]. Below, we will show that the asymptotic behaviour $\mathsf{R} \sim \mathsf{B}$ at long wavelengths is associated with a hydrostatic state because topographic relaxation is perfectly balanced by buoyancy. At short wavelengths, the behaviour $\mathsf{R} \gg \mathsf{B}$ can lead to diminished surface topography because the presence of deviatoric stresses causes the topographic relaxation to exceed the buoyancy forcing (Figure 2). For the purpose of computing the solutions below, we regularise the functions R and B (Appendix A). However, we show that the solutions are well-defined despite the singularities.

(d) Scaling

Now, we introduce a scaling for the model equations (2.35) and (2.37). We scale the variables according to

$$\begin{aligned} &h = Hh', \quad s = Hs', \quad x = Hx', \quad k_x = H^{-1}k_x', \quad t = t_\mathrm{r}t', \\ &\mathsf{R} = {t_\mathrm{r}}^{-1}\mathsf{R}', \quad \mathsf{B} = {t_\mathrm{r}}^{-1}\mathsf{B}', \quad m = {t_\mathrm{r}}^{-1}Hm', \end{aligned} \tag{2.38}$$

where the viscous relaxation time scale t_r is defined in (2.34). Omitting primes, equations (2.35) and (2.37) become

$$\frac{\partial \widehat{h}}{\partial t} + [ik_x\alpha + \mathsf{R}]\,\widehat{h} = -\mathsf{B}\delta\widehat{s} \tag{2.39}$$

$$\frac{\partial \widehat{s}}{\partial t} + [ik_x\alpha + \delta\mathsf{R}]\widehat{s} = \widehat{m} - \mathsf{B}\widehat{h}, \tag{2.40}$$

where the parameter

$$\alpha = \frac{\bar{u}t_\mathrm{r}}{H} \tag{2.41}$$

is the background flow speed relative the the characteristic velocity scale H/t_r. Below, we show that solutions evolve over a longer time scale $t_\mathrm{e} \equiv t_\mathrm{r} \times 2\left(1 + \delta^{-1}\right)$, which coincides with the characteristic time scale in shallow approximations of ice-shelf flow [54].

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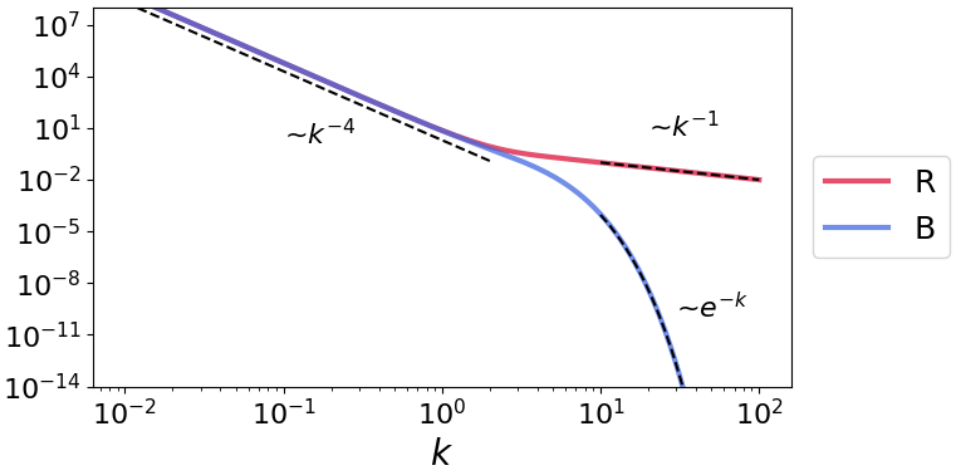

Figure 2. Graphs of R (2.32) and B (2.33) as functions of k (nondimensional), scaled by t_{r}^{-1}, along with their asymptotic behaviours as $k \to 0$ and $k \to \infty$ (dashed lines).

3. Analysis

Next, we derive and analyse general solutions to the problem (2.39)-(2.40) via a spectral decomposition. In particular, we quantify when solutions can be expected to deviate from the perfect flotation condition, $h = -\delta s$. We then derive steady solution formulas and find a Green's function for a one-dimensional melt channel in the absence of background advection ($\alpha = 0$). While our focus is on topography, velocity solutions can be derived an analysed once the elevations h and s are obtained. We provide expressions for the vertical and horizontal velocity solutions in Appendix B and C, respectively.

(a) General solutions

We rewrite the system (2.39)-(2.40) as

$$\frac{\partial \boldsymbol{y}}{\partial t} = \mathbf{A}\boldsymbol{y} + \boldsymbol{b}, \tag{3.1}$$

where

$$\boldsymbol{y} = \begin{bmatrix} \widehat{h} \\ \widehat{s} \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 0 \\ \widehat{m} \end{bmatrix}, \quad \mathbf{A} = -\begin{bmatrix} ik_x\alpha + \mathsf{R} & \delta\mathsf{B} \\ \mathsf{B} & ik_x\alpha + \delta\mathsf{R} \end{bmatrix}. \tag{3.2}$$

To derive solutions and analyse their basic properties, we diagonalise the matrix $\mathbf{A}$ according to

$$\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}, \quad \mathbf{P} = \begin{bmatrix} | & | \\ \boldsymbol{\varphi}_- & \boldsymbol{\varphi}_+ \\ | & | \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} \lambda_- & 0 \\ 0 & \lambda_+ \end{bmatrix}. \tag{3.3}$$

The eigenvalues ($\lambda_\pm$) and eigenvectors ($\boldsymbol{\varphi}_\pm$) of $\mathbf{A}$ in (3.3) are

$$\lambda_\pm = -\left(ik_x\alpha - \frac{\delta+1}{2}\mathsf{R} \mp \frac{\mu}{2}\right) \tag{3.4}$$

$$\boldsymbol{\varphi}_\pm = \begin{bmatrix} \frac{\mathsf{R}(1-\delta)\mp\mu}{2\mathsf{B}} \\ 1 \end{bmatrix} \tag{3.5}$$

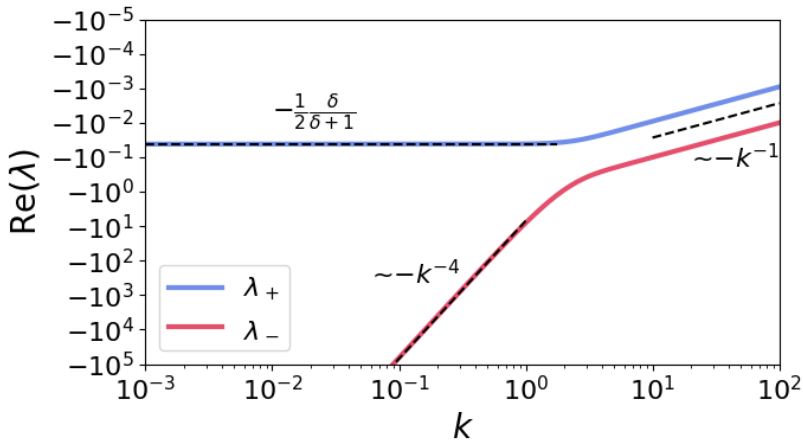

Figure 3. Real parts of the eigenvalues ($\lambda_\pm$) of **A** along with their asymptotic behaviours as $k \to 0$ and $k \to \infty$ (dashed lines).

where we have defined

$$\mu = \sqrt{4\delta \mathrm{B}^2 + \mathrm{R}^2(\delta - 1)^2}. \tag{3.6}$$

The eigenvector corresponding to the dominant eigenvalue λ_+ has the limit $\boldsymbol{\varphi}_+/\|\boldsymbol{\varphi}_+\| \to [-\delta, 1]^T$ as $k \to 0$, which drives the solution towards the perfect flotation condition $h = -\delta s$ (Figure 3 and Figure 4). The other eigenpair encourages deviations from perfect flotation, but this effect decays as $k \to 0$ since λ_- becomes very negative (Figure 3). However, we show below that these effects can be important at short wavelengths. The normalised eigenvectors approach $\boldsymbol{\varphi}_+/\|\boldsymbol{\varphi}_+\| \to [0, 1]^T$ and $\boldsymbol{\varphi}_-/\|\boldsymbol{\varphi}_-\| \to [1, 0]^T$ as $k \to \infty$, meaning that the surface and basal responses becomes decoupled at short wavelengths because perturbations are not transmitted through the ice column.

Steady-state solutions are stable because the eigenvalues of **A** have negative real part. Equation (3.4) implies that $\mathrm{Re}(\lambda_-) < 0$ and $\mathrm{Re}(\lambda_+) < 0$ if and only if $\mathrm{R} > \mathrm{B}$, which is satisfied for all $k > 0$ (Figure 2). However, other physics not considered here such as an extensional background state can lead to instability [47]. The long-wavelength limit of the dominant eigenvalue is

$$\lim_{k \to 0} \lambda_+ = -\frac{1}{2}\left(\frac{\delta}{\delta + 1}\right), \tag{3.7}$$

although this limit is approximately attained for $k \leq 1$ (Figure 3). This limit shows that solutions will tend to evolve over the longer timescale t_e defined by

$$t_\mathrm{e} \equiv t_\mathrm{r} \times 2\left(1 + \frac{1}{\delta}\right) = \frac{4\eta}{\Delta\rho\, g H}\frac{\rho_\mathrm{w}}{\rho_\mathrm{i}}, \tag{3.8}$$

which is a characteristic timescale in shallow approximations of ice-shelf flow [54]. The timescale t_e (3.8) properly accounts for the competition between topographic relaxation and buoyancy in contrast to the original timescale t_r.

Assuming no initial perturbation, the solution to (3.1) is

$$\boldsymbol{y} = \int_0^t e^{\mathbf{A}(t-\tilde{t})}\boldsymbol{b}(\tilde{t})\,\mathrm{d}\tilde{t}, \tag{3.9}$$

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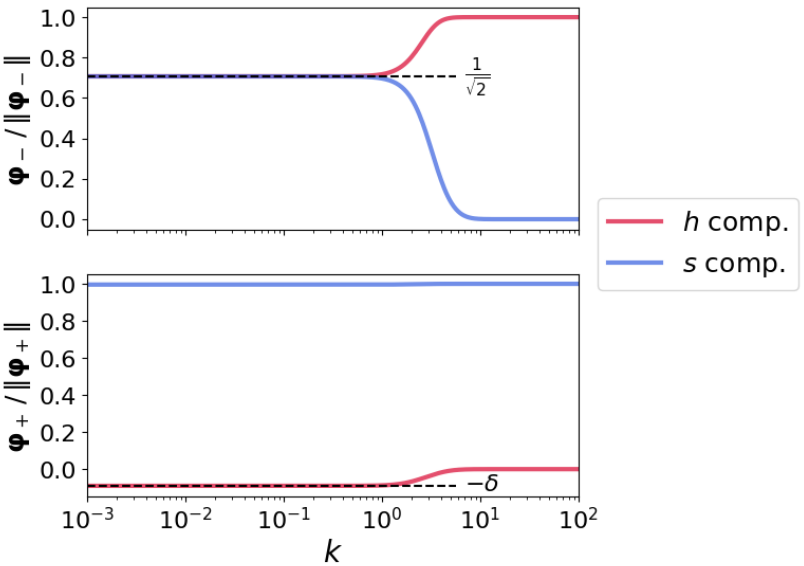

Figure 4. Normalised eigenvectors ($\boldsymbol{\varphi}_\pm$) of $\mathbf{A}$. 'h comp.' is the first vector component and 's comp.' is the second component in accordance with (3.2). Dashes lines highlight analytical limits.

where we compute the matrix exponential via $e^{\mathbf{A}t} = \mathbf{P}e^{\mathbf{D}t}\mathbf{P}^{-1}$. Equation (3.9) shows that the solution for the ice-surface elevation anomaly is

$$\widehat{h} = \int_0^t \mathsf{K}_h(t-\tilde{t})\widehat{m}(\tilde{t})\,\mathrm{d}\tilde{t},$$
$$\mathsf{K}_h(t) \equiv -\frac{\delta \mathsf{B}}{\mu}(e^{\lambda_+ t} - e^{\lambda_- t}). \tag{3.10}$$

Similarly, the solution for the elevation anomaly at the ice-water interface is

$$\widehat{s} = \int_0^t \mathsf{K}_s(t-\tilde{t})\widehat{m}(\tilde{t})\,\mathrm{d}\tilde{t},$$
$$\mathsf{K}_s(t) \equiv \frac{1}{2\mu}[(\mu + (1-\delta)\mathsf{R})e^{\lambda_+ t} + (\mu - (1-\delta)\mathsf{R})e^{\lambda_- t}]. \tag{3.11}$$

The formulas (3.10) and (3.11) show that $\widehat{h}$ will be close to $-\delta\widehat{s}$ provided that K_h is close to $-\delta\mathsf{K}_s$ and $|\widehat{m}|$ is not too large. We show the kernels K_s and $-\mathsf{K}_h/\delta$ in Figure 5, highlighting that $\mathsf{K}_h(k,t) \approx -\delta\mathsf{K}_s(k,t)$ when $k \in (0,1)$ and $t \in (10^{-1}, \infty)$. Therefore, significant deviations from perfect flotation only occur on fast timescales relative to the relaxation timescale or short wavelengths relative to the ice thickness. Finally, advection from the background flow, which is described by the α parameter, produces asymmetry and smaller surface perturbations. In the fast-flow limit ($\alpha \to \infty$), the perturbations decay to zero because the integral kernels in (3.10) and (3.11) become highly oscillatory due to the imaginary parts of the eigenvalues in (3.4). This limit occurs when the ice advects with the background flow too quickly to be disrupted by basal melting or freezing perturbations. We explore all of these effects in the examples below.

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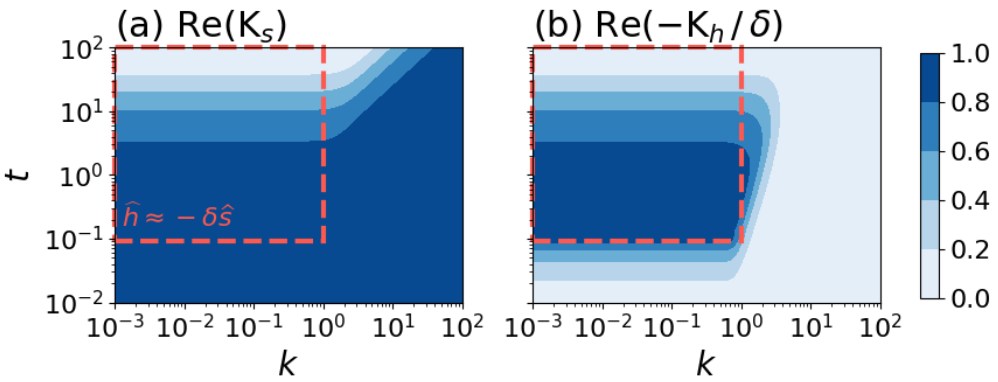

Figure 5. Real parts of the kernels K_s (3.11) and $-\mathsf{K}_h/\delta$ (3.10) of the solution operators. The dashed boxes denote the region in (k,t) space where the flotation thickness is approximately satisfied since $\mathsf{K}_h \approx -\delta\mathsf{K}_s$.

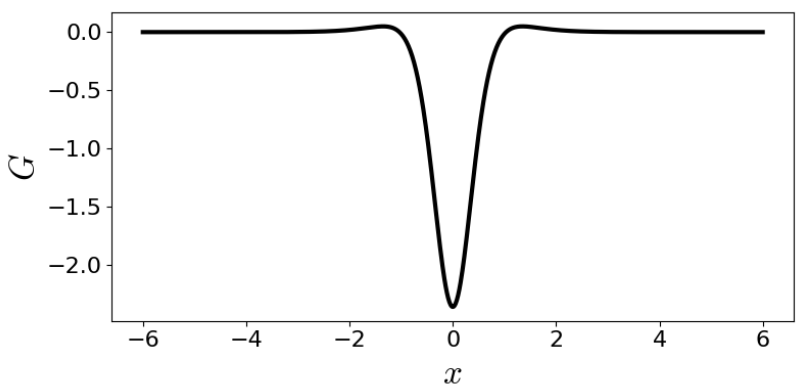

Figure 6. Graph of the free-space Green's function G given by equation (3.17).

(b) Steady states

We compute steady solutions via $[\widehat{h}_{\mathrm{e}}, \widehat{s}_{\mathrm{e}}]^{\mathrm{T}} = -\mathbf{A}^{-1}\boldsymbol{b}$ to obtain

$$\widehat{h}_{\mathrm{e}} = -\frac{\delta \mathsf{B}}{\delta(\mathsf{R}^2 - \mathsf{B}^2) + ik_x\alpha(\delta+1)\mathsf{R} - k_x^2\alpha^2}\,\widehat{m} \tag{3.12}$$

$$\widehat{s}_{\mathrm{e}} = \frac{\mathsf{R} + ik_x\alpha}{\delta(\mathsf{R}^2 - \mathsf{B}^2) + ik_x\alpha(\delta+1)\mathsf{R} - k_x^2\alpha^2}\,\widehat{m}. \tag{3.13}$$

We derive a free-space Green's function for the steady problem assuming no background advection ($\alpha = 0$) and restricting the problem to one horizontal dimension (i.e., $k = |k_x|$). In the limit $\alpha \to 0$ we find

$$\widehat{h}_{\mathrm{e}} = \widehat{G}\,\widehat{m}. \tag{3.14}$$

where

$$\widehat{G} \equiv -\frac{\mathsf{B}}{\mathsf{R}^2 - \mathsf{B}^2} = -k^2\coth(k)\mathrm{csch}(k) - k\mathrm{csch}(k). \tag{3.15}$$

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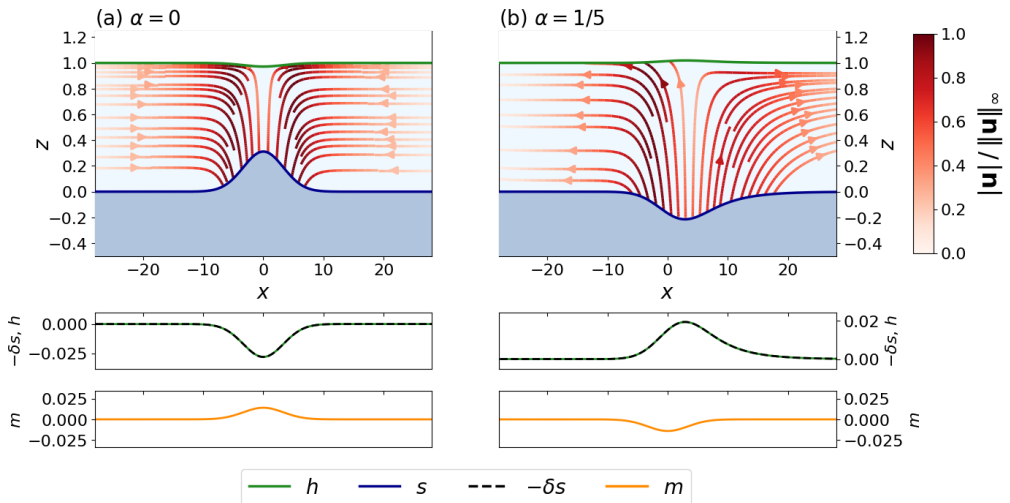

Figure 7. Steady ice-shelf profiles highlighting the flow field and topography. The profiles are taken along the $y = 0$ centerline. Normalised velocity streamlines are plotted on the deformed grid defined by $z \mapsto (1 - z)s + zh$. The elevation (green) and perfect flotation elevation $-\delta s$ (dashed black) are shown for each example, along with the melting or freezing rate m (orange). For both examples, the standard deviation of the melting rate is $\sigma = 10/3$.

Restricting the problem to one horizontal dimension (i.e., $k = |k_x|$), we use the convolution theorem to find the steady surface elevation via

$$h_e(x) = \int_{-\infty}^{+\infty} G(x - \tilde{x}) m(\tilde{x})\, \mathrm{d}\tilde{x}, \tag{3.16}$$

where the free-space Green's function is

$$G(x) = \frac{\pi}{4} \operatorname{sech}^2\left(\frac{\pi x}{2}\right)\left[\pi x \tanh\left(\frac{\pi x}{2}\right) - 3\right]. \tag{3.17}$$

In two horizontal dimensions, G corresponds to the steady elevation anomaly that arises when the melt rate is a line source (Figure 6).

4. Examples

Finally, we explore some steady-state solutions. In all of these examples, we choose a Gaussian-shaped melting or freezing anomaly that does not vary with time. Therefore, we set

$$m(x, y) = \pm m_0 \exp\left(-\frac{1}{2}(x^2 + y^2)/\sigma^2\right) \tag{4.1}$$

and choose an amplitude of $m_0 = 0.014$, which corresponds to a melt rate of 5 m/yr when $H = 500$ m, $\bar{\eta} = 10^{14}$ Pa s, and $\rho_i = 917$ kg/m^3. In all examples, we set $\delta = 0.09$, which corresponds to a water density of $\rho_w = 1000$ kg/m^3. We explore the effect of varying the advection parameter α, the width of the anomaly via σ, and the sign of the anomaly (Figures 7 and 8). We set either $\alpha = 0$ or $\alpha = 1/5$, corresponding to no background flow or a background flow of $\bar{u} \approx 77$ m/yr with the parameters cited above, respectively. We also show streamlines of the steady flow fields for each example. Expressions for the vertical and horizontal velocity fields are provided in Appendix B and Appendix C, respectively.

When the melting anomaly is wide relative to the ice thickness ($\sigma = 10/3$), we find that the steady surface elevation closely matches the perfect flotation condition (Figure 7a). Introducing background advection leads to asymmetry in the surfaces and flow fields (Figure 7b). We find

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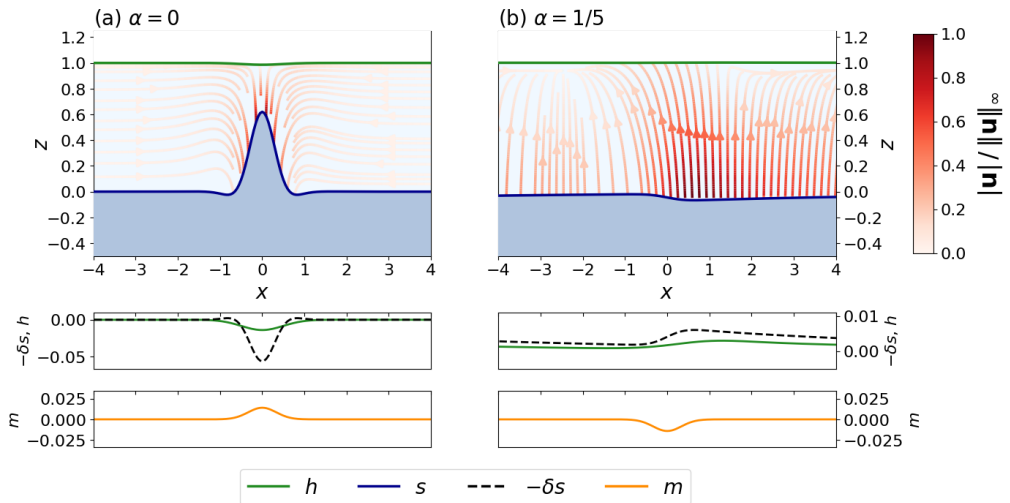

Figure 8. Same as Figure 7 except that the standard deviation of the melting rate is set to $\sigma = 1/3$. The ice-surface elevations (green lines) deviate from the perfect flotation condition (dashed lines).

that deviations from perfect flotation ($h \neq -\delta s$) occur when the melting anomaly width reaches the ice thickness ($\sigma = 1/3$) or smaller. In this regime, the elevation anomaly is diminished relative to the perfect flotation elevation (Figure 8a). At these smaller length scales, advection from the background flow produces pronounced asymmetry and the surface expression of the basal anomaly can be greatly diminished (Figure 8b).

We also compare the one-dimensional convolution solution (3.16) to a solution obtained by solving the fully nonlinear problem (2.1)-(2.7) with a finite element method [55]. In the flow law (2.3), we set the stress exponent to $n = 4$, the ice hardness to $B = 10 \times 10^8$ Pa s$^{1/n}$ [56], and the viscosity regularization parameter ν to coincide with the background viscosity $\bar{\eta} = 10^{14}$ Pa s at zero strain rate with these parameters. On the side-walls of the computational domain, we set the horizontal velocity and vertical shear stress to zero. We set the melting rate to be a one-dimensional Gaussian analogous to (4.1) with $y = 0$. The computational domain is defined by a height of $H = 500$ m and width of 40 km, which is discretised by a uniform mesh with an element width of 100 m. The sea level is set to $\ell = (\rho_i/\rho_w)H$ in the lower-surface boundary condition (2.5). The velocity and pressure are approximated by Taylor-Hood elements [55]. To allow for steady-state solutions in this confined setting, we set a constant accumulation rate a in (2.6) that is equal to the average melt rate [39]. We have implemented the finite element method in FEniCSx and made it openly available as described in the Data Availability section [57–59].

We find that the nonlinear solution reaches steady state by $t = 5\,t_e$. For the parameters listed above, the finite element solution closely matches the Green's function solution (3.16) when the melt rate is small, e.g., 1 m/yr in dimensional terms (Figure 9a). When the melt rate is larger, there can be deviations between the Green's function solution and finite element solution due to nonlinearities in the original problem (2.1)-(2.7) and the lateral confinement in the numerical solution (Figure 9b). Deviations from an isoviscous state in the nonlinear model is controlled by the rheology parameters B and n in the flow law (2.3), which influence the agreement with the linearised model. Linearisation of the boundary conditions onto the simplified domain also contributes to the accuracy of the small-perturbation solutions when the channel incises through a significant fraction of the ice thickness (Figure 9b).

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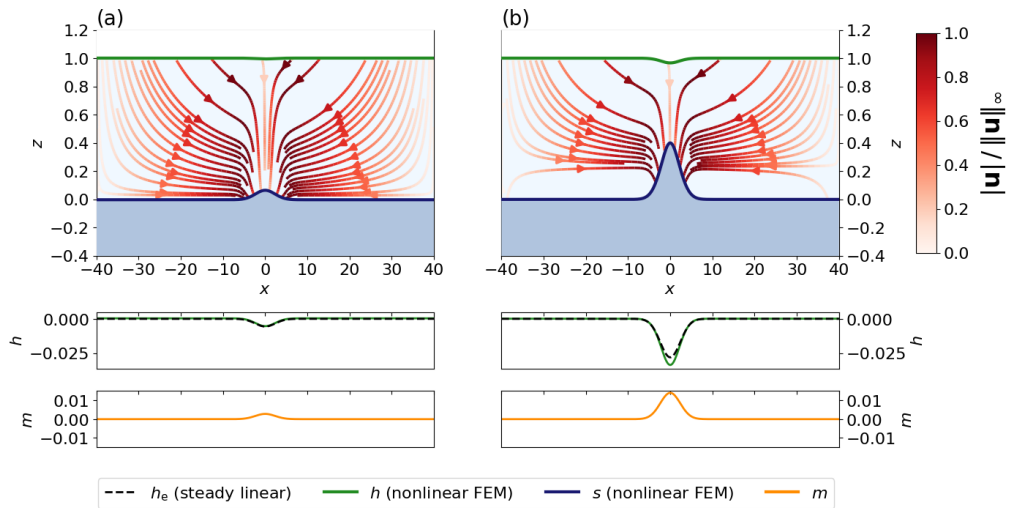

Figure 9. Nonlinear solutions obtained with a finite element method in one horizontal dimension for two different basal melting rates. The melting rate in panel (b) is five times larger than the melting rate in panel (a). The finite element solutions are shown at time $t = 5\,t_e$ and have reached steady state. Steady state Green's function solutions h_e given by (3.16)-(3.17) are shown as dashed lines.

5. Discussion

The linearised analysis shows that short wavelength topography can deviate from perfect flotation even in steady state. These non-hydrostatic effects are due to the presence of non-negligible deviatoric (bridging) stresses. As the ice thickness is typically assumed to be in hydrostatic balance, non-hydrostatic effects can influence the accuracy basal melt-rate estimates [38–40]. For example, recent numerical modelling showed that the true melt rate and hydrostatic melt-rate estimate can have a pointwise mismatch of up to 25%, although the total effect was less than 1% when integrated over the entire domain [39]. Incorporating hydrostatic imbalance into estimates of ice thickness and discharge could help to sharpen melt-rate estimates. The linearised model developed herein could potentially be used to invert for the basal melt rate from altimetry data while allowing for hydrostatic imbalance. For example, equation (3.10) suggests that the inverse problem would take the form of a spatiotemporal deconvolution. Further validation against nonlinear models and comparison with existing estimation methods would be necessary steps in applying the linearised model to estimate basal melt rates.

Several nonlinear effects not considered in our analysis may also influence the evolution of melt-generated ice shelf topography. For example, ice deformation effects such as proximity to the grounding line [22,60], varying ice-shelf geometry, extensional stress regimes and fracture [47,48], or viscoelasticity [45] may produce different non-hydrostatic effects relative to the purely viscous shelves considered here. We have also assumed that the melting or freezing rate is a prescribed forcing function in this model. While considering non-hydrostatic effects can help to improve estimates of basal melting from surface observations, models that include oceanic physics coupled to ice deformation are needed to fully understand the physical processes governing ice-ocean interaction and predict the ice-shelf response to ocean warming [12,13,61–64]. Observations also suggest complex relations between ocean conditions, melting rate, and basal ice properties. For example, ice-penetrating radar and autonomous underwater vehicles have shown that melting depends on the ice roughness or slope of the ice-water interface [11,15,65]. Further modelling and analysis would be useful for elucidating the effects that these processes have on the evolution of ice-shelf topography that is generated by melting at the base.

In our linearised analysis, we excluded perturbations in ice-surface mass balance for simplicity. Localised changes in mass balance at the ice-shelf surface are often related to hydrology [66–70]. In particular, the drainage of ponded water on ice shelves can lead to flexure and potential instability [71–74]. For example, supraglacial lake drainage has been linked to the breakup of the Larsen B Ice Shelf [75,76]. As observations suggest that basal channels and surface hydrology can be coupled [77], implementing surface hydrology in ice-shelf models is a valuable avenue for future work [78,79]. While including surface mass balance perturbations in our linearised analysis is straightforward, we would need to modify our model so that the normal stress condition at the surface accounts for the load associated with ponded water and include additional physics to describe the redistribution of surface waters [80]. We could explore the effects of lateral variations in ice density due to variations in firn properties in a similar way [81]. We leave considerations of perturbations related to surface hydrology or ice density variations for future work.

Finally, the theory developed herein could potentially be extended to analyse the evolution of topography on icy satellites that are thought to have subsurface oceans, like Europa and Enceladus [82–84]. Our analysis diverges from previous treatments of icy satellite topography wherein pressure gradients are assumed to balance shear stresses similar to the shallow-ice approximation for land-based glaciers [85,86]. Further analysis of the physics governing topographic evolution on icy satellites and terrestrial ice shelves would be value for elucidating connections with subsurface thermomechanical processes at the ice-water interface [87,88].

6. Conclusions

Here, we have derived and analysed a linearised model for the evolution of ice-shelf topography that is generated by melting or freezing at the base. We derived a spectral decomposition of the linearised problem to elucidate the dominant modes and time scales governing channel evolution. The analysis shows that the flotation condition is the dominant mode at longer wavelengths, while shorter wavelength topography can persist in hydrostatic imbalance indefinitely. We illustrated through several examples the variety of topographic expressions that can result from a Gaussian-shaped melting rate when advection or the width of the anomaly are modified. We validated the linearised approach for the examples herein by comparing steady solutions obtained via a Green's function to solutions of the fully nonlinear problem obtained with a finite element method. Future work should focus on including hydrostatic imbalance in basal melting estimates, further constraining the ice-ocean interactions that lead to channel incision, and determining the influence that channelisation has on ice-shelf stability.

Data Accessibility. No new data are presented in this study. The code is openly available at `https://github.com/agstub/linear-shelf-melt` and will be archived with Zenodo prior to publication.

Authors' Contributions. A.G.S. conceived the study, developed the analysis, conducted numerical simulations, and drafted the manuscript. C.R.M. and M.G.W. provided valuable insight in the analysis and helped revise the manuscript.

Competing Interests. The authors declare no competing interests.

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Appendix A. Regularisation

To remove the singularities in R and B for the purpose of computation, we define the regularised relaxation and buoyancy functions

$$\mathsf{R}_\theta = (\theta + \mathsf{R}^{-1})^{-1}, \qquad \mathsf{B}_\theta = (\theta + \mathsf{B}^{-1})^{-1}, \tag{6.1}$$

where θ is a small regularisation parameter. We choose θ such that the analytical limit $\mathsf{R}_\theta - \mathsf{B}_\theta \to \frac{1}{4}$ as $k \to 0$ is attained at $k = 10^{-3}$, resulting in $\theta \approx 2.5 \times 10^{-14}$. The regularisation results in other analytical limits in the problem being respected at small k (Figure 3 and Figure 4). These regularisations are only used to facilitate computation of the solutions; we show in the main text that the solutions are well-defined without any regularisation. Analogous regularisations are employed for the velocity response functions derived in Appendix B and Appendix C.

Appendix B. Vertical velocity

Velocity solutions can be obtained after solving for the elevations h and s. The surface elevation equations remedy the singularity of the system (2.28) in the limit $k \to 0$ by determining the long-wavelength component. In steady state, equations (2.12) and (2.13) imply that the long-wavelength limits of the vertical velocity at the surface and base are

$$\lim_{k\to 0} \widehat{w}|_{z=H} = 0, \qquad \lim_{k\to 0} \widehat{w}|_{z=0} = -\lim_{k\to 0} \widehat{m}. \tag{6.2}$$

The elevation solutions derived in the main text result in these limits being satisfied in (2.31) and (2.36). We apply the limits (6.2) to (2.24) to find that the the long-wavelength component of the vertical velocity on the interior of the ice is

$$\lim_{k\to 0} \widehat{w} = -\left(1 - \frac{z}{H}\right) \lim_{k\to 0} \widehat{m}. \tag{6.3}$$

The particular expression for $\widehat{w}$ (2.24) at an arbitrary depth is given by

$$\widehat{w} = -\mathsf{W}_h \widehat{h} - \mathsf{W}_s \delta\widehat{s}, \tag{6.4}$$

where the functions W_h and W_s are

$$\begin{aligned}
\mathsf{W}_h &= \frac{1}{t_\mathrm{r}}\Big[\Big(k'z'\left(\left(2k'+1\right)e^{2k'}-1\right)+k'+\left(k'+1\right)e^{2k'}- \\
&\quad \left(k'z'\left(2k'+e^{2k'}-1\right)-\left(k'+1\right)e^{2k'}-k'+1\right)e^{2k'z'}-1\Big)e^{k'(1-z')}\Big]/\mathsf{L} \qquad (6.5)\\
\mathsf{W}_s &= \frac{1}{t_\mathrm{r}}\Big[\Big(\left(k'z'\left(\left(2k'+1\right)e^{2k'}-1\right)-\left(2k'^2+2k'+1\right)e^{2k'}+1\right)e^{2k'z'} \\
&\quad -\left(k'z'\left(2k'+e^{2k'}-1\right)-2k'^2+2k'+e^{2k'}-1\right)e^{2k'}\Big)e^{-k'z'}\Big]/\mathsf{L} \qquad (6.6)\\
\mathsf{L} &= k'(e^{4k'}-2(1+2k'^2)e^{2k'}+1) \qquad (6.7)
\end{aligned}$$

These response functions reduce to R and B when evaluated at the upper and lower surfaces in accordance with equations (2.31) and (2.36).

Appendix C. Horizontal velocity

We now outline a derivation of expressions for the horizontal velocities. We derive only the component in the x direction, u; the derivation of the y component, v, is analogous except with k_x

replaced by k_y. The transformed horizontal momentum equation (2.19) has the general solution

$$\widehat{u}(z) = \frac{ik_x}{2\bar{\eta}k}\left[e^{kz}\int_0^z \widehat{p}(\tilde{z})e^{-k\tilde{z}}\,\mathrm{d}\tilde{z} - e^{-kz}\int_0^z \widehat{p}(\tilde{z})e^{k\tilde{z}}\,\mathrm{d}\tilde{z}\right] + d_1e^{kz} + d_2e^{-kz}, \tag{6.8}$$

where d_1 and d_2 depend on k. Using the identity $\widehat{p} = \bar{\eta}\left(k^{-2}\widehat{w}_{zzz} - \widehat{w}_z\right)$, we find that equation (6.8) reduces to

$$\widehat{u}(z) = \frac{ik_x}{k^2}P(z) + d_1e^{kz} + d_2e^{-kz} \tag{6.9}$$

$$P(z) = \widehat{w}_z - \widehat{w}_z|_{z=0}\cosh(kz) - \frac{1}{k}\widehat{w}_{zz}|_{z=0}\sinh(kz). \tag{6.10}$$

Equation (6.9) and the vanishing shear-stress conditions, $\widehat{u}_z = -ik_x\widehat{w}$, at $z = H$ and $z = 0$ lead to the linear system

$$\begin{bmatrix} e^{k'} & -e^{-k'} \\ 1 & -1 \end{bmatrix}\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = -\frac{ik_x}{k^2}\begin{bmatrix} k\widehat{w}|_{z=H} + \frac{1}{k}P_z(H) \\ k\widehat{w}|_{z=0} \end{bmatrix}, \tag{6.11}$$

where $\widehat{w}|_{z=H}$ and $\widehat{w}|_{z=0}$ are given by (2.31) and (2.36), respectively. After solving the system (6.11) for the coefficients d_1 and d_2, we can write the horizontal velocity in (6.9) as

$$\widehat{u} = ik_x\left(\mathsf{U}_h\widehat{h} + \mathsf{U}_s\delta\widehat{s}\right), \tag{6.12}$$

where the response functions U_h and U_s are

$$\mathsf{U}_h = \frac{1}{t_\mathrm{r}}\left[z'((1-2k'-e^{2k'})e^{2k'z'}+1) + \left(1-z'(1+2k')\right)e^{2k'} + e^{2k'(z'+1)}(e^{2k'}-1) - 1\right] \times e^{k'(1-z')}/\mathsf{L} \tag{6.13}$$

$$\mathsf{U}_s = \frac{1}{t_\mathrm{r}}\left[z'(e^{2k'z'} - e^{4k'} - (1+2k')e^{2k'(z'+1)}) + 2k'e^{2k'(z'+1)} + \left(-2k'z' + 2k' + z'\right)e^{2k'}\right] \times e^{-k'z'}/\mathsf{L}, \tag{6.14}$$

where L is defined in (6.7). As with the vertical velocity, the coupling with the elevation solutions renders the horizontal velocity well-defined in the limit $k \to 0$ even though the system (6.11) is singular. We find that that $\widehat{u} \to 0$ as $k \to 0$ for the examples herein, which is the expected behavior for a symmetric flow field with vanishing horizontal integral.

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