River Network HyperGraphs and Transportation Network HyperGraphs: A Graph-Theoretic Approach for Geoscientific and Civil Applications

Takaaki Fujita [1] *
[1] Independent Researcher, Shinjuku, Shinjuku-ku, Tokyo, Japan. Email: Takaaki.fujita060@gmail.com

Abstract

River Network Graphs and Transportation Network Graphs are classical models that represent river systems and transportation infrastructures as vertices and edges, respectively, and underpin applications in hydrological simulation, watershed management, shortest-path computation, and urban traffic analysis. In this paper, we extend these graph-based models into the hypergraph and superhypergraph domains by introducing two new structures—River Network HyperGraph and Transportation Network HyperGraph—and their hierarchical generalizations, River Network SuperHyperGraph and Transportation Network SuperHyperGraph. These enhanced representations enable multi-scale, hierarchical modeling of waterway and transportation networks, offering a unified framework for advanced analysis and management of complex infrastructure systems.

Keywords: Superhypergraph, Hypergraph, River Network HyperGraph, Transportation Network HyperGraph

1 Preliminaries

We begin by fixing notation and recalling foundational definitions that will be used throughout this paper. Unless otherwise specified, all graphs are assumed to be finite. For more extensive treatments, see the referenced works.

1.1 Hypergraphs and SuperHyperGraphs

Graph theory is the mathematical study of networks of vertices and edges representing relationships or connections [1–3]. A *hypergraph* extends the notion of a graph by permitting each *hyperedge* to join any nonempty subset of vertices at once [4–7]. Hypergraphs have been applied in a wide range of domains, and various mathematical properties and graph algorithms have been developed to analyze them [8–11].

A *SuperHyperGraph* further builds a hierarchy by iterating the powerset construction, thus capturing nested, multi-scale relationships among vertices and edges [12–18].

Definition 1.1 (Base set)**.** Let V_0 be a finite set, called the *base set*. All subsequent vertex and edge collections are drawn from V_0 or its iterated powersets.

Definition 1.2 (Powerset)**.** For any set X, its *powerset* is

$$\mathcal{P}(X) = \{ A : A \subseteq X \}.$$

Definition 1.3 (Hypergraph)**.** [4, 6] A *hypergraph* is a pair $H = (V, E)$ where

- V is a finite set of *vertices*, and
- $E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$ is a finite collection of nonempty subsets of V, called *hyperedges*.

Example 1.4 (Urban Bus Network Hypergraph)**.** Consider a small downtown bus system with four major stops:

$$V = \{\mathit{Central}, \ \mathit{Museum}, \ \mathit{CityHall}, \ \mathit{Airport}\}.$$

The bus routes serve these stops as follows:

$$e_1 = \{\mathit{Central}, \ \mathit{Museum}, \ \mathit{CityHall}\},$$
$$e_2 = \{\mathit{Central}, \ \mathit{CityHall}, \ \mathit{Airport}\},$$
$$e_3 = \{\mathit{Museum}, \ \mathit{CityHall}, \ \mathit{Airport}\}.$$

Then the hypergraph

$$H = (V, \ E), \quad E = \{ e_1, e_2, e_3\},$$

models the overlapping coverage of the three bus routes in the urban core.

Definition 1.5 (Iterated powerset)**.** [19–21] Define recursively for $k \geq 0$:

$$\mathcal{P}^0(V_0) = V_0, \quad \mathcal{P}^{k+1}(V_0) = \mathcal{P}\big(\mathcal{P}^k(V_0)\big).$$

We write $\mathcal{P}_n(V_0)$ for $\mathcal{P}^n(V_0)$ and denote by $\mathcal{P}_n^*(V_0)$ its collection of nonempty subsets.

Example 1.6 (Iterated powerset for the Tokaido Shinkansen)**.** Let the base set of key stations on the Tokaido Shinkansen be

$$V_0 = \{\text{Tokyo, Nagoya, Kyoto, Shin-Osaka}\}.$$

Then the first iterated powerset

$$\mathcal{P}^1(V_0) = \mathcal{P}(V_0)$$

is the collection of all nonempty station subsets, for example

$$\{\text{Tokyo}\}, \quad \{\text{Nagoya, Kyoto}\}, \quad \{\text{Tokyo, Shin-Osaka}\}, \quad \ldots$$

Here, {Nagoya, Kyoto} might represent a service stopping only at Nagoya and Kyoto.

Next, the second iterated powerset

$$\mathcal{P}^2(V_0) = \mathcal{P}\big(\mathcal{P}(V_0)\big)$$

consists of all nonempty collections of these station-sets. For instance,

$$\big\{\{\text{Tokyo, Nagoya}\}, \ \{\text{Kyoto, Shin-Osaka}\}\big\} \ \in \ \mathcal{P}^2(V_0),$$

which can be interpreted as grouping two distinct service patterns: one linking Tokyo–Nagoya and another linking Kyoto–Shin-Osaka.

Definition 1.7 (n–SuperHyperGraph)**.** [22, 23] An *n–SuperHyperGraph* is a pair

$$\text{SuHyG}^{(n)} = (V, E), \quad V, \, E \subseteq \mathcal{P}^n(V_0),$$

where each element of V is called an *n–supervertex* and each element of E an *n–superedge*.

Example 1.8 (River Network 2-SuperHyperGraph of the Amazon Basin)**.** Let the base set of river confluences be

$$V_0 = \{\text{RioNegro, Madeira, Solimões, Amazonas}\}.$$

In the underlying River Network HyperGraph, we have the hyperedges

$$\begin{aligned} e_{RNA} &= \{\text{RioNegro, Amazonas}\}, \\ e_{MAA} &= \{\text{Madeira, Amazonas}\}, \\ e_{SAA} &= \{\text{Solimões, Amazonas}\}, \\ e_{\text{Main}} &= \{\text{RioNegro, Madeira, Solimões, Amazonas}\}. \end{aligned}$$

Typical mean annual discharges (in m³/s) and channel lengths (in km) are

$$\varphi(e_{RNA}) = 2.83 \times 10^4, \quad \ell(e_{RNA}) = 2430,$$

$$\varphi(e_{MAA}) = 3.15 \times 10^4, \quad \ell(e_{MAA}) = 3200,$$

$$\varphi(e_{SAA}) = 1.60 \times 10^4, \quad \ell(e_{SAA}) = 1600,$$

$$\varphi(e_{\text{Main}}) = 2.83 \times 10^4 + 3.15 \times 10^4 + 1.60 \times 10^4 = 7.58 \times 10^4, \quad \ell(e_{\text{Main}}) = \max\{2430, 3200, 1600\} = 3200.$$

For $n = 2$, the 2-SuperHyperGraph is

$$V_2 = \mathcal{P}^2(V_0), \qquad E_2 = \big\{ \, \mathcal{P}(e) \setminus \{\emptyset\} \, \big| \, e \in \{e_{RNA}, e_{MAA}, e_{SAA}, e_{\text{Main}}\}\big\},$$

with propagated labelings

$$\varphi^{(2)}\big(\mathcal{P}(e) \setminus \{\emptyset\}\big) = \varphi(e), \quad \ell^{(2)}\big(\mathcal{P}(e) \setminus \{\emptyset\}\big) = \ell(e).$$

Two representative superedges are:

- **Tributary superedge at Rio Negro:**

$$\mathcal{P}(e_{RNA}) \setminus \{\emptyset\} = \big\{\{\text{RioNegro}\},\ \{\text{Amazonas}\},\ \{\text{RioNegro, Amazonas}\}\big\},$$

  carrying $\varphi^{(2)} = 2.83 \times 10^4\ m^3/s$ and $\ell^{(2)} = 2430$ km.

- **Main-stem superedge of the basin:**

$$\mathcal{P}(e_{\text{Main}}) \setminus \{\emptyset\} = \Big\{\{\text{RioNegro}\}, \{\text{Madeira}\}, \{\text{Solimões}\}, \{\text{Amazonas}\}, \dots,$$

$$\{\text{RioNegro, Madeira, Solimões, Amazonas}\}\Big\},$$

  carrying $\varphi^{(2)} = 7.58 \times 10^4\ m^3/s$ and $\ell^{(2)} = 3200$ km.

Example 1.9 (Urban Road Network 2-SuperHyperGraph of the Times–Herald–Union Triangle)**.** We model three major intersections in Midtown Manhattan as the Transportation Network Graph

$$G = (V,\ E,\ \tau,\ \ell,\ \kappa),$$

with

$$V = \{\text{TimesSq, HeraldSq, UnionSq}\},$$

$$E = \{(\text{TimesSq, HeraldSq}),\ (\text{HeraldSq, UnionSq}),\ (\text{TimesSq, UnionSq})\},$$

where each road is bidirectional. Typical parameters on each undirected link are:

$$\tau(\text{TimesSq, HeraldSq}) = 2 \text{ min}, \quad \ell(\text{TimesSq, HeraldSq}) = 0.4 \text{ km}, \quad \kappa(\text{TimesSq, HeraldSq}) = 1000 \text{ veh/h},$$

$$\tau(\text{HeraldSq, UnionSq}) = 3 \text{ min}, \quad \ell(\text{HeraldSq, UnionSq}) = 0.6 \text{ km}, \quad \kappa(\text{HeraldSq, UnionSq}) = 800 \text{ veh/h},$$

$$\tau(\text{TimesSq, UnionSq}) = 5 \text{ min}, \quad \ell(\text{TimesSq, UnionSq}) = 0.9 \text{ km}, \quad \kappa(\text{TimesSq, UnionSq}) = 1200 \text{ veh/h}.$$

Its Transportation Network HyperGraph has hyperedges

$$\mathcal{E} = \big\{\{\text{TimesSq, HeraldSq}\},\ \{\text{HeraldSq, UnionSq}\},\ \{\text{TimesSq, UnionSq}\},\ \{\text{TimesSq, HeraldSq, UnionSq}\}\big\},$$

where the last set is the junction hyperedge at Herald Square (all three roads meet).

For $n = 2$, the 2-SuperHyperGraph is

$$V_2 = \mathcal{P}^2(V), \qquad E_2 = \big\{\ \mathcal{P}(e) \setminus \{\emptyset\}\ \big|\ e \in \mathcal{E}\big\},$$

with weight-functions inherited from level 1:

$$\tau^{(2)}\big(\mathcal{P}(e) \setminus \{\emptyset\}\big) = \tau_H(e), \quad \ell^{(2)}\big(\mathcal{P}(e) \setminus \{\emptyset\}\big) = \ell_H(e), \quad \kappa^{(2)}\big(\mathcal{P}(e) \setminus \{\emptyset\}\big) = \kappa_H(e).$$

Two illustrative level-2 superedges are:

(a) Binary connection at Times–Herald:

$$\mathcal{P}\big(\{\text{TimesSq, HeraldSq}\}\big) \setminus \{\emptyset\} = \{\{\text{TimesSq}\},\ \{\text{HeraldSq}\},\ \{\text{TimesSq, HeraldSq}\}\},$$

carrying $\tau^{(2)} = 2$ min, $\ell^{(2)} = 0.4$ km, $\kappa^{(2)} = 1000$ veh/h.

(b) Junction at HeraldSq:

$$\mathcal{P}\big(\{\text{TimesSq, HeraldSq, UnionSq}\}\big) \setminus \{\emptyset\} = \{\{\text{TimesSq}\}, \{\text{HeraldSq}\}, \{\text{UnionSq}\},$$
$$\{\text{TimesSq, HeraldSq}\}, \{\text{TimesSq, UnionSq}\}, \{\text{HeraldSq, UnionSq}\},$$
$$\{\text{TimesSq, HeraldSq, UnionSq}\}\},$$

with $\tau^{(2)} = \max\{2, 3, 5\} = 5$ min, $\ell^{(2)} = \max\{0.4, 0.6, 0.9\} = 0.9$ km, $\kappa^{(2)} = 1000 + 800 + 1200 = 3000$ veh/h.

2 River Network Graph and Their Extensions

River networks are interconnected branching systems of channels and streams that transport water and sediments across drainage basin landscapes continuously (cf. [24–27]). A River Network Graph is a graph that represents rivers and their tributary connections as vertices and edges, used in flow simulations and watershed management (cf. [28–30]). We extend this concept using hypergraphs and superhypergraphs.

Definition 2.1 (River Network Graph)**.** A *River Network Graph* is a directed acyclic graph

$$G = (V, E, \varphi, \ell),$$

equipped with

- a finite set of *nodes* V, whose elements are
  - *sources* (head-water confluences or springs),
  - *junctions* (confluence of two or more upstream channels), and
  - *sinks* (river mouth or gauging station).
- a set of *directed edges* $E \subseteq V \times V$, where $(u, v) \in E$ if and only if there is a river channel carrying flow from node u downstream to node v. The requirement that G is acyclic (no directed cycles) encodes the fact that water cannot flow uphill back into an upstream channel.
- a function $\varphi : E \to \mathbb{R}_{>0}$ assigning each edge $e = (u, v)$ a *stream discharge capacity* $\varphi(e)$ (e.g. mean annual flow).
- a function $\ell : E \to \mathbb{R}_{>0}$ assigning each edge e its *channel length* $\ell(e)$.

We may represent G by its *adjacency matrix* $A \in \{0, 1\}^{|V|\times|V|}$ with

$$A_{uv} = \begin{cases} 1, & (u, v) \in E, \\ 0, & \text{otherwise}, \end{cases}$$

and by weighted matrices $\Phi, \Lambda \in \mathbb{R}^{|V|\times|V|}$ with $\Phi_{uv} = \varphi(u, v)$ and $\Lambda_{uv} = \ell(u, v)$.

Example 2.2 (River Network Graph of the Mississippi–Ohio System)**.** Consider the major confluences in the Mississippi–Ohio watershed (cf. [31–33]). We model this as the directed acyclic graph

$$G = (V, E, \varphi, \ell),$$

where

$$\begin{aligned} V = \{\, & \text{Allegheny Source (Lake Pleasant, NY)}, \\ & \text{Monongahela Source (Fairmont, WV)}, \\ & \text{Mississippi Source (Lake Itasca, MN)}, \\ & \text{Pittsburgh (Allegheny + Monongahela)}, \\ & \text{Cairo (Ohio + Mississippi)}, \\ & \text{Gulf Mouth (Gulf of Mexico)} \,\}, \\ E = \{\, & (\text{Allegheny Source}, \text{Pittsburgh}),\ (\text{Monongahela Source}, \text{Pittsburgh}), \\ & (\text{Pittsburgh}, \text{Cairo}),\ (\text{Mississippi Source}, \text{Cairo}),\ (\text{Cairo}, \text{Gulf Mouth}) \,\}. \end{aligned}$$

Stream discharge capacities (mean annual flow, in m³/s) and channel lengths (in km) are recorded by

$$\varphi(e) = \begin{cases} 560, & e = (\text{Allegheny Source}, \text{Pittsburgh}), \\ 340, & e = (\text{Monongahela Source}, \text{Pittsburgh}), \\ 7970, & e = (\text{Pittsburgh}, \text{Cairo}), \\ 16790, & e = (\text{Mississippi Source}, \text{Cairo}), \\ 17100, & e = (\text{Cairo}, \text{Gulf Mouth}), \end{cases}$$

$$\ell(e) = \begin{cases} 981, & e = (\text{Allegheny Source, Pittsburgh}), \\ 499, & e = (\text{Monongahela Source, Pittsburgh}), \\ 1579, & e = (\text{Pittsburgh, Cairo}), \\ 3766, & e = (\text{Mississippi Source, Cairo}), \\ 1045, & e = (\text{Cairo, Gulf Mouth}). \end{cases}$$

Definition 2.3 (River Network HyperGraph)**.** Let $G = (V, E, \varphi, \ell)$ be a River Network Graph. We define its *River Network HyperGraph*

$$H = (V, \mathcal{E}, \varphi_H, \ell_H)$$

as follows:

- The vertex set of H is the same as that of G: $V(H) = V$.
- The hyperedge set $\mathcal{E}$ consists of two types of nonempty subsets of V:
  1. For each directed channel $(u, v) \in E$ in G, include the *binary hyperedge*
  $$e_{u,v} = \{u, v\}.$$
  2. For each junction node $j \in V$ with at least two immediate upstream neighbors,
  $$U_j = \{u \in V : (u, j) \in E\}, \quad |U_j| \geq 2,$$
  include the *junction hyperedge*
  $$e_j = U_j \cup \{j\}.$$
- The capacity-labeling $\varphi_H : \mathcal{E} \to \mathbb{R}_{>0}$ is given by
  $$\varphi_H(e) = \begin{cases} \varphi(u, v), & e = \{u, v\}, \\ \sum_{u \in U_j} \varphi(u, j), & e = e_j, \end{cases}$$
  i.e. binary edges inherit the channel capacity, and junction hyperedges aggregate upstream capacities.
- The length-labeling $\ell_H : \mathcal{E} \to \mathbb{R}_{>0}$ is given by
  $$\ell_H(e) = \begin{cases} \ell(u, v), & e = \{u, v\}, \\ \max_{u \in U_j} \ell(u, j), & e = e_j, \end{cases}$$
  i.e. junction hyperedges carry the maximum upstream reach length.

Example 2.4 (River Network HyperGraph of the Mississippi–Ohio System)**.** We model the key confluences in the Mississippi–Ohio watershed as the hypergraph

$$H = (V, \mathcal{E}, \varphi_H, \ell_H),$$

where

$$V = \{\text{Allegheny Source, Monongahela Source, Mississippi Source, Pittsburgh, Cairo, Gulf Mouth}\},$$

$$\mathcal{E} = \{\{\text{Allegheny Source, Pittsburgh}\}, \{\text{Monongahela Source, Pittsburgh}\}, \{\text{Pittsburgh, Cairo}\}, \{\text{Mississippi Source, Cairo}\}, \{\text{Allegheny Source, Monongahela Source, Pittsburgh}\}, \{\text{Pittsburgh, Mississippi Source, Cairo}\}\}.$$

The capacity-labeling $\varphi_H : \mathcal{E} \to \mathbb{R}_{>0}$ is

$$\varphi_H(e) = \begin{cases} 560, & e = \{\text{Allegheny Source, Pittsburgh}\}, \\ 340, & e = \{\text{Monongahela Source, Pittsburgh}\}, \\ 7970, & e = \{\text{Pittsburgh, Cairo}\}, \\ 16790, & e = \{\text{Mississippi Source, Cairo}\}, \\ 560 + 340, & e = \{\text{Allegheny Source, Monongahela Source, Pittsburgh}\}, \\ 7970 + 16790, & e = \{\text{Pittsburgh, Mississippi Source, Cairo}\}. \end{cases}$$

The length-labeling $\ell_H : \mathcal{E} \to \mathbb{R}_{>0}$ is

$$\ell_H(e) = \begin{cases} 981, & e = \{\text{Allegheny Source, Pittsburgh}\}, \\ 499, & e = \{\text{Monongahela Source, Pittsburgh}\}, \\ 1579, & e = \{\text{Pittsburgh, Cairo}\}, \\ 3766, & e = \{\text{Mississippi Source, Cairo}\}, \\ \max\{981, 499\}, & e = \{\text{Allegheny Source, Monongahela Source, Pittsburgh}\}, \\ \max\{1579, 3766\}, & e = \{\text{Pittsburgh, Mississippi Source, Cairo}\}. \end{cases}$$

Theorem 2.5. *For any River Network Graph $G = (V, E, \varphi, \ell)$, the structure H of the Definition is a hypergraph that* generalizes *G:*

1. *H satisfies the hypergraph axioms (each hyperedge is a nonempty subset of V).*
2. *The* 2-section *(or clique-expansion) of H recovers the undirected version of G: every $(u, v) \in E$ appears as an edge in the graph whose edges are all unordered pairs contained in some $e \in \mathcal{E}$.*

Proof. We verify each claim in turn.

(1) Hypergraph axioms. By construction, each binary hyperedge $e_{u,v} = \{u, v\}$ is nonempty since $(u, v) \in E$ implies $u \neq v$. Each junction hyperedge

$$e_j = U_j \cup \{j\}$$

is nonempty because $|U_j| \geq 2$ by hypothesis, and $j \notin U_j$ by acyclicity of G. Hence $\mathcal{E} \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$, so H is a hypergraph.

(2) Generalization via 2-section. Define the *2-section* (or clique-expansion) graph $\Gamma(H) = (V, E')$ by

$$E' = \big\{\{x, y\} \subseteq V \mid \exists\, e \in \mathcal{E},\ \{x, y\} \subseteq e\big\}.$$

- Every original channel $(u, v) \in E$ yields the binary hyperedge $e_{u,v}$, so $\{u, v\} \subseteq e_{u,v}$. Hence $\{u, v\} \in E'$.
- Conversely, if $\{u, v\} \in E'$, then there exists some hyperedge e with $\{u, v\} \subseteq e$. If $e = e_{x,y}$ is binary, then $(x, y) \in E$ and $\{u, v\} = \{x, y\}$. If $e = e_j$ is a junction hyperedge, then $u, v \in U_j \cup \{j\}$. In this case at least one of $\{u, v\}$ is of the form $\{u', j\}$ with $(u', j) \in E$, so the directed edge $(u', j) \in E$ implies $\{u', j\} \in E'$. Thus every pair in E' corresponds to some original channel in E.

Therefore $\Gamma(H)$ is exactly the (undirected) edge-set of G. This shows that H indeed *generalizes G* via clique-expansion. □

Definition 2.6 (River Network n-SuperHyperGraph)**.** Let $G = (V, E, \varphi, \ell)$ be a River Network Graph and let

$$\mathcal{E} = \{\, e \subseteq V : e \text{ is a hyperedge of the River Network HyperGraph of } G\}$$

be its hyperedge family. For any integer $n \geq 1$, the *River Network n-SuperHyperGraph* is

$$\mathrm{RNHG}^{(n)} = \big(V_n,\ E_n,\ \varphi^{(n)},\ \ell^{(n)}\big),$$

where for each $k = 1, \dots, n$:

$$V_k = \mathcal{P}^k(V), \qquad E_k = \big\{\mathcal{P}^{k-1}(e) \,\big|\, e \in \mathcal{E}\big\} \subseteq \mathcal{P}(V_k) \setminus \{\emptyset\},$$

and the capacity- and length-labelings are propagated by

$$\varphi^{(k)}\big(\mathcal{P}^{k-1}(e)\big) = \varphi(e), \qquad \ell^{(k)}\big(\mathcal{P}^{k-1}(e)\big) = \ell(e).$$

Example 2.7 (River Network 2-SuperHyperGraph of the Mississippi–Ohio System)**.** Let $G = (V, E, \varphi, \ell)$ and $\mathcal{E}$ be as in Example 3.2. For $n = 2$, the River Network 2-SuperHyperGraph is

$$V_2 = \mathcal{P}^2(V), \qquad E_2 = \left\{ \mathcal{P}(e) \setminus \{\emptyset\} \;\middle|\; e \in \mathcal{E} \right\},$$

with labelings

$$\varphi^{(2)}\big(\mathcal{P}(e) \setminus \{\emptyset\}\big) = \varphi(e), \quad \ell^{(2)}\big(\mathcal{P}(e) \setminus \{\emptyset\}\big) = \ell(e).$$

We illustrate two representative superedges:

(a) Binary channel at Pittsburgh:
For the channel hyperedge

$$e_{AP} = \{\text{Allegheny Source}, \text{ Pittsburgh}\},$$

its level-2 superedge is

$$\mathcal{P}\big(e_{AP}\big) \setminus \{\emptyset\} = \big\{\{\text{Allegheny Source}\}, \{\text{Pittsburgh}\}, \{\text{Allegheny Source, Pittsburgh}\}\big\}.$$

This inherits capacity and length

$$\varphi^{(2)}\big(\mathcal{P}(e_{AP}) \setminus \{\emptyset\}\big) = \varphi(e_{AP}) = 560, \qquad \ell^{(2)}\big(\mathcal{P}(e_{AP}) \setminus \{\emptyset\}\big) = \ell(e_{AP}) = 981.$$

(b) Junction at Pittsburgh:
For the junction hyperedge

$$e_P = \{\text{Allegheny Source}, \text{ Monongahela Source}, \text{ Pittsburgh}\},$$

its level-2 superedge is

$$\begin{aligned}\mathcal{P}(e_P) \setminus \{\emptyset\} = \big\{&\{\text{Allegheny Source}\}, \{\text{Monongahela Source}\}, \{\text{Pittsburgh}\},\\ &\{\text{Allegheny Source, Monongahela Source}\}, \{\text{Allegheny Source, Pittsburgh}\},\\ &\{\text{Monongahela Source, Pittsburgh}\}, \{\text{Allegheny Source, Monongahela Source, Pittsburgh}\}\big\}.\end{aligned}$$

This superedge carries

$$\varphi^{(2)}\big(\mathcal{P}(e_P) \setminus \{\emptyset\}\big) = \varphi(e_P) = 560 + 340 = 900, \quad \ell^{(2)}\big(\mathcal{P}(e_P) \setminus \{\emptyset\}\big) = \max\{981, 499\} = 981.$$

Theorem 2.8. *For each $n \geq 1$,* $\mathrm{RNHG}^{(n)}$ *is an n-SuperHyperGraph and* generalizes *the River Network HyperGraph in the sense that*

$$\mathrm{RNHG}^{(1)} \;=\; \big(V_1, E_1, \varphi^{(1)}, \ell^{(1)}\big) \;\cong\; \big(V, \mathcal{E}, \varphi, \ell\big).$$

Proof. We verify both claims:

(i) n-SuperHyperGraph structure. By construction,

$$V_n = \mathcal{P}^n(V) \quad \text{and} \quad E_n \subseteq \mathcal{P}(V_n) \setminus \{\emptyset\},$$

so $\mathrm{RNHG}^{(n)}$ exactly matches the definition of an n-SuperHyperGraph.

(ii) Generalization of the HyperGraph. When $n = 1$, note that $\mathcal{P}^0(e) = e$ for each hyperedge $e \in \mathcal{E}$. Hence

$$V_1 = \mathcal{P}^1(V) = \mathcal{P}(V), \qquad E_1 = \{\mathcal{P}^0(e) \mid e \in \mathcal{E}\} = \mathcal{E},$$

and the labelings satisfy $\varphi^{(1)}(e) = \varphi(e)$, $\ell^{(1)}(e) = \ell(e)$. Thus $\mathrm{RNHG}^{(1)} = (V_1, E_1, \varphi^{(1)}, \ell^{(1)})$ coincides exactly with the River Network HyperGraph $(V, \mathcal{E}, \varphi, \ell)$. This shows that for arbitrary $n \geq 1$, the construction extends and indeed generalizes the level-1 hypergraph to a fully hierarchical, n-layer superhypergraph. □

3 Transportation Network Graph and Their Extensions

Transportation networks represent intersections (nodes) and road or rail links (edges) to optimize travel connectivity (cf. [34–37]). A Transportation Network Graph is a graph in which vertices represent intersections and edges represent road segments, widely applied in shortest-path computation, traffic congestion analysis, and urban planning (cf. [38–40]). We extend this concept using hypergraphs and superhypergraphs.

Definition 3.1 (Transportation Network Graph)**.** A *Transportation Network Graph* is a weighted directed graph

$$G = \big(V,\ E,\ \tau,\ \ell,\ \kappa\big),$$

equipped with

- a finite set of *nodes* V, each representing a junction, intersection, station, or terminal;
- a set of *directed edges* $E \subseteq V \times V$, where $(u, v) \in E$ if and only if there is a viable transportation link (road segment, rail line, airway corridor, etc.) carrying traffic from node u to node v;
- a *travel-time function* $\tau\colon E \to \mathbb{R}_{>0}$, assigning each edge $e = (u, v)$ its typical traversal time $\tau(e)$;
- a *length (distance) function* $\ell\colon E \to \mathbb{R}_{>0}$, assigning each edge e its physical length $\ell(e)$;
- a *capacity function* $\kappa\colon E \to \mathbb{R}_{>0}$, assigning each edge e its maximum flow capacity $\kappa(e)$ (e.g. vehicles per hour).

We may represent G by its *adjacency matrix* $A \in \{0, 1\}^{|V|\times|V|}$ with

$$A_{uv} = \begin{cases} 1, & (u, v) \in E,\\ 0, & \text{otherwise,} \end{cases}$$

and by weighted matrices $\mathcal{T}, \Lambda, \mathcal{K} \in \mathbb{R}^{|V|\times|V|}$ with

$$\mathcal{T}_{uv} = \begin{cases} \tau(u, v), & (u, v) \in E,\\ 0, & \text{otherwise,} \end{cases} \quad \Lambda_{uv} = \begin{cases} \ell(u, v), & (u, v) \in E,\\ 0, & \text{otherwise,} \end{cases} \quad \mathcal{K}_{uv} = \begin{cases} \kappa(u, v), & (u, v) \in E,\\ 0, & \text{otherwise.} \end{cases}$$

Example 3.2 (Transportation Network Graph of the I-90 Corridor)**.** We model the segment of Interstate 90 (I-90) in the northeastern United States (cf. [41, 42]), connecting Boston, Worcester, Springfield, and Albany, as the directed graph

$$G = \big(V,\ E,\ \tau,\ \ell,\ \kappa\big),$$

where

$$V = \{\text{Boston, Worcester, Springfield, Albany}\},$$

$$E = \{(\text{Boston, Worcester}),\ (\text{Worcester, Boston}),\ (\text{Worcester, Springfield}),$$
$$(\text{Springfield, Worcester}),\ (\text{Springfield, Albany}),\ (\text{Albany, Springfield})\}.$$

Typical travel times (in hours), distances (in kilometers), and capacities (vehicles per hour) are specified by

$$\tau(e) = \begin{cases} 1.25, & e = (\text{Boston, Worcester}),\\ 1.25, & e = (\text{Worcester, Boston}),\\ 1.00, & e = (\text{Worcester, Springfield}),\\ 1.00, & e = (\text{Springfield, Worcester}),\\ 1.50, & e = (\text{Springfield, Albany}),\\ 1.50, & e = (\text{Albany, Springfield}), \end{cases} \qquad \ell(e) = \begin{cases} 113, & e = (\text{Boston, Worcester}),\\ 113, & e = (\text{Worcester, Boston}),\\ 88, & e = (\text{Worcester, Springfield}),\\ 88, & e = (\text{Springfield, Worcester}),\\ 137, & e = (\text{Springfield, Albany}),\\ 137, & e = (\text{Albany, Springfield}), \end{cases}$$

$$\kappa(e) = \begin{cases} 2000, & e = (\text{Boston, Worcester}),\\ 2000, & e = (\text{Worcester, Boston}),\\ 1800, & e = (\text{Worcester, Springfield}),\\ 1800, & e = (\text{Springfield, Worcester}),\\ 1500, & e = (\text{Springfield, Albany}),\\ 1500, & e = (\text{Albany, Springfield}). \end{cases}$$

Definition 3.3 (Transportation Network HyperGraph)**.** Let $G = (V, E, \tau, \ell, \kappa)$ be a Transportation Network Graph. Its *Transportation Network HyperGraph* is

$$H = \big(V,\ \mathcal{E},\ \tau_H,\ \ell_H,\ \kappa_H\big),$$

where

$$\mathcal{E} = \Big\{ \{u, v\} : (u, v) \in E \Big\} \cup$$
$$\Big\{ \{i\} \cup \{ j : (i, j) \in E\} : \big|\{ j : (i, j) \in E\}\big| \geq 2 \Big\} \cup$$
$$\Big\{ \{i\} \cup \{ j : (j, i) \in E\} : \big|\{ j : (j, i) \in E\}\big| \geq 2 \Big\}.$$

Each hyperedge $e \in \mathcal{E}$ is assigned

$$\tau_H(e) = \begin{cases} \tau(u, v), & e = \{u, v\}, \\ \max\limits_{(i,j)\in E,\ j\in e} \tau(i, j), & e = \{i\} \cup \{j : (i, j) \in E,\ |\cdot| \geq 2\}, \\ \max\limits_{(j,i)\in E,\ j\in e} \tau(j, i), & e = \{i\} \cup \{j : (j, i) \in E,\ |\cdot| \geq 2\}, \end{cases}$$

$$\ell_H(e) = \begin{cases} \ell(u, v), & e = \{u, v\}, \\ \max\limits_{(i,j)\in E,\ j\in e} \ell(i, j), & e = \{i\} \cup \{j : (i, j) \in E,\ |\cdot| \geq 2\}, \\ \max\limits_{(j,i)\in E,\ j\in e} \ell(j, i), & e = \{i\} \cup \{j : (j, i) \in E,\ |\cdot| \geq 2\}, \end{cases}$$

$$\kappa_H(e) = \begin{cases} \kappa(u, v), & e = \{u, v\}, \\ \sum\limits_{(i,j)\in E,\ j\in e} \kappa(i, j), & e = \{i\} \cup \{j : (i, j) \in E,\ |\cdot| \geq 2\}, \\ \sum\limits_{(j,i)\in E,\ j\in e} \kappa(j, i), & e = \{i\} \cup \{j : (j, i) \in E,\ |\cdot| \geq 2\}. \end{cases}$$

Example 3.4 (Transportation Network HyperGraph of the I-90 Corridor)**.** Let $G = (V, E, \tau, \ell, \kappa)$ be the I-90 segment graph from Example 4.2. The corresponding Transportation Network HyperGraph

$$H = \big(V,\ \mathcal{E},\ \tau_H,\ \ell_H,\ \kappa_H\big)$$

is given by

$$V = \{\text{Boston, Worcester, Springfield, Albany}\},$$
$$\mathcal{E} = \big\{\{\text{Boston, Worcester}\},\ \{\text{Worcester, Springfield}\},\ \{\text{Springfield, Albany}\},$$
$$\{\text{Worcester, Boston, Springfield}\},\ \{\text{Springfield, Worcester, Albany}\}\big\}.$$

The first three are binary hyperedges corresponding to directed links in E, and the last two are junction hyperedges at nodes with two outgoing roads.

The travel-time labeling τ_H is

$$\tau_H(e) = \begin{cases} 1.25, & e = \{\text{Boston, Worcester}\}, \\ 1.00, & e = \{\text{Worcester, Springfield}\}, \\ 1.50, & e = \{\text{Springfield, Albany}\}, \\ \max\{1.25,\ 1.00\} = 1.25, & e = \{\text{Worcester, Boston, Springfield}\}, \\ \max\{1.00,\ 1.50\} = 1.50, & e = \{\text{Springfield, Worcester, Albany}\}. \end{cases}$$

The distance labeling ℓ_H is

$$\ell_H(e) = \begin{cases} 113, & e = \{\text{Boston, Worcester}\}, \\ 88, & e = \{\text{Worcester, Springfield}\}, \\ 137, & e = \{\text{Springfield, Albany}\}, \\ \max\{113,\ 88\} = 113, & e = \{\text{Worcester, Boston, Springfield}\}, \\ \max\{88,\ 137\} = 137, & e = \{\text{Springfield, Worcester, Albany}\}. \end{cases}$$

The capacity labeling κ_H is

$$\kappa_H(e) = \begin{cases} 2000, & e = \{\text{Boston, Worcester}\}, \\ 1800, & e = \{\text{Worcester, Springfield}\}, \\ 1500, & e = \{\text{Springfield, Albany}\}, \\ 2000 + 1800 = 3800, & e = \{\text{Worcester, Boston, Springfield}\}, \\ 1800 + 1500 = 3300, & e = \{\text{Springfield, Worcester, Albany}\}. \end{cases}$$

Theorem 3.5. *The structure $H = (V, \mathcal{E}, \tau_H, \ell_H, \kappa_H)$ of Definition 3.3 is a hypergraph that* generalizes *the Transportation Network Graph G in the sense that its 2-section recovers the undirected version of G.*

Proof. First, each $e \in \mathcal{E}$ is by construction a nonempty subset of V, so H is a hypergraph.

Next, form the 2-section (clique-expansion) graph $\Gamma(H) = (V, E')$ with

$$E' = \big\{\{x, y\} \subseteq V : \exists\, e \in \mathcal{E},\ \{x, y\} \subseteq e\big\}.$$

— If $(u, v) \in E$, then $\{u, v\} \in \mathcal{E}$, so $\{u, v\} \in E'$.

— Conversely, if $\{u, v\} \in E'$, then some $e \in \mathcal{E}$ contains $\{u, v\}$. If $e = \{u, v\}$ arises from a graph edge, we recover $(u, v) \in E$ (or (v, u)). If e is a junction hyperedge at node i, then $\{u, v\} \subseteq \{i\} \cup N(i)$, so at least one of $\{u, v\}$ corresponds to an original directed link into or out of i. Hence every pair in E' arises from some $(x, y) \in E$.

Therefore $\Gamma(H)$ coincides with the underlying undirected graph of G, proving that H indeed generalizes G. □

Definition 3.6 (Transportation Network n-SuperHyperGraph)**.** Let $G = (V, E, \tau, \ell, \kappa)$ be a Transportation Network Graph and let

$$\mathcal{E} = \big\{ e \subseteq V : e \text{ is a hyperedge of the Transportation Network HyperGraph of } G \big\}.$$

For any integer $n \geq 1$, the *Transportation Network n-SuperHyperGraph* is the tuple

$$\text{TNHG}^{(n)} = \big(V_n,\ E_n,\ \tau^{(n)},\ \ell^{(n)},\ \kappa^{(n)}\big),$$

where:

$$V_n = \mathcal{P}^n(V), \qquad E_n = \Big\{\mathcal{P}^{n-1}(e) \,\Big|\, e \in \mathcal{E}\Big\} \subseteq \mathcal{P}(V_n) \setminus \{\emptyset\},$$

and the weight-functions on each hyperedge $\mathcal{P}^{n-1}(e) \in E_n$ are propagated from the level-1 HyperGraph by

$$\tau^{(n)}\big(\mathcal{P}^{n-1}(e)\big) = \tau_H(e), \quad \ell^{(n)}\big(\mathcal{P}^{n-1}(e)\big) = \ell_H(e), \quad \kappa^{(n)}\big(\mathcal{P}^{n-1}(e)\big) = \kappa_H(e),$$

where $(\tau_H, \ell_H, \kappa_H)$ are the weight-functions of the Transportation Network HyperGraph.

Example 3.7 (Transportation Network 2-SuperHyperGraph of the I-90 Corridor)**.** Let $G = (V, E, \tau, \ell, \kappa)$ be the I-90 segment graph and let

$$\mathcal{E} = \{\{u, v\} : (u, v) \in E\} \ \cup\ \{\{i\} \cup \{j : (i, j) \in E : |\cdot| \geq 2\}\} \ \cup\ \{\{i\} \cup \{j : (j, i) \in E : |\cdot| \geq 2\}\}$$

be its hyperedge family (as in Example 4.5). For $n = 2$, the Transportation Network 2-SuperHyperGraph is

$$V_2 = \mathcal{P}^2(V), \qquad E_2 = \big\{\, \mathcal{P}(e) \setminus \{\emptyset\} \ \big|\ e \in \mathcal{E}\big\},$$

with weight-functions

$$\tau^{(2)}\big(\mathcal{P}(e) \setminus \{\emptyset\}\big) = \tau_H(e), \quad \ell^{(2)}\big(\mathcal{P}(e) \setminus \{\emptyset\}\big) = \ell_H(e), \quad \kappa^{(2)}\big(\mathcal{P}(e) \setminus \{\emptyset\}\big) = \kappa_H(e).$$

We highlight two representative superedges:

(a) Binary link between Boston and Worcester:
For the hyperedge

$$e_{BW} = \{\text{Boston}, \text{Worcester}\},$$

its level-2 superedge is

$$\mathcal{P}(e_{BW}) \setminus \{\emptyset\} = \big\{\{\text{Boston}\},\ \{\text{Worcester}\},\ \{\text{Boston}, \text{Worcester}\}\big\}.$$

This inherits

$$\tau^{(2)}(\mathcal{P}(e_{BW}) \setminus \{\emptyset\}) = \tau_H(e_{BW}) = 1.25, \quad \ell^{(2)}(\cdot) = 113, \quad \kappa^{(2)}(\cdot) = 2000.$$

(b) Junction at Worcester:
For the junction hyperedge

$$e_W = \{\text{Boston}, \text{Worcester}, \text{Springfield}\},$$

its level-2 superedge is

$$\begin{aligned}\mathcal{P}(e_W) \setminus \{\emptyset\} = \{&\{\text{Boston}\}, \{\text{Worcester}\}, \{\text{Springfield}\},\\ &\{\text{Boston}, \text{Worcester}\}, \{\text{Boston}, \text{Springfield}\}, \{\text{Worcester}, \text{Springfield}\},\\ &\{\text{Boston}, \text{Worcester}, \text{Springfield}\}\}.\end{aligned}$$

This superedge carries

$$\tau^{(2)}(\mathcal{P}(e_W) \setminus \{\emptyset\}) = \tau_H(e_W) = \max\{1.25, 1.00\} = 1.25,$$

$$\ell^{(2)}(\cdot) = \ell_H(e_W) = \max\{113, 88\} = 113, \quad \kappa^{(2)}(\cdot) = \kappa_H(e_W) = 2000 + 1800 = 3800.$$

Example 3.8 (Transportation Network 2-SuperHyperGraph of the Amtrak Northeast Corridor)**.** Let $G = (V, E, \tau, \ell, \kappa)$ model the Amtrak Northeast Corridor with

$$V = \{\text{Washington},\ \text{Philadelphia},\ \text{NewYork},\ \text{Boston},\ \text{Albany}\},$$

$$E = \{(\text{Washington}, \text{Philadelphia}),\ (\text{Philadelphia}, \text{NewYork}),\ (\text{NewYork}, \text{Boston}),\ (\text{NewYork}, \text{Albany})\}.$$

Typical parameters are

$$\tau(e) = \begin{cases} 1.5, & e = (\text{Washington}, \text{Philadelphia}),\\ 1.25, & e = (\text{Philadelphia}, \text{NewYork}),\\ 4.00, & e = (\text{NewYork}, \text{Boston}),\\ 2.50, & e = (\text{NewYork}, \text{Albany}),\end{cases}$$

$$\ell(e) = \begin{cases} 225, & e = (\text{Washington}, \text{Philadelphia}),\\ 150, & e = (\text{Philadelphia}, \text{NewYork}),\\ 338, & e = (\text{NewYork}, \text{Boston}),\\ 260, & e = (\text{NewYork}, \text{Albany}),\end{cases}$$

$$\kappa(e) = \begin{cases} 20, & e = (\text{Washington}, \text{Philadelphia}),\\ 25, & e = (\text{Philadelphia}, \text{NewYork}),\\ 15, & e = (\text{NewYork}, \text{Boston}),\\ 10, & e = (\text{NewYork}, \text{Albany}).\end{cases}$$

The hyperedge set for G is

$$\mathcal{E} = \big\{\{\text{Washington}, \text{Philadelphia}\},\ \{\text{Philadelphia}, \text{NewYork}\},$$

$$\{\text{NewYork}, \text{Boston}\},\ \{\text{NewYork}, \text{Albany}\},\ \{\text{NewYork}, \text{Boston}, \text{Albany}\}\big\}.$$

For $n = 2$, the 2-SuperHyperGraph is

$$V_2 = \mathcal{P}^2(V),$$

$$E_2 = \big\{\mathcal{P}(e) \setminus \{\emptyset\} \mid e \in \mathcal{E}\big\},$$

with weight-functions propagated by

$$\tau^{(2)}\big(\mathcal{P}(e) \setminus \{\emptyset\}\big) = \tau_H(e),$$

$$\ell^{(2)}\big(\mathcal{P}(e) \setminus \{\emptyset\}\big) = \ell_H(e),$$

$$\kappa^{(2)}\big(\mathcal{P}(e) \setminus \{\emptyset\}\big) = \kappa_H(e).$$

Two illustrative superedges are:

- For $e_{PhNY} = \{\text{Philadelphia, NewYork}\}$,

$$\mathcal{P}(e_{PhNY}) \setminus \{\emptyset\} = \{\{\text{Philadelphia}\},\ \{\text{NewYork}\},\ \{\text{Philadelphia, NewYork}\}\},$$

  carrying $\tau^{(2)} = 1.25$, $\ell^{(2)} = 150$, $\kappa^{(2)} = 25$.

- For the junction $e_{NY} = \{\text{NewYork, Boston, Albany}\}$,

$$\mathcal{P}(e_{NY}) \setminus \{\emptyset\} = \{\{\text{NewYork}\}, \{\text{Boston}\}, \{\text{Albany}\}, \ldots, \{\text{NewYork, Boston, Albany}\}\},$$

  carrying $\tau^{(2)} = \max\{4.00, 2.50\} = 4.00$, $\ell^{(2)} = \max\{338, 260\} = 338$, $\kappa^{(2)} = 15 + 10 = 25$.

Theorem 3.9. *For each integer $n \geq 1$, $\mathrm{TNHG}^{(n)}$ is an n-SuperHyperGraph, and it* generalizes *the Transportation Network HyperGraph in the sense that*

$$\mathrm{TNHG}^{(1)} \ \cong\ \big(V,\ \mathcal{E},\ \tau_H,\ \ell_H,\ \kappa_H\big).$$

Proof. (i) n-SuperHyperGraph structure: By construction,

$$V_n = \mathcal{P}^n(V) \quad \text{and} \quad E_n \subseteq \mathcal{P}(V_n) \setminus \{\emptyset\}.$$

Thus $\mathrm{TNHG}^{(n)} = (V_n, E_n, \tau^{(n)}, \ell^{(n)}, \kappa^{(n)})$ satisfies the definition of an n-SuperHyperGraph.

(ii) Generalization of the HyperGraph: When $n = 1$, note that $\mathcal{P}^0(e) = e$ for each $e \in \mathcal{E}$. Hence

$$V_1 = \mathcal{P}^1(V) = \mathcal{P}(V), \qquad E_1 = \{\mathcal{P}^0(e) \mid e \in \mathcal{E}\} = \mathcal{E},$$

and $\tau^{(1)}(e) = \tau_H(e)$, $\ell^{(1)}(e) = \ell_H(e)$, $\kappa^{(1)}(e) = \kappa_H(e)$. There is a natural bijection $\iota : V \to \{\{v\} : v \in V\} \subseteq V_1$, identifying each original node with its singleton. Under this identification, the level-1 superhypergraph $\mathrm{TNHG}^{(1)} = (V_1, E_1, \tau^{(1)}, \ell^{(1)}, \kappa^{(1)})$ is isomorphic to the Transportation Network HyperGraph $(V, \mathcal{E}, \tau_H, \ell_H, \kappa_H)$.

Therefore, for all $n \geq 1$, the sequence of structures $\mathrm{TNHG}^{(n)}$ forms a hierarchy of valid n-SuperHyperGraphs that at $n = 1$ recover and thus generalize the level-1 Transportation Network HyperGraph. □

4 Conclusion and Future Work

In this work, we have generalized traditional river and transportation network graphs into the hypergraph and superhypergraph realms by defining two novel structures—River Network HyperGraph and Transportation Network HyperGraph—and their hierarchical extensions, the River Network SuperHyperGraph and Transportation Network SuperHyperGraph.

Looking ahead, we plan to enrich these models by incorporating uncertainty frameworks such as Fuzzy Sets [43–45], Intuitionistic Fuzzy Sets [46,47], HyperFuzzy Sets [48–50], Vague sets [51,52], Hesitant Fuzzy Sets [53,54], Picture Fuzzy Sets [55–58], Neutrosophic Sets [59,60], and Plithogenic Sets [61,62]. We also hope that further progress will be made in computational experiments and algorithmic investigations.

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Data Availability

No data were generated or analyzed in the course of this theoretical investigation. We encourage future empirical studies to test and extend the ideas presented here.

Ethical Approval

Not applicable.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Disclaimer

The concepts introduced in this paper are theoretical and have not yet been validated empirically. Readers should independently verify all references and may encounter inadvertent errors. The views expressed here are solely those of the authors and do not necessarily reflect the positions of their affiliated organizations.

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